# On k-Partite Subgraphs

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**Abstract**

Initiated by a recent question of Erdös, we give lower bounds on the size of a largest $k$-partite subgraph of a graph. Also the corresponding problem for uniform hypergraphs is considered.

# 1 Introduction

A graph $G = (V_1 \cup\!\!\!\cdot \cdots \cup\!\!\!\cdot\, V_k, E)$ with vertex set $V_1 \cup\!\!\!\cdot \cdots \cup\!\!\!\cdot\, V_k$ and edge set $E$ is called *$k$-partite* if every vertex set $V_i$ is an independent set.
Inspired by the recent question of Erdös [6], whether every graph with $\binom{3n}{2}$ edges contains a 3-partite subgraph with $3n^2$ edges, we are looking for large $k$-partite subgraphs of graphs and hypergraphs. Indeed, the question of Erdös can be answered in the affirmative, as has been independently observed by Alon [1]. As a byproduct we obtain a simple proof of a theorem of Edwards [3] on the size of the largest bipartite subgraph of a graph. Recently, short proofs for this result have also been given by Alon [1], Poljak and Tuza [11], and Erdös, Gyárfás and Kohayakawa [7].
Finally, we give lower bounds on the size of a largest $k$-partite subhypergraph of a given $k$-uniform hypergraph in terms of its strong chromatic number.

# 2 Graphs

Let $G = (V, E, w)$ be an edge weighted graph, i.e., $w: E \longrightarrow \mathbb{R}$. For a partition $V = V_1 \cup\!\!\!\cdot \cdots \cup\!\!\!\cdot\, V_k$ of the vertex set, the *cutsize* $cut(V_1, \ldots, V_k)$ is the sum of the weights of all crossing edges. We call an edge $\{v, w\}$ *crossing* if $v \in V_i$ and $w \notin V_i$ for some $i$. Thus,

$$cut(V_1, \ldots, V_k) = \sum_{e \in E;\ e \text{ crossing}} w(e) .$$

In the unweighted case, i.e., $w(e) = 1$ for all edges $e$, $cut(V_1, \ldots, V_k)$ is the number of edges of the $k$-partite subgraph determined by $V_1, \ldots, V_k$.

The *chromatic number* $\chi(G)$ is the minimum number of colors needed to properly color the vertices of $G$, i.e., the least $t$ such that there exists a coloring $c: V \longrightarrow \{1, \ldots, t\}$ and $c(v) \neq c(w)$ for every edge $\{v, w\} \in E$. The following has been observed independently by Alon [1].

**Theorem 2.1** *Let $k, n$ be positive integers. Every graph $G$ with $\binom{kn}{2}$ edges contains a $k$-partite subgraph with $\binom{k}{2}n^2$ edges.*

For the proof of Theorem 2.1, we use the following lemma, which has been observed by Locke [10], cf. also [2]. Since we need its simple proof in section 3, we include it here.

**Lemma 2.2** *Let $k, n$ be positive integers. Let $G = (V, E, w)$ be a weighted graph. If $\chi(G) \leq kn$, then there exists a partition $V = V_1 \cup \cdots \cup V_k$ such that*

$$cut(V_1, \ldots, V_k) \geq \frac{k-1}{k} \cdot \left(1 + \frac{1}{kn-1}\right) \cdot \sum_{e \in E} w(e) . \qquad (1)$$

*Proof:* We first prove the lemma for the case $|V| \leq kn$. By possibly adding isolated dummy vertices we can assume that $|V| = kn$. For any partition of $V$ into $k$ equally sized subsets $V_1, \ldots, V_k$, a portion of $\frac{n^2\binom{k}{2}}{\binom{kn}{2}} = \frac{k-1}{k} \cdot \left(1 + \frac{1}{kn-1}\right)$ vertex pairs $\{i, j\}$ is crossing. Thus, the expected cutsize of such a randomly chosen partition is equal to the right hand side of (1) and at least one partition with at least this cutsize exists.
Now let $|V|$ be arbitrary, but $\chi(G) \leq kn$. Every color class $c^{-1}(i)$ is an independent set. For every such class, we collapse its vertices to a super-vertex and also collapse edges between super-vertices by adding up their weights.
To the new graph on the super-vertices, we apply the above arguments and obtain a partition $V_1', \ldots, V_k'$. We then replace every super-vertex $v$ in $V_i'$ by the vertices of the corresponding color class to obtain the desired partition $V = V_1 \cup \cdots \cup V_k$. □

**Lemma 2.3** *Every graph $G$ contains at least $\binom{\chi(G)}{2}$ edges.*

*Proof:* For every $i, j$ with $1 \leq i < j \leq \chi(G)$, there must be an edge between the two sets of vertices with colors $i$ and $j$. □

Now Theorem 2.1 easily follows: As $G$ contains $\binom{kn}{2}$ edges, we infer with Lemma 2.3 that $\chi(G) \leq kn$. By Lemma 2.2, $G$ contains a $k$-partite subgraph with at least $\frac{k-1}{k} \cdot (1 + \frac{1}{kn-1}) \cdot \binom{kn}{2} = \binom{k}{2} \cdot n^2$ edges. □

Using somewhat involved arguments, Edwards [3] proved a lower bound on the largest size of a bipartite subgraph in any graph with $m$ edges, cf. [4]. However, Lemma 2.2 and 2.3 yield a short proof of his theorem. Now, the following is a straightforward generalization of Edwards' result.

**Corollary 2.4** *Let $k$ be a positive integer. Let $G = (V, E)$ be a graph with $m$ edges and let the integer $t$ be defined by $\binom{t}{2} \leq m < \binom{t+1}{2}$.*
*Let $j \equiv t \bmod k$, $0 \leq j \leq k-1$. There exists a partition $V = V_1 \uplus \cdots \uplus V_k$ with*

$$cut(V_1, \ldots, V_k) \geq \begin{cases} \frac{k-1}{k} \cdot m + \frac{k-1}{k} \cdot \frac{\sqrt{8m+1}+1}{4} & \text{if } t \equiv 0 \bmod k \\ \frac{k-1}{k} \cdot m + \frac{k-1}{k} \cdot \frac{2m}{\sqrt{8m+1}+2(k-j)-1} & \text{if } t \not\equiv 0 \bmod k. \end{cases}$$

*Proof:* By Lemma 2.3, we infer $\chi(G) \leq t$. By choice of $t$, we have $t \leq 1/2 \cdot (\sqrt{8m+1} + 1)$. If $t \equiv 0 \bmod k$, Lemma 2.2 proves the existence of a $k$-partite subgraph with at least $\frac{k-1}{k} \cdot m + \frac{k-1}{k} \cdot \frac{\sqrt{8m+1}+1}{4}$ edges. If $t \not\equiv 0 \bmod k$, we add $(k-j)$ dummy vertices and apply Lemma 2.2 to obtain a $k$-partite subgraph with at least $\frac{k-1}{k} \cdot m + \frac{k-1}{k} \cdot \frac{2m}{\sqrt{8m+1}+2(k-j)-1}$ edges. □

Notice that for $k = 2$, Corollary 2.4 is Edwards' result.

# 3 Hypergraphs

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a hypergraph. $\mathcal{G}$ is called *k-uniform* if $|E| = k$ for every hyperedge $E \in \mathcal{E}$. For a partition $\mathcal{V} = \mathcal{V}_1 \uplus \cdots \uplus \mathcal{V}_k$, a hyperedge $E \in \mathcal{E}$ is called *crossing* if $|E \cap \mathcal{V}_i| = 1$ for $i = 1, \ldots, k$.
In [6], the question was raised, whether for every 3-uniform hypergraph with $\binom{3n}{3}$ hyperedges, its vertex set can be partitioned into three sets so that the number of crossing hyperedges is at least $n^3$.
The *strong chromatic number* $\chi_S(\mathcal{G})$ of a hypergraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined as the least $t$ such that there exists a proper vertex coloring $c: \mathcal{V} \longrightarrow \{1, \ldots, t\}$ such that no color occurs more than once in any hyperedge, that is $|E \cap c^{-1}(i)| \leq 1$ for $i = 1, \ldots, t$ and every $E \in \mathcal{E}$.

**Proposition 3.1** *Let $k, n$ be positive integers. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$ be a weighted $k$-uniform hypergraph with $\chi_S(\mathcal{G}) \leq kn$, where $w: \mathcal{E} \longrightarrow \mathbb{R}$. Then there exists a partition $\mathcal{V} = \mathcal{V}_1 \uplus \cdots \uplus \mathcal{V}_k$ such that*

$$\sum_{E \in \mathcal{E};\ E \text{ crossing}} w(E) \geq \frac{n^k}{\binom{kn}{k}} \cdot \sum_{E \in \mathcal{E}} w(E)\,. \qquad (2)$$

If the hypergraph $\mathcal{G}$ from Proposition 3.1 in addition fulfills $w(E) = 1$ for each hyperedge $E \in \mathcal{E}$, then the existence of $\binom{kn}{k}$ hyperedges guarantees the existence of a partition $\mathcal{V} = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_k$ such that the number of crossing hyperedges $E \in \mathcal{E}$ is at least $n^k$.
We remark that Erdös and Kleitman already showed in [8] by a counting argument that the vertex set of every $k$-uniform hypergraph can be partitioned into $k$ sets such that the proportion of all crossing hyperedges is at least $\frac{k!}{k^k}$. However, in (2) we take care of lower order terms.

*Proof:* The proof is analogous to the proof of Theorem 2.1. This time, we have $\binom{kn}{k}$ possibilities to choose $k$ distinct vertices and $n^k$ of those possibilities are crossing. □

Proposition 3.1 only partially answers the question of Erdös for 3-uniform hypergraphs with $\binom{3n}{3}$ hyperedges, namely when the strong chromatic number is at most $3n$. We remark however, that such 3-uniform hypergraphs can have strong chromatic number as large as $c \cdot n^{3/2}$. This can be seen by using Wilson's results on designs, in particular:

**Theorem 3.2** [12] [13] *Let $k$ be a fixed positive integer. There exists a positive integer $N_0(k)$, such that for all positive integers $N \geq N_0(k)$ with $(k-1)|(N-1)$ and $k \cdot (k-1)|N \cdot (N-1)$ there exists a $k$-uniform hypergraph $\mathcal{G}$ on vertex set $\mathcal{V}$ with $|\mathcal{V}| = N$ and $m = \binom{N}{2}/\binom{k}{2}$ hyperedges, where each two-element subset of $\mathcal{V}$ is contained in exactly one hyperedge of $\mathcal{G}$.*

Clearly, such a hypergraph $\mathcal{G}$ as guaranteed by Wilson's result satisfies $\chi_S(\mathcal{G}) = N$, i.e., $\chi_S(\mathcal{G}) \geq 1/2 + \sqrt{1/4 + k \cdot (k-1) \cdot m}$. Moreover, we have a matching upper bound:

**Lemma 3.3** *Let $\mathcal{G}$ be a $k$-uniform hypergraph with $m$ hyperedges. Then*

$$\chi_S(\mathcal{G}) \leq 1/2 + \sqrt{1/4 + k \cdot (k-1) \cdot m}\,.$$

*Proof:* For a given $k$-uniform hypergraph $\mathcal{G}$ we form a graph $G = (V, E)$ on the same vertex set as $\mathcal{G}$ and with edges obtained by replacing every hyperedge of $\mathcal{G}$ by a complete graph on $k$ vertices. Then $|E| \leq \binom{k}{2} \cdot m$. By Lemma 2.3 we infer $\chi(G) \leq 1/2 + \sqrt{k(k-1)m + 1/4}$. By construction, we have $\chi_S(\mathcal{G}) = \chi(G)$. □

For fixed $k$, let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a $k$-uniform hypergraph with $m$ hyperedges. Define $N$ as the smallest integer which is not smaller than $\chi_S(\mathcal{G})$ and divisible by $k$. By Lemma 3.3, $N = O(1) + \sqrt{k(k-1)m}$.

By (2) (replacing $kn$ by $N$), there exists a partition $\mathcal{V} = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_k$ such that the number of crossing hyperedges $E \in \mathcal{E}$ is at least

$$m \cdot \frac{\left(\frac{N}{k}\right)^k}{\binom{N}{k}} = \frac{k!}{k^k} \cdot m \cdot \frac{N^k}{N \cdot (N-1) \cdots (N-k+1)} \tag{3}$$

$$\geq \frac{k!}{k^k} \cdot m \cdot \frac{1}{1 - \binom{k}{2}/N + O(1/N^2)} \geq \frac{k!}{k^k} \cdot m \cdot \left(1 + \frac{\binom{k}{2}}{N} - O\left(\frac{1}{N^2}\right)\right) \tag{4}$$

Since expression (3) is decreasing in $N$, we can insert the upper bound for $N$ into (4) to obtain the following lower bound on the number of crossing hyperedges:

$$\frac{k!}{k^k} \cdot m \cdot \left(1 + \frac{1}{2} \cdot \sqrt{\frac{k(k-1)}{m}} - O\left(\frac{1}{m}\right)\right)$$

We remark that earlier considerations of the authors focussing on algorithmic aspects of problems related to those considered in this paper can be found in [9].

**Acknowledgment:** The authors would like to thank the referee for pointing out Wilson's theorem.

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