On Pancyclic Claw-Free Graphs

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ABSTRACT. In this paper, we show that if G is a connected SN_2-locally connected claw-free graph with $\delta(G) \geq 3$, which does not contain an induced subgraph H isomorphic to either G_1 or G_2 such that $N_1(x, G)$ of every vertex x of degree 4 in H is disconnected, then every N_2-locally connected vertex of G is contained in a cycle of all possible lengths and so G is pancyclic. Moreover, G is vertex pancyclic if G is N_2-locally connected.

1 Introduction

In this paper we deal with finite simple graphs. Let G be a graph of order n. We denote by $\delta(G)$ the minimum degree of G. For a vertex v of G, the neighborhood of v, defined in the obvious sense, i.e., as the induced subgraph on the set of all vertices that are adjacent to v, will be called the neighborhood of the first type of v in G and denoted by $N_1(v, G)$ or briefly, $N_1(v)$. We say that an edge $xy \in E(G)$ is adjacent to v if $x \neq v \neq y$ and x or y is adjacent to v. We define the neighborhood of the second type of v in G (denoted by $N_2(v, G)$, or briefly, $N_2(v)$) as the edge-induced subgraph on the set of all edges that are adjacent to v. We say that a vertex v is locally connected if its neighborhood $N_1(v)$ is a connected graph. G is called locally connected if every vertex of G is locally connected. G is called S-locally connected if every vertex-cut of G contains a locally connected vertex. Obviously, every locally connected graph is S-locally connected. Analogously, a vertex v is N_2-locally connected if the second-type neighborhood $N_2(v)$ is connected. G is N_2-locally connected if every vertex in G is N_2-locally connected. G is SN_2-locally connected if every vertex-cut of G contains an N_2-locally connected vertex. Obviously, every N_2-locally connected graph is SN_2-locally connected, every locally connected graph is N_2-locally connected, and every S-locally connected graph is SN_2-locally connected. G is called claw-free if it does not contain a copy of $K_{1,3}$ as an induced subgraph. G is called pancyclic if G contains a cycle of all possible lengths. G is vertex pancyclic if every vertex of G is contained in a cycle of all possible lengths. A cycle C in G is extendable if there exists a cycle C' in G such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)|+1$. G is called full cycle extendable if G contains at least one cycle and every nonhamiltonian cycle in G is extendable and every vertex of G lies on a triangle of G. Obviously, if G is full cycle extendable then G is vertex pancyclic.

For a subgraph H of a graph G and a subset S of $V(G)$, we denote by $G-H$ and $G[S]$ the induced subgraphs of G by $V(G)-V(H)$ and S, respectively, and we denote by $N_H(S)$ the set of all vertices v in H adjacent to some vertex of S. Let $d_H(S) = |N_H(S)|$. For a cycle C with a fixed orientation, and two vertices x and y on C, we define the segment $C[x,y]$ to be the set of vertices on C from x to y (including x and y) according to the orientation. Let $C(x,y) = C[x,y] - \{x,y\}$, and x^+ and x^- denote the successor and predecessor of x according to the orientation, respectively. We say that xy is a chord on C if x, $y \in V(C)$, $x \neq y^+$, y^-, y and $xy \in E(G)$. A cycle C of G is called chord-free if there is no chord on C. We call x and y on C consecutive vertices if $x = y^+$ or $x = y^-$. Other notation and terminology not defined here can be found in [1].

There have been many papers dealing with hamiltonicity in claw-free graphs. M.M. Matthews and D. P. Sumner [7] proved the following result.

Theorem A. (Mattthews and Sumner, [7]). *Every 2-connected claw-free graph G of order n contains a cycle of length at least* $\min\{2\delta(G)+4, n\}$, *and is hamiltonian if* $n \le 3\delta(G)+2$.

Let D be the set of all the graphs defined as follows:
Any graph H of order at most $(9\delta(H))/2-1$ in D can be decomposed into three disjoint hamiltonian subgraphs H_1, H_2 and H_3 such that $E_H(H_i, H_j) = \{u_iu_j, v_iv_j\}$ for $i \neq j$ and $i, j = 1,2,3$ (where $u_i \neq v_i \in V(H_i)$ for $i = 1,2,3$, and $E_H(H_i, H_j)$ denotes the set $\{xy \in E(H) : x \in V(H_i) \text{ and } y \in V(H_j)\}$) and at most one subgraph H_i has at most $2\delta(H)-2$ vertices.

The author [5] improved Theorem A and proved the following result in 1992.

Theorem B. (M.Li, [5]). *Every 2-connected claw-free graph $G \notin D$ of order n contains a cycle of length at least* $\min\{3\delta(G)+2, n\}$, *and is hamiltonian if* $n \le 4\delta(G)$.

Flandrin, Fournier and Germa [3] proved that a graph G satisfying the conditions of Theorem A is pancyclic. R. Shi [9] improved this result as follows.

Theorem C. (R.Shi, [9]). *Every 2-connected claw-free graph G of order $n(\geq 100)$ with $\sum_{i=1}^{3} d(v_i) \geq n-2$ for any three nonadjacent vertices v_1, v_2 and v_3 of G is pancyclic.*

The author [6] improved this result and showed the following result.

Theorem D. (M. Li, [6]). *Every hamiltonian claw-free graph G of order $n(\geq 100)$ with $\max\{d(u), d(v), d(w)\} \geq (n-2)/3$ for any three nonadjacent vertices u, v and w of G is pancyclic.*

Let G be a connected claw-free graph on at least three vertices. Oberly and Sumner [8] proved that G is hamiltonian if G is locally connected. Clark [2] showed that G is vertex pancyclic and Hendry [4] proved that G is fully cycle extendable if G is locally connected, and Zhang [11] proved that G is vertex pancyclic if G is S-locally connected. Let G be a connected claw-free graph without vertices of degree 1 which does not contain an induced subgraph H isomorphic to either G_1 or G_2 (Figure 1) such that $N_1(x, G)$ of every vertex x of degree 4 in H is disconnected. Z.Ryjacek [10] proved that G is Hamiltonian if G is N_2-locally connected. In this paper, we prove that G is Hamiltonian if G is SN_2-locally connected, G is pancyclic if G is SN_2-locally connected and $\delta(G) \geq 3$, and G is vertex pancyclic if G is N_2-locally connected and $\delta(G) \geq 3$.

2 Lemmas

In this section, we assume that G is a Hamiltonian, SN_2-locally connected claw-free graph of order n with $\delta(G) \geq 3$. Obviously, every vertex of G is contained in a cycle of length 3. Suppose that there exists an N_2-locally connected vertex in G such that it is contained in a cycle of length m but is not contained in a cycle of length $m+1$. Then we have $3 \leq m \leq n-2$.

In order to prove our main theorems, we need to prove the following two preliminary results.

Lemma 1. *Let u be an N_2-locally connected vertex in G such that G has a cycle C of length m which contains the vertex u but has no cycle of length $m+1$ which contains u, let $x_0 \notin V(C)$ and $x_0u \in E(G)$, and let P be a shortest path in $N_2(u)$ from x_0 to u^+ and $x_0, x_1, ..., x_k(= u^+)$ be vertices of P. Then we have*

(1) $x_ix_j \notin E(G)$ *for* $|i-j| > 1$,

(2) $u^+u^- \in E(G)$ *and* $u^+x_0, u^-x_0 \notin E(G)$.

Proof: From the minimality of P we immediately obtain (1). Since G is claw-free, by the choice of C and $G[u, u^+, u^-, x_0] \neq K_{1,3}$, we immediately know that (2) holds.

Lemma 2. *Let u, x_0, C and P satisfy the conditions of Lemma 1 and let C and x_0 be chosen so that $P = x_0x_1...x_k(= u^+)$ is shortest possible in $N_2(u)$. Then we have the following*

(1) *x_{k-1} is not adjacent to u,*

(2) *x_{k-1} is the only vertex of P that is nonadjacent to u,*

(3) *$2 \leq k \leq 3$,*

(4) *If $k = 3$, then either $x_1, x_2 \in V(C)$ or $x_1, x_2 \notin V(C)$,*

(5) *If $k = 2$, then $x_1 \notin V(C)$,*

(6) *If $k = 3$ and $x_1x_2 \in E(C)$ and $x_2^+ = x_1$, then we have $x_1^-x_1^+, x_2^-u^+, x_3^+x_2, x_3x_2^- \in E(G)$ but $ux_3^+, x_1x_2^-, u^-x_2, x_0x_2^-, x_0(x_1^+)^+, x_1^+x_2^-, x_1^+u^+, x_1u^+ \notin E(G)$.*

Proof: (1). Suppose that $x_{k-1}u \in E(G)$. By the choice of C and x_0, we obtain that x_{k-1} is on C. Clearly, $ux_{k-1}^+, ux_{k-1}^- \notin E(G)$, otherwise, let $ux_{k-1}^- \in E(G)$. Then replacing on C the edge $x_{k-1}^-x_{k-1}$ by the path $x_{k-1}^-ux_{k-1}$ and the path u^-uu^+ by the edge u^+u^- we obtain a cycle C' of same length as C, and such that if we denote $u' = x_{k-1}$ then u' is a neighbor of u on C' and in $N_2(u)$ exists a path from x_0 to u' shorter than P. This is a contradiction with the choice of C and P. Similarly, $ux_{k-1}^+ \notin E(G)$. Since $G[x_{k-1}^+, x_{k-1}^-, u, x_{k-1}] \neq K_{1,3}$, $x_{k-1}^+x_{k-1}^- \in E(G)$. Replacing on C the path $x_{k-1}^+x_{k-1}x_{k-1}^-$ by the edge $x_{k-1}^+x_{k-1}^-$ and the edge uu^+ by the path $ux_{k-1}u^+$ we again obtain a contradiction. So (1) is proved.

(2). Let x_j $(1 \leq j \leq k-2)$ be nonadjacent to u. Then $j \leq k-3$, since otherwise the edge $x_{k-2}x_{k-1}$ can not be in $N_2(u)$. By Lemma 1 (1), we have $G[x_{j-1}, x_{k-2}, x_k, u] = K_{1,3}$, a contradiction. So (2) is proved.

(3). Since $x_0u^+, x_0u^- \notin E(G)$, $k \geq 2$. If $k \geq 4$, then, by Lemma 1(1) and from (2), we have that $G[x_0, x_2, u^+, u] = K_{1,3}$, a contradiction. So (3) is proved.

(4). By (2), we have $ux_1 \in E(G)$. If $x_1 \in V(C)$ and $x_2 \notin V(C)$, then $x_1^-x_1^+ \in E(G)$, which implies that G contains a cycle C' of length $m+1$ containing u. Namely, $C' = C[u^+, x_1^-]C[x_1^+, u]x_1x_2u^+$, a contradiction. If $x_2 \in V(C)$ and $x_1 \notin V(C)$, then replacing the edge uu^+ by the path $ux_1x_2u^+$ and the path $x_2^-x_2x_2^+$ on C by the $x_2^-x_2^+$, we obtain a cycle of length $m+1$ containing u, a contradiction. So (4) is proved.

(5). Assume that $x_1 \in V(C)$, then $x_1^-x_1^+ \in E(G)$. Replacing the path $x_1^-x_1x_1^+$ on C by the edge $x_1^-x_1^+$ and the edge uu^+ by the path $ux_0x_1u^+$, we obtain a cycle of length $m+1$ containing u, a contradiction. Hence (5) is proved.

(6). Clearly, $ux_3^+ \notin E(G)$ since otherwise G has a cycle C' of length $m+1$ containing u, namely, $C' = C[x_3^+, x_2]x_3C^-[u^-, x_1]x_0ux_3^+$ (where $C^-[u^-, x_1]$ denotes a traversal of the $C[x_1, u^-]$ in the opposite sense according to the orientation of C), a contradiction. Similarly, $x_1x_2^-, u^-x_2$, $x_0x_2^-, x_0(x_1^+)^+, x_1^+x_2^-, x_1^+u^+, x_1u^+ \notin E(G)$. Since $G[u^+, x_1, x_2^-, x_2] \neq K_{1,3}$, $u^+x_2^- \in E(G)$. Similarly, $x_1^-x_1^+, x_3^+x_2, x_3x_2^- \in E(G)$. Hence (6) is proved and thereby the proof of the lemma is completed.

In the proof of our main theorems, we use the following lemma.

Lemma 3. [9]. *Let C be a cycle in a connected graph G and $|V(C)| = t$. If P is a path in $G - C$ and $s = |V(P)| \geq 1$ such that v has consecutive neighbors on C for any vertex v of P, then G has a cycle of length r for each r (where $t \leq r \leq s + t$).*

3 Main Results

In this section, we will prove our main results.

Theorem 1. *Let G be a connected, SN_2-locally connected claw-free graph without vertices of degree 1, which does not contain an induced subgraph H isomorphic to either G_1 or G_2 (Figure 1) such that $N_1(x, G)$ of every vertex x of degree 4 in H is disconnected. Then G is Hamiltonian.*

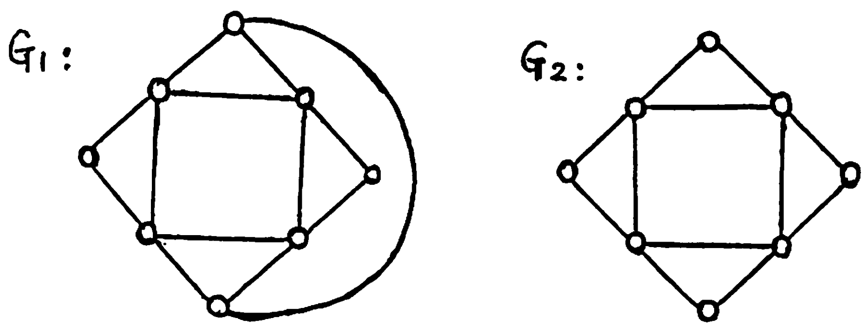

Figure 1

Proof: The proof of Theorem 1 is a straightforward extension of the main result of [10] using analogous approach and ideas to those of [10] and [11]. The details are therefore left to the reader. In fact, its proof is also similar to the following one of Theorem 2.

Theorem 2. *Let G be a graph of order n satisfying the conditions of Theorem 1 with $\delta(G) \geq 3$. Then every N_2-locally connected vertex of G is contained in a cycle of all possible lengths.*

Proof: Assume that the Theorem does not hold. Since G is claw-free and $\delta(G) \geq 3$, every vertex of G is contained in a cycle of length 3. Suppose that an N_2-locally connected vertex v in G is contained in a cycle C of length m but is not contained in any cycle of length $m+1$. Then we have $3 \leq m \leq n-2$. Since G is SN_2-locally connected, we can find a vertex u on C such that u is N_2-locally connected and an edge ux_0 such that x_0 is not on C, which implies that we can find a shortest path P in $N_2(u)$ from x_0 to one of u^+ or u^-. Without loss of generality assume that P is a path from x_0 to u^+ and that $u^- \notin V(P)$. Let the cycle C and x_0u be chosen so that $P = x_0x_1...x_k(=u^+)$ is shortest possible in $N_2(u)$. From Lemma 2(3), we know that $k=2$ or $k=3$. So we next consider two cases.

Case 1. $k=2$.

From Lemma 2(5), we have $x_1 \notin V(C)$. Replacing the edge uu^+ on C by the path $ux_0x_1u^+(=x_2)$, we obtain a cycle $C' = C[u^+(=x_2), u^-]ux_0x_1u^+$ of length $m+2$ containing v. Let the orientation of C' be the same as that of C. Recall ux_2, u^-x_2, ux_2^+ are edges in $E(G)$. Let $R = G - C$ and $R' = G - C'$. In order to prove this case, we first verify the following eight claims.

Claim 1. *For any vertex $x(\neq v)$ on C', we have $x^+x^- \notin E(G)$.*

Proof: Otherwise, G has a cycle of length $m+1$ containing v, a contradiction.

Claim 2. *For any vertex $x(\neq v, u, u^+(=x_2))$ on C, we have $d_{R'}(x) = 0$.*

Proof: Otherwise, let $y_0 \in R'$ and $y_0x \in E(G)$. By Claim 1, we know $y_0x^- \in E(G)$ or $y_0x^+ \in E(G)$. Without loss of generality assume that $y_0x^+ \in E(G)$. Then replacing the edge xx^+ on C by the path xy_0x^+, we obtain a cycle of length $m+1$ containing v, a contradiction.

Claim 3. *There is no chord on C whose one end-vertex is v.*

Proof: Otherwise, let $yv \in E(G)$ such that $x_0, x_1 \notin C'(y, v)$ and $vx \notin E(G)$ for any vertex $x(\neq v^-)$ on $C'(y, v)$. Then we have the following fact:

$$\text{The cycle } C'' = C'[y, v]y \text{ is chord-free.}$$

Indeed, let $ab \in E(G)$ such that $a, b \in C[y, v)$ and the cycle $C_0 = C'[a, b]a$ is chord-free. By Claims 1 and 2 and $\delta(G) \geq 3$, we know that for any vertex $w \in C'(a, b)(w \neq x_2)$, there is a vertex $w' \in C'(b, a)$ such that $ww' \in E(G)$. Again by Claim 1, we obtain that w has consecutive neighbors on $C'(b, a)$ since G is claw-free.

If $x_2 \in C'(a, b)$, then $x_2 = a^+$ and $x_2u, x_2u^- \in E(G)$, that is, x_2 has consecutive neighbors on $C'(b, a)$. From Lemma 3, we know that G has a cycle of length $m+1$ containing v, a contradiction. Hence $C'' = C'[y, v]y$ is chord-free.

To prove Claim 3, we obtain a contradiction using a similar argument to C'' (instead of C_0).

Claim 4. *There is no segment (say $C'(a,b)$) on C' such that $ab \in E(G), v, x_0, x_1 \notin C'(a,b), |C'(a,b)| \geq 1$ and $C'' = aC'[a^+, b]a$ is chord-free.*

Proof: Otherwise, by Claim 3, we know that $xv \notin E(G)$ for any $x \in C'(a,b)$. By Claims 1 and 2, we obtain that for any vertex $x \in C'(a,b)(x \neq x_2)$ there is a vertex y on $C'(b,a)$ such that $xy \in E(G)$. Again from Claim 1, we must have $xy^- \in E(G)$ or $xy^+ \in E(G)$. Hence x has consecutive neighbors on $C'(b,a)$, and so has x_2 if $x_2 \in C'(a,b)$. By Lemma 3, we get that G has a cycle of length $m+1$ containing v. Thus the Claim holds.

Claim 5. *There is no segment (say $C'(a,b)$) on C' such that $|C'(a,b)| \geq 2$, awb is a path (where $w \in V(R')$), $v, x_0, x_1 \notin C'(a,b)$ and $C'[a,b]wa$ is a chord-free cycle.*

Proof: Otherwise, by a similar argument as in the proof of Claim 3, we get that each x on $C'(a,b)$ has consecutive neighbors on $C'(b,a)$. By Lemma 3 and $|C'(a,b)| \geq 2$, we obtain that G has a cycle of length $m+1$ containing v, a contradiction.

Claim 6. $d_{C'}(x_0) = d_{C'}(x_1) = 2$.

Proof: Let $d_{C'}(x_1) \geq 3$. Then we can choose a vertex w on C' such that $wx_1 \in E(G)$, $C'(w, x_1^-)$ does not contain the vertex v and $w'x_1 \notin E(G)$ for any vertex w' on $C'(w, x_1^-)$. By $\delta(G) \geq 3$ and Claims 1, 2 and 4, we get that for any vertex q ($\neq u$) on $C'(w, x_0)$ there is a vertex q' on $C'(x_1, w)$ such that $qq' \in E(G)$. Again from Claims 1 and 3 we have $q' \neq v$ and $q'^-q \in E(G)$ or $q'^+q \in E(G)$. Hence q has consecutive neighbors on $C'(x_1, w)$. Clearly, u has consecutive neighbors on $C'(x_1, w)$. Let $P = C'(w, x_0)$. Then by Lemma 3, we can easily get a contradiction. Thus $d_{C'}(x_1) = 2$. Similarly, $d_{C'}(x_0) = 2$. Hence the claim is proved.

Claim 7. *We have $u \neq v$.*

Proof: If $u = v$, then $m \geq 5$, otherwise, we have $m = 3$, or $m = 4$. If $m = 3$, then $C'' = ux_0x_1u^+$ is a cycle of length 4 containing v. Hence $m = 4$. Since $\delta(G) \geq 3$, by Claim 6, there is a vertex w in R' such that $wx_0 \in E(G)$. Since $G[x_0, w, u, x_1] \neq K_{1,3}$, we have $uw \in E(G)$ or $x_1w \in E(G)$. It follows that there is a cycle of length 5 containing v.

Since $m \geq 5$, there is at least one vertex x on C' such that $x \notin \{u^-, u, x_0, x_1, x_2, x_2^+\}$. By Claims 1, 2 and 6 and $\delta(G) \geq 3$, there exists a vertex q on C' such that $C'(x,q)$ (or $C'(q,x)$) contains no vertices of $\{x_0, x_1, u\}$ and $xq \in E(G)$. Choose q as close to x as possible. Then, by Claim 1, we have $|C'(x,q)| \geq 1$ (or $|C'(q,x)| \geq 1$). By a similar argument to Claim 4 and by Lemma 3,we know that the cycle $C'' = C'[x,q]x$ (or $C'[q,x]q$) is chord-free, which contradicts Claim 4. So Claim 7 is proved.

Claim 8. *$v \notin \{u, u^-, x_2, x_2^+\}$ and $v^+v^- \in E(G)$.*

Proof: Since ux_2, u^-x_2, $ux_2^+ \in E(G)$, v is not one of the u, u^-, x_2, x_2^+ on C'. By Claim 3 and Claim 6, $d_{C'}(v) = 0$. Since $\delta(G) \geq 3$, there is v' such that $v'v \in E(G)$ and $v' \in V(R')$. From $G[v', v, v^-, v^+] \neq K_{1,3}$, we have $v^+v^- \in E(G)$.

We next complete the proof of this case.

By Claims 3 and 6, there is a vertex $y_0 \in V(R')$ such that $y_0 v \in E(G)$. Since v is N_2-locally connected, we can find a shortest path Q in $N_2(v)$ from y_0 to one of v^+ or v^-. We may assume that without loss of generality that Q is a path from y_0 to v^+ and that $v^- \notin V(Q)$. Let $Q = y_0 y_1 y_2 \ldots y_h$. From the minimality of Q it follows that no y_i, y_j can be adjacent for $|i-j| > 1$. Clearly, by Claim 6, $v^+, v^- \notin \{x_0, x_1\}$ and $h \geq 2$. By a similar argument to Lemma 2(3), we have $h \leq 3$. Clearly, $(v^+)^+ \notin \{x_0, x_1\}$. Furthermore, we have $y_{h-1} \in V(C')$ (otherwise, since $v(v^+)^+ \notin E(G)$ and $G[v^+, y_{h-1}, v, (v^+)^+] \neq K_{1,3}$, $vy_{h-1} \in E(G)$ or $(v^+)^+ y_{h-1} \in E(G)$, it follows that G has a cycle of length $m+1$ containing v since the vv^+ on C is replaced by the path $vy_{h-1}v^+$ or the edge $v^+(v^+)^+$ on C is replaced by the path $v^+ y_{h-1} (v^+)^+$).

(1). $y_{h-1} \in C'(x_1, v)$.
Otherwise, let $y_{h-1} \in C'(v^+, x_0)$. Suppose first that $y_{h-1} = (v^+)^+$. Then, if $h = 2$, then $y_{h-1}^+ y_{h-1}^- \in E(G)$ because $G[y_{h-1}, y_{h-1}^+, y_{h-1}^-, y_0] \neq K_{1,3}$ and $y_{h-1}^+ y_0, y_{h-1}^- y_0 \notin E(G)$. Replacing on C' the path $y_{h-1}^+ y_{h-1} y_{h-1}^-$ by the edge $y_{h-1}^+ y_{h-1}^-$, we obtain a cycle of length $m+1$ containing v, a contradiction. If $h = 3$, by Claim 3, we have $y_{h-1}v \notin E(G)$. Since $y_1 y_2 \in E(N_2(v))$, $y_1 v \in E(G)$. It follows that $y_1 \notin V(C')$ by Claims 3 and 6. Replacing on C the path vv^+y_2 by the path $vy_0y_1y_2$, we obtain a cycle of length $m+1$ containing v, a contradiction. Hence we get $y_{h-1} \neq (v^+)^+$.

Since $y_{h-1}v^+ (= y_h) \in E(G)$ and $\delta(G) \geq 3$, by Lemma 3 and a similar argument to Claim 4, we can get that the cycle $C'' = C'[v^+, y_{h-1}]v^+$ is chord-free, which contradicts Claim 4.

(2). If $h = 3$, then $y_2 v \notin E(G)$ by Claim 3, which implies $y_1 v \in E(G)$ since otherwise we have that $y_1 y_2 \notin N_2(v)$.

(3). If $h = 3$, then $y_1 \notin V(C')$ by (2) and Claim 3.

(4). If $h = 3$, then $y_2 = x_2$, otherwise, since $y_2^- y_2^+ \notin E(G)$ and $G[y_2, y_1, y_2^+, y_2^-] \neq K_{1,3}$, $y_2^- y_1 \in E(G)$ or $y_2^+ y_1 \in E(G)$, say $y_2^- y_1 \in E(G)$. Then replacing the edge $y_2^- y_2$ on C by the path $y_2^- y_1 y_2$, we get a cycle of length $m+1$ containing v, a contradiction. Note that $y_2 \neq u$ by (1), and $y_2 \neq x_0, x_1$ by Claim 6.

(5). By a similar argument to (4), we obtain that if $h = 2$, then $y_1 = x_2$.

From (1)-(5), we know that there is a path $Q' = vv'x_2$ such that $v' \in V(R')$ (where $v' = y_1$ or y_0).

Clearly, $x_0, x_1, v \notin C'(x_2, v)$. If there is an edge $ab \notin E(C)$ in $E(G)$ such that $a, b \in C'[x_2, v]$, then, by Lemma 3 and a similar argument to Claim 4, we know that the cycle $C'' = C'[a, b]a$ (or $C'[b, a]b$) is chord-free, which contradicts Claim 4. Hence for any two distinct vertices a and b ($\neq a^-, a^+$) we have $ab \notin E(G)$ on $C'[x_2, v]$. If $|C'(x_2, v)| = 1$, then $v^- = x_2^+$. Replacing the path $x_2v^-vv^+$ on C by the path $x_2v'vv^-v^+$, we obtain a cycle of length $m+1$ containing v. Hence $|C'(x_2, v)| \geq 2$. Since $\delta(G) \geq 3$, by Lemma 3 and a similar argument to Claim 4, we get that the cycle $C'' = C'[x_2, v]v'x_2$ is chord-free, which contradicts Claim 5. So the Case is proved.

Case 2. $k = 3$.
From Lemma 2(4), we know that either $x_1, x_2 \in V(C)$ or $x_1, x_2 \notin V(C)$. If $x_2, x_1 \notin V(C)$, clearly $x_2u \notin E(G)$ (since otherwise replacing the edge ux_3 on C by the path ux_2x_3, we obtain a cycle of length $m+1$ containing v). Since $x_1x_2 \in N_2(u)$, $x_1u \in E(G)$. So G has a cycle of length $m+2$ containing v. By a similar proof to Case 1, we can get a contradiction. Hence let $x_2, x_1 \in V(C)$.

Then we can assume without loss of generality that $x_1x_2 \in E(C)$.

Suppose, on the contrary, that $x_1x_2 \notin E(C)$. We only consider this case: $x_2 \in C(u, x_1)$ (and the case: $x_2 \in C(x_1, u)$ is similar). Clearly, $x_2^+ \neq x_1$ and $x_1^-x_1^+ \in E(G)$. If $x_2^-x_2^+ \in E(G)$, then replacing on C the path $x_1^-x_1x_1^+$ by the edge $x_1^-x_1^+$, the path $x_2^-x_2x_2^+$ by the edge $x_2^-x_2^+$ and the edge uu^+ by the path $ux_0x_1x_2u^+$, a cycle of length $m+1$ containing v should arise; So $x_2^-x_2^+ \notin E(G)$. Since $G[x_1, x_2, x_2^+, x_2^-] \neq K_{1,3}$, x_1 is adjacent either to x_2^+ or to x_2^-, say $x_1x_2^- \in E(G)$. Then replacing the edge $x_2^-x_2$ on C by the path $x_2^-x_1x_2$ and the path $x_1^-x_1x_1^+$ on C by the edge $x_1^-x_1^+$, we obtain a cycle C' of same length as C containing v and such that $x_1x_2 \in E(C')$.

Obviously $x_2 = x_1^-$ since otherwise deleting from C the edges $x_1x_1^+$ and ux_3 and adding the edge $x_3x_1^+$ and the path ux_0x_1, we obtain a cycle of length $m+1$ containing v.

By Lemma 2(6), we know that the induced subgraph of G on the set $\{x_0, x_1, x_2, x_1^+, x_2^-, u^+, u, u^-\}$ is isomorphic to either G_1 or to G_2 in Figure 1 (see Figure 4). It remains to prove that the first type neighborhoods of the vertices u, x_1, x_2 and u^+ are disconnected.

(a). $N_1(u)$ is disconnected since if it were connected then we could obtain a contradiction in the same way as in the proof of the main Theorem of [2].

(b). The disconnectedness of $N_1(x_1)$ can be verified in the same way as in (a) considering x_1 instead of u.

Let $P_1 = C[x_1^+, u^-]$ and $P_2 = C[u^+, x_2^-]$. Then

(c). If $y \in V(P_1)$ is adjacent to both u^+ and x_2^-, then $y^+y^- \in E(G)$. If $y \in V(P_2)$ is adjacent to both x_1^+ and x_1, then y^+ and y^- are adjacent.

Indeed, if $y^-x_2^- \in E(G)$, then deleting from C the edges $x_2^-x_2$, $x_1x_1^+$, y^-y, uu^+, adding the edges $y^-x_2^-, x_2x_1^+, yu^+$ and the path ux_0x_1 we could obtain a cycle of length $m+1$ containing v, a contradiction. So $y^-x_2^- \notin E(G)$. Similarly, $y^+x_2^- \notin E(G)$. Since $G[y, y^+, y^-, x_2^-] \neq K_{1,3}$, $y^+y^- \in E(G)$. Similarly, we can prove the remainder.

(d). We show that $N_1(x_2)$ is disconnected.

Suppose that, on the contrary, that $N_1(x_2)$ is connected. Since $x_1x_1^+$ and $u^+x_2^- \in N_1(x_2)$, there is a path in $N_1(x_2)$ that joins one of x_1, x_1^+ with one of x_2^-, u^+. Let Q be a shortest path in $N_1(x_2)$ from x_1 or x_1^+ to x_2^- or u^+ and denote by $y_0, y_1, ..., y_p$ its vertices (i.e., $y_0 = x_1$ or $y_0 = x_1^+$ and $y_p = x_2^-$ or $y_p = u^+$). From the minimality of Q it follows that no y_i, y_j are adjacent for $|i-j| > 1$ and hence $p \leq 3$ (otherwise $\{y_0, y_2, y_p, x_2\}$ should induce $K_{1,3}$). On the other hand, $p \geq 2$, since by Lemma 2(6), none of x_1, x_1^+ can be adjacent to any of x_2^-, u^+. So either $p=2$ or $p=3$. Next, consider two cases.

Case d_1 . $p=2$.
Obviously, $y_1 \in V(C)$ since otherwise we get a cycle of length $m+1$ containing v and y_1.

Suppose first that $y_1^+y_1^- \in E(G)$. Then, if $Q = x_1y_1x_2^-$ and $y_1 \in V(P_1)$, then $C' = C[x_1^+, y_1^-]C[y_1^+, u]x_0x_1y_1C^-[x_2^-, u^+]x_2x_1^+$ is a cycle of length $m+1$ containing v, a contradiction, where $C^-[x_2^-, u^+]$ is a traversal of the $C[u^+, x_2^-]$ in the opposite sense according to the orientation. Similarly, we can get a contradiction in the remaining cases (i.e. $Q = x_1y_1u^+$, $Q = x_1^+y_1x_1^-$ and $Q = x_1^+y_1u^+$ and also for $y_1 \in V(P_2)$). Hence $y_1^+y_1^- \notin E(G)$.

Obviously, either $y_1x_1, y_1x_1^+ \in E(G)$ or $y_1u^+, y_1x_2^- \in E(G)$ since otherwise (say $y_1x_1, y_1u^+ \notin E(G)$) $G[y_1, x_1, u^+, x_2] = K_{1,3}$. Hence if $y_1u^+, y_1x_2^- \in E(G)$, then by (c) and $y_1^+y_1^- \notin E(G)$ we have $y_1 \in V(P_2)$. If $y_1x_1, y_1x_1^+ \in E(G)$, then by (c) we have $y_1 \in V(P_1)$. Namely, we obtain the following two possibilities:

(*i*) y_1 is on P_1 and is adjacent to both x_1 and x^+.

(*ii*) y_1 is on P_2 and is adjacent to both x_2^- and u^+.

(*i*). Since $y_1 \in V(P_1)$, it divides P_1 into two subpaths P_1' (containing u^-) and P_1'' (containing x_1^+), each of them having evidently at least two edges. Since $\{y_1^-, y_1^+, y_1, x_2\}$ cannot induce $K_{1,3}$ and $y_1^-y_1^+ \notin E(G)$, $y_1^-x_2 \in E(G)$ or $y_1^+x_2 \in E(G)$. Simultaneously, $y_1x_2^- \in E(G)$ or $y_1u^+ \in E(G)$. Hence we have four cases. Now we can only consider the case: $y_1^-x_2, x_2^-y_1 \in E(G)$ (and the other three cases are similar and left to the reader). Deleting from C the edges $y_1^-y_1, uu^+, x_2^-x_2, x_2x_1$ and adding the edges $y_1^-x_2, y_1x_2^-, u^+x_2$ and the path ux_0x_1, we could obtain a cycle of length $m+1$ containing v. Hence (*i*) is proved.

(*ii*). This implies a contradiction in the same way as the preceding one (details are left to the reader). So Case d_1 is proved.

Case d_2. $p=3$.
Let $Q = y_0y_1y_2y_3 (y_0 = x_1 \text{ or } x_1^+; y_3 = u^+ \text{ or } x_2^-)$. Then $y_1x_1, y_1x_1^+, y_2x_2^-, y_2u^+$

are edges in $E(G)$ since if, e.g., $y_1x_1 \notin E(G)$, then $G[x_1, y_1, u^+, x_2] = K_{1,3}$. Hence without loss of generality assume that $y_0 = x_1^+$ and $y_3 = x_2^-$. Obviously, $y_1 \in V(C)$ and $y_2 \in V(C)$.

Clearly, $y_1^+y_1^- \notin E(G)$ since otherwise we could replace on C the path $y_1^-y_1y_1^+$ by the edge $y_1^-y_1^+$ and edge $x_1^-x_1$ by the path $x_1^-y_1x_1$, and would have obtained the (impossible) case $p = 2$.

Similarly, $y_2^-y_2^+ \notin E(G)$. By (c), we have that $y_1 \in V(P_1)$ and $y_2 \in V(P_2)$.

Next we end the proof of Case d_2.

Denote again by P_1', P_1'' the subpaths of P_1 determined by y_1, and by y_1^- and y_1^+ on them. Analogously, define the subpaths P_2' and P_2'' of P_2 and the vertices y_2^-, y_2^+ on them. Excluding the case $y_1^- = x_1^+$, $y_2^+ = x_2^-$, and $y_2^- = u^+$ and observing the induced $K_{1,3}$ on $\{y_1, y_1^-, x_2, y_1^+\}$, we obtain $y_1^+x_2 \in E(G)$ or $y_1^-x_2 \in E(G)$.

If $y_1^+x_2 \in E(G)$, then the cycle $C' = C[x_1, y_1]C[y_2, x_2^-]C^-[y_2^-, x_3]x_2 C[y_1^+, u]x_0x_1$ shows $y_2^-x_2^- \notin E(G)$ and the cycle $C'' = C[x_1, y_1]C^-[y_2^-, x_3] C^-[x_2^-, y_2]x_2C[y_1^+, u]x_0x_1$ shows $y_2^-y_1 \notin E(G)$. Note that $y_1y_3(= x_2^-) \notin E(G)$, and $x_3x_2^- \in E(G)$ by Lemma 2(6). Hence $G[y_2, y_2^-, y_1, x_2^-] = K_{1,3}$. This contradiction shows $y_1^+x_2 \notin E(G)$, and so $y_1^-x_2 \in E(G)$.

The cycle $C' = C[x_1, y_1^-]x_2C[x_3, y_2^-]C^-[x_2^-(= y_3), y_2]C[y_1, u]x_0x_1$ shows that $y_2^-x_2^- \notin E(G)$. The cycle $C'' = C[x_1, y_1^-]x_2C[y_2, x_2^-(= y_3)]C[x_3, y_2^-] C[y_1, u]x_0x_1$ shows $y_2^-y_1 \notin E(G)$. Hence $G[y_1, y_2^-, y_2, x_2^-(= y_3)] = K_{1,3}$. This contradiction shows Case d_2 is proved. Hence $N_1(x_2)$ is disconnected.

Analogously, using u^+ instead of x_2 and the edges $x_2x_2^-$ and uu^- instead of $x_1x_1^+$ and $u^+x_2^-$, we can prove the following.

(e). $N_1(u^+)$ is disconnected.

Therefore Case 2 is proved. So the proof of the Theorem is completed.

Theorem 3. *Let G be a graph satisfying the conditions of Theorem 1 and $\delta(G) \geq 3$. Then G is pancyclic.*

Proof: Since G contains some N_2-locally connected vertex, by Theorem 2, G is pancyclic.

Theorem 4. *Let G be a connected, N_2-locally connected claw-free graph with $\delta(G) \geq 3$, which does not contain an induced subgraph H isomorphic to either G_1 or G_2 such that $N_1(x, G)$ of every vertex x of degree 4 in H is disconnected. Then G is vertex pancyclic.*

Proof: Since every vertex of G is N_2-locally connected, by Theorem 2, G is vertex pancyclic.

We make the following conjecture.

Conjecture 5. *Every 3-connected, SN_2-locally connected claw-free graph is vertex pancyclic.*

Remark 1. The condition: "$\delta(G) \geq 3$" of Theorem 2 is necessary. For example, the graph in Figure 2 is not vertex pancyclic. Also the graph of Figure 3 is not pancyclic.

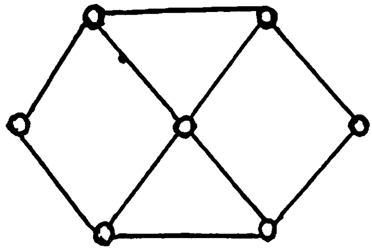

Figure 2

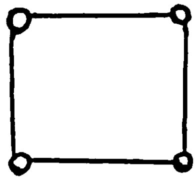

Figure 3

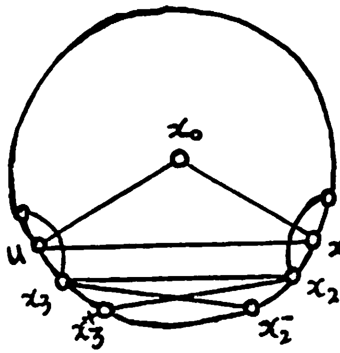

Figure 4

Remark 2. The graph in Figure 5 is an example of a claw-free graph which is SN_2-locally connected but is not N_2-locally connected. (The vertices v_1 and v_2 are not N_2-locally connected.)

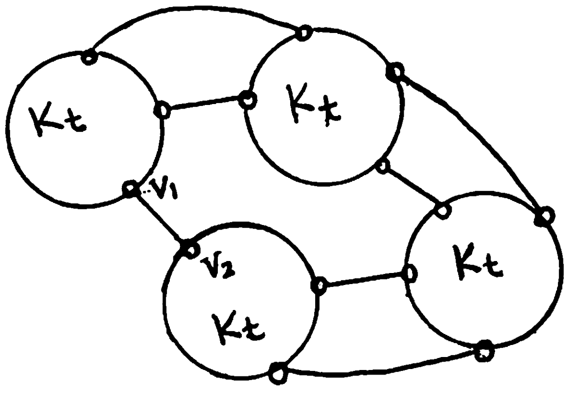

Figure 5

Remark 3. The assumptions of Theorem 4 do not imply that G is full cycle extendable. For example, a graph G obtained by joining two vertex disjoint cliques K_1, K_2 of the same size with a perfect matching satisfies the assumptions of Theorem 4 but e.g. any cycle C with $V(C) = V(K_1)$ is nonextendable in G.

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