Snarks and Reducibility

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ABSTRACT. A snark is a simple, cyclically 4-edge connected, cubic graph with girth at least 5 and chromatic index 4. We give a complete list of all snarks of order less than 30. Motivated by the long standing discussion on trivial snarks (i.e. snarks which are *reducible*), we also give a brief survey on different reduction methods for snarks. For all these reductions we give the complete numbers of irreducible snarks of order less than 30 and the number of nonisomorphic 3-critical subgraphs of these graphs. The results are obtained with the aid of a computer.

1 Introduction

We consider proper edge colorings of cubic graphs. It follows from a famous theorem of Vizing that a cubic graph has chromatic index 3 or 4. Isaacs [9] called cubic graphs with chromatic index 3 *colorable* graphs and those with chromatic index 4 *uncolorable* graphs. He also introduced the first notion of a trivial uncolorable graph for those with bridge, a cycle separating 2- or 3-cut or with a square.

Gardner [6] proposed to call nontrivial (in this sense) uncolorable graphs snarks. Since then, reductions of snarks have attracted considerable discussion in the literature and different types of reductions, and corresponding notions of nontriviality, were investigated, see e. g. [3], [6], [9], [12] and [15].

Some lists of small snarks already occur in the literature: Chetwynd and Wilson [4] gave a list of snarks of order less than 32 known in 1981. At that time only ten small snarks were known. The only known snark of order 20 was Isaacs' flower snark J_5 (c.f. [9]) and of order 22 only the two Loupekine snarks were known. In 1989 Watkins [16] showed that there are twelve snarks of order 22, and he asserted that there are eight snarks of order 20. This was shown to be wrong by the complete list of snarks with up to 22 vertices given by Holton and Sheehan [8] in 1993. They show that there are six snarks of order 20 and twenty of order 22. Furthermore they show that the Petersen graph P is the only snark with less than 18 vertices, and that there are exactly two snarks of that kind of order 18. These two latter results are also proved theoretically by a lot of authors, showing that there are no snarks of order 12, 14 and 16, and that there is one snark of order 10 and two of order 18, c.f. [14].

Gardners' definition of a snark has quite some disadvantages in the study of reducibility of snarks. So we define a *weak snark* to be a simple, cubic, bridgeless, uncolorable graph. Obviously, every snark is also a weak snark.

We describe different notions of reducibility of (weak) snarks. With respect to these notions we investigate the reducibility of all snarks of order less than 30. The results are given in a table in section 3. In this table we also give the number of nonisomorphic 3-critical subgraphs.

2 Reductions of Snarks

2.1 Vertex-Reduction

A cubic graph G is *vertex-reducible* to a simple cubic graph G', if G' can be obtained from G by removing two vertices together with all incident edges from G and adding edges to obtain G'. This reduction divides the class of weak snarks, denoted by $\mathcal{G}$, into three classes.

$$\mathcal{B} := \{G \in \mathcal{G} | G \text{ is not vertex-reducible to a weak snark}\}$$
$$\mathcal{S} := \{G \in \mathcal{G} | G \text{ is not vertex-reducible to a colorable graph}\}$$
$$\mathcal{C} := \mathcal{G} \setminus (\mathcal{B} \cup \mathcal{S})$$

Vertex-reductions of weak snarks are investigated in detail in [15], where structural characterizations of the induced classes are given. It is also shown that these classes are not empty, and that each weak snark with 2-, 3- or 4-edge cut is vertex-reducible to a smaller one.

Let $\mathcal{G}_{30}$ denote the set of snarks of order less than 30.

Theorem 2.1 *1. P is the only snark in $\mathcal{B} \cap \mathcal{G}_{30}$.*

2. $\mathcal{S} \cap \mathcal{G}_{30} = \emptyset$.

3. All snarks with less than 30 vertices (except P) are in $\mathcal{C}$.

Clearly, 3 is a consequence of 1 and 2. Together with Theorem 5.2 in [15] this result implies that all snarks in $\mathcal{G}_{30}$ have a 2-factor with exactly two odd components.

Theorem 2.1 is proved with the aid of a computer. Details are given in section 3.

2.2 Edge-Cut Reduction I

Isaacs [9] approach was taken in [3] and [7].

A cyclically 4- or 5-edge connected graph G is called *(edge-cut-1) reducible*, if after removing a cycle separating cut in G, both of the resulting components G_0, G_1 are subgraph of weak snarks G'_0, G'_1 with in case of a cyclical 4-cut $|G'_0| < |G|$, $|G'_1| < |G|$ and $|G'_0| = |G_0|$, $|G'_1| \leq |G_1| + 2$. In case of a cyclical 5-cut, $|G'_0| < |G|$, $|G'_1| \leq |G|$ and $|G'_0| \leq |G_0| + 5$, $|G'_1| \leq |G_1| + 5$ is required.

It is shown that weak snarks having an edge cut of size 4 [3, 7] as well as weak snarks having an edge cut of size 5 [3] are reducible.

This approach has the disconcerting consequence that each cyclically 5-edge connected snark G containing a 5-cycle, except P, is reducible by taking a trivial edge cut and regarding the 5-cycle as a subgraph of P and the other component as a subgraph of G itself.

So cyclically 5-edge connected snarks that only have trivial cycle separating 5-cuts are interesting too. The following theorem was obtained with the help of a computer:

Theorem 2.2 *The only cyclically 5-edge connected snarks with less than 30 vertices that have only trivial cycle separating 5-cuts are P and the flower snark J_5.*

2.3 Edge-Cut Reduction II

Following [12] we use the following definitions. A weak snark G is *k-edge-cut-irreducible* if all components obtained from G after removing an edge-cut of cardinality less than k are 3-colorable; $k \in \mathbb{N}$. A weak snark G is *critical* if $G - \{v, w\}$ is 3-colorable for any two adjacent vertices v, w of G, it is *cocritical* if $G - \{v, w\}$ is 3-colorable for any two nonadjacent vertices v, w of G, and it is *bicritical* if $G - \{v, w\}$ is 3-colorable for any two vertices v, w of G.

In [12] it is shown that a weak snark is critical if and only if it is k-edgecut-irreducible for $k = 5$ and $k = 6$, and it is bicritical if and only if it is k-edgecut-irreducible for every k. In [15] it is shown that all hypohamiltonian weak snarks (c.f. [5]) are bicritical (that is: e.g. the flower snarks J_{2n+1}, $n \geq 2$ are bicritical). In section 3 we give the numbers of snarks of order less than 30 which are bicritical, critical but not cocritical, cocritical but not critical and those which are neither critical nor cocritical,

3 List of small Snarks

In this section we give the numbers snarks with less than 30 vertices and some data related to the reduction methods discussed before. All graphs mentioned can be obtained in binary form from the authors.

The lists of snarks were obtained by filtering the output of the program *minibaum* (see [1], [2]) which is a fast generation program for cubic graphs. The program was checked in various cases by independently checking the generated graphs for being non-isomorphic (see [10]) and comparing the number of generated graphs to theoretically obtained results ([13]) and other generation programs (e.g. [11]). The filtering and testing programs were independently programmed by some students. For the small cases, the results were independly checked using these programs.

$\|V\|$	cy. con.	#Snarks	#3-c. Subg.	bic.	c.,¬coc.	¬c.,coc.	¬c.,¬coc.
10	≥ 5	1	1	1	0	0	0
12	≥ 4	0	0	0	0	0	0
14	≥ 4	0	0	0	0	0	0
16	≥ 4	0	0	0	0	0	0
18	≥ 4	2	10	2	0	0	0
20	≥ 4	6	55	1	0	0	5
20	≥ 5	1	3	1	0	0	0
22	≥ 4	20	189	2	0	0	18
22	≥ 5	2	9	2	0	0	0
24	≥ 4	38	471	0	0	2	36
24	≥ 5	2	18	0	0	2	0
26	≥ 4	280	3853	111	0	2	167
26	≥ 5	10	126	8	0	0	2
28	≥ 4	2900	48239	33	0	2	2865
28	≥ 5	75	1440	1	0	2	72
28	≥ 6	1	3	1	0	0	0

For each order mentioned there are no snarks of higher cyclical connectivity than mentioned in the list.

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