Several new lower bounds for football pool systems

Uri Blass and Simon Litsyn
Tel-Aviv University
Department of Electrical Engineering – Systems
Ramat-Aviv 69978, Israel

ABSTRACT. We derive several new lower bounds on the size of ternary covering codes of lengths 6, 7 and 8 and with covering radii 2 or 3.

1 Introduction

Ternary covering code C of length n and radius R is a collection of ternary vectors of length n possessing the property that every ternary vector of length n differs from at least one codeword in at most R coordinates. We wish to minimize the size of C. The search for ternary covering codes attracted a big deal of attention due to its equivalence to constructing systems for football pools. A survey of what is known on the problem can be found in [8, 7, 11]. Along with constructing such codes, lower bounds on their size are extensively studied [2, 4, 5, 6, 9, 12, 13]. The known lower bounds give reasonably good results in cases when R is relatively small in comparison with n. In the paper we present an approach giving several better lower bounds in some situations when R is relatively big.

Let $K_3(n, R)$ be the minimal size of a ternary code of length n and covering radius R. We prove the following bounds:

$$K_3(6,2) \geq 14,\ K_3(7,3) \geq 9,\ K_3(8,2) \geq 54,\ K_3(8,3) \geq 14,$$

thus improving the best previously known bounds of 12, 7, 52 and 13.

We denote the space of ternary vectors of length n by F_3^n. Let $d(\cdot,\cdot)$ stand for the Hamming distance. Let $C(n, M)_3R$ be a ternary covering code C of length n, covering radius R and size M. Let $B_3(x, r)$ be the ternary Hamming ball of radius r with the center at x, $V_3(n, r)$ stand for the size of Hamming ball of radius r in F_3^n,

$$V_3(n,r) = \sum_{i=0}^{r} \binom{n}{i} 2^i.$$

We say that a vector v provides the distance vector $(d_1, \ldots, d_M)$ on the code $C = \{c_1, \ldots, c_M\}$, if $d(v, c_i) \geq d_i$, $i = 1, \ldots, M$.

We say that $C = \{c_1, \ldots, c_M\}$ is a *generalized covering code* for the vector $(d_1, \ldots, d_M)$ if

$$\cup V(c_i, d_i) = F_3^n.$$

In other words, the generalized covering code is a collection of M Hamming balls, maybe having different radii, centered at the codewords, and covering the space. Such codes were considered in [1, 3, 14]. Clearly, if all d_i's are equal we get the standard definition of covering code. Our approach is based on an analysis of bounds for such generalized covering codes.

2 New bounds

Let $N(d_1, d_2, \ldots, d_M) = N$ be the minimal length such that for every ternary code $\{x_1, x_2, \ldots, x_M\}$ of length N there can be found a vector $h \in F_3^N$ such that $d(h, x_i) \geq d_i$. It is clear, that if we have $d_1 = \ldots = d_M = R + 1$ then there does not exist a code of length N with covering radius R and of size M. In what follows we will demonstrate how to use properties of the function N to derive lower bounds on the size of covering codes.

First, we give some properties of the function N.

Property 1 Let $\sigma(d_1, \ldots, d_M)$ be a permutation of $(d_1, \ldots, d_M)$. Then

$$N(d_1, \ldots, d_M) = N(\sigma(d_1, \ldots, d_M)).$$

Property 2 If

$$\sum_{i=1}^{M} V_3(N, d_i - 1) < 3^N,$$

then

$$N(d_1, \ldots, d_M) \leq N.$$

Now we proceed with particular cases. We will give a detailed proof only for the case of $n = 7$ and $R = 3$. For the other situations the proofs are similar, and we omit details giving only sketches of the proofs.

Theorem 1

$$K_3(7,3) \geq 9.$$

Proof. To prove nonexistence of $(7,8)_3 3$ code, it is enough to show $N(4^8) \leq 7$. We proceed in several successive steps.

Assume there exists a $(7,8)_3 3$ code. Let $(a_1, \ldots, a_8)$ be the first column of the code. In most of the cases we will prove that there exists a vector (a, h) providing a sought distance vector.

a) $N(32^7) \leq 4$. W.l.o.g. we may assume that the first row of the code consists of identical symbols a_1. Consider the thirty two ternary vectors containing symbol a_1 exactly once. It is easy to check that at most four such vectors can be within a Hamming ball of radius 1. So, there are at least four such vectors that are not covered by seven spheres of radius one. Each of them can be chosen as the vector providing the sought distance vector.

b) $N(3^5 2^3) \leq 5$. In one of the five first rows of the code one of the three symbols of the first column, say a_1, appears at most once. By a) and Property 1 for the code C constituted by the four last columns we may always find a vector h being at distance at least 2 from all the vectors, and at distance 3 from the row of C corresponding to a_1 in the first column of the code. The vector (a_1, h) then guarantees the result.

c) $N(43^2 2^5) \leq 5$. If in a column the first three coordinates are not pairwise different, the result follows from a). So, by permutation of the symbols we may assume that the first row of the code consists only of a_1's, the second row of a_2's, and the third row of a_3's. Consider all 30 vectors having one a_1, two a_2's and two a_3's. Every such vector is at distance 4 from the first row and is at distance 3 from the second and the third rows of the code. Notice, that at most three such vectors may occur in a Hamming sphere of radius one. So, at most 15 such vectors can be at distance at most one from one of the five last rows of the code.

d) $N(4^2 3^6) \leq 6$. Assume $a_1 \neq a_2$. If either a_1 or a_2 appear in the last six coordinates of the first column at most once, then the claim follows from c). Otherwise, a_3 appears in the last six coordinates at most twice, and the claim follows from b). If $a_1 = a_2$ then there exists a symbol (other than a_1) that appears at most three times in the last six coordinates. Then the claim follows from b).

e) $N(4^8) \leq 7$. At least one of the three symbols appears at most twice in the first column. The result now follows from d).

The last claim is equivalent to the statement of the theorem. □

Theorem 2

$$K_3(6,2) \geq 14.$$

Sketch of proof.

a) $N(2^8 1^5) \leq 4$. Follows from Property 2.

b) $N(32^k1^{31-4k}) \leq 4$ for $k = 0, \ldots, 7$. Exactly like in case a) of Theorem 1.

c) $N(3^3 2^{10}) \leq 5$. We proceed in cases. If $a_1 = a_2 = a_3$ then we choose as a the symbol different from a_1 that appears less in the last ten rows, and the claim follows from a). If $a_1 = a_2 \neq a_3$ we have two possibilities. If a_3 appears at least five times in the last ten rows, we choose $a \neq a_1$, $a \neq a_3$, and the result follows from a). Otherwise, we choose $a = a_3$, and the result follows from b) for $k \leq 6$. If a_1, a_2 and a_3 are pairwise different then we choose as a the symbol which appears less in the first column, and the result follows from b) for $k \leq 5$.

d) $N(3^{13}) \leq 6$. If there exists a column where one of the symbols appears at most three times then the result follows from c). Now assume that a_2 is the symbol that appears exactly five times in every column. We can assume that not all the rows that contain a_2 are identical. Then we have a row that contains a_2 and another symbol different from a_2, say a_0. Without loss of generality, the first row begins $(a_2, a_0, \ldots)$. There are altogether four rows containing a_0 in the second column. In these four rows, one of the symbols different from a_2 appears at most once in the first column. If it does not appear at all, the result follows from a); if it appears once, the result follows from b) (for $k = 6$). □

Theorem 3

$$K_3(8,2) \geq 54.$$

Sketch of proof. Assume there exists a $(8,53)_3 2$ code. If in some column one of the three symbols appears less than 17 times, then the result follows from $N(3^{16}2^{37}) \leq 7$, that we get from Property 2. So, we may assume that in all the columns every symbol appears at least 17 times. Define $N^1(d_1, \ldots, d_{53})$ to be the corresponding value of N if in all the columns every symbol appears at least 17 times. Now,

a) $N^1(3^{5-k}2^{26+2k}1^{22-k}) \leq 6$ for $k = 0, \ldots, 5$. Follows from Property 2.

b) $N^1(3^{17}2^{36}) \leq 7$. Choose as a in (a, h) the symbol appearing less than the others in the first 17 coordinates of the first column. Then the result follows from a).

c) $N(3^{53}) \leq 8$. Either there exists a column where some symbol appears less than 17 times, or one of the symbols necessarily appears exactly 17 times, and the result follows from b). □

Theorem 4

$$K_3(8,3) \geq 14.$$

Sketch of proof. We prove $N(4^{13}) \leq 8$.

a) $N(3^4 2^4 1^5) \leq 5$. If among a_1, a_2, a_3 and a_4 there are at most two different symbols then the result follows from a) of Theorem 2. Otherwise, we pick the two symbols that appear less than twice among $a_1, \ldots, a_4$, and as a use the one of them that appears less in all the column. Now the statement follows from b) of Theorem 2.

Define $N^1(d_1, \ldots, d_{13})$ to be the corresponding value of N if in all the columns every symbol appears at least four times.

b) $N^1(4 3^6 2^6) \leq 6$. If there is a symbol different from a_1 which appears less than three times among $a_2, \ldots, a_7$, we use it as a, and the statement follows from c) of Theorem 2. Otherwise, each of the two symbols different from a_1 appears exactly three times among $a_2, \ldots, a_7$. We choose the one that appears less in all the column. The statement now follows from a).

c) $N^1(4^4 3^9) \leq 7$. If there is at most two different symbols among $a_1, \ldots, a_4$ then the statement follows from Theorem 2. Otherwise, it follows from b) if we take as a one of the symbols that appears exactly once among $a_1, \ldots, a_4$ and exactly four times in all the column. □

Acknowledgement. The authors are grateful to the anonymous referee for helpful suggestions.

References

[1] J. M. van der Akker, J. H. Koolen and R. J. M. Vaessens, Perfect codes with distinct protective radii, *Discrete Math.*, **81** (1990), 103–109.

[2] W. Chen and I. S. Honkala, Lower bounds for q-ary covering codes, *IEEE Trans. Inform. Theory* **36** (1990), 664–671.

[3] G. D. Cohen and B. Montaron, Empilements parfaits de boules dans les espaces vectoriels binaires, *C.R. Acad. Sc. Paris*, **288** (1979), 578–582.

[4] L. Habsieger, Lower bounds for q-ary coverings by spheres of radius 1, *Journal of Combinatorial Theory*, Ser. A, **67** (1994), 199–222.

[5] L. Habsieger, Binary codes with covering radius one from a linear programming point of view: some new lower bounds, submitted.

[6] L. Habsieger, A new lower bound for the football pool problem for 7 matches, preprint.

[7] H. Hämäläinen and S. Rankinen, Upper bounds for football pool problems and mixed covering codes, *J. Combin. Theory Ser. A* **56** (1991), 84–95.

[8] H. Hämäläinen, I.Honkala, S.Litsyn and P. R. J. Östergård, Football pools - a game for mathematicians, *the American Mathematical Monthly* **102**, No. 7 (1995), 579–588.

[9] J. H. van Lint Jr. and G. J. M. van Wee, Generalized bounds on binary/ternary mixed packing- and covering codes, *J. Combin. Theory Ser. A* **57** (1991), 130–143.

[10] C.Lo and Z.Zhang, New lower bounds for $K_3(n, R)$ from linear inequalities, submitted.

[11] P. R. J. Östergård and H. O. Hämäläinen, A new table of binary/ternary mixed covering codes, submitted.

[12] G. J. M. van Wee, Bounds on packings and coverings by spheres in q-ary and mixed Hamming spaces, *J. Combin. Theory Ser. A* **57** (1991), 117–129.

[13] G. J. M. van Wee, Some new lower bounds for binary and ternary covering codes, *IEEE Trans. on Inform. Theory*, **39** (1993), 1422–1424.

[14] V.A. Zinoviev and G.L. Katsman, Universal codes families, *Problemy Peredachi Informatsii*, **29**, No. 2 (1993), 3–8, Translated in: *Problems of Inform. Transm.*, **29**, No.2 (1993), 95–100.