Properly Coloured Hamiltonian Paths in Edge-coloured Complete Graphs without Monochromatic Triangles

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ABSTRACT. We show that every complete graph K_n with an edge-colouring without monochromatic triangles has a properly coloured hamiltonian path.

1 Introduction

In this note we consider simple complete graphs whose edges are coloured. Solving a problem by P. Erdős, M. Bankfalvi and Zs. Bankfalvi [3] obtained a necessary and sufficient condition for the existence of a properly coloured Hamiltonian cycle in a 2-edge-coloured complete graph. (A cycle or a path Q is *properly coloured* if adjacent edges of Q have distinct colours.) Another, essentially equivalent, condition for properly coloured Hamiltonian cycles was proved by R. Saad in [7]. A characterization of 2-edge-coloured complete graphs having properly coloured Hamiltonian paths is obtained in [2].

So far we are not aware of any characterization of m-edge-coloured complete graphs having properly coloured Hamiltonian cycles or paths respectively when $m \geq 3$. Moreover, the time complexity of the problems is unknown as J. Bang-Jensen and G. Gutin points out in [2]. Thus, sufficient conditions are of interest. The following was conjectured in [4]. Here and later K_n denotes an edge-coloured complete graph while Δ_c is the maximum number of edges of the same colour incident to a vertex.

Conjecture 1.1: *Every K_n with $\Delta_c \leq \lfloor \frac{n}{2} \rfloor - 1$ has a properly coloured Hamiltonian cycle.*

Some partial results on the conjecture were obtained in [1], [4], [5], [6] and [8].

In this note we give a simple sufficient condition for the properly coloured Hamiltonian path problem. Our condition is somewhat analogous to the well-known Redei theorem on Hamiltonian paths in tournaments which has a considerably number of applications in the study of directed and properly coloured paths or cycles.

2 Main result

Theorem 2.1. *Every complete graph K_n with an edge-colouring without monochromatic triangles has a properly coloured hamiltonian path.*

Proof: By induction on n. If $n = 1$ the result is trivial. Assume the theorem to be true for $n = k - 1$. Consider a K_k without monochromatic triangles. Let $\{v_1, v_2, \dots, v_k\}$ be the vertices of K_k. By the induction hypothesis the subgraph of K_k induced by $\{v_1, v_2, \dots, v_{k-1}\}$ has a properly coloured Hamiltonian path, $v_1 v_2 \dots v_{k-1}$ say.

Assume that K_k has no properly coloured Hamiltonian path. Then are the edges $v_k v_1$ and $v_1 v_2$ of the same colour since otherwise K_k would contain the properly coloured Hamiltonian path $v_k v_1 v_2 \dots v_{k-1}$. Since there are no monochromatic triangles in K_k, the edge $v_k v_2$ must have a different colour from $v_k v_1$. But the colour of $v_k v_2$ must be the same as of $v_2 v_3$, otherwise $v_1 v_k v_2 \dots v_{k-1}$ would be a properly coloured Hamiltonian path in K_k. By the same argument we get that every pair of edges $v_k v_i$ and $v_i v_{i+1}$ has the same colour when $i \in \{1, 2, \dots, k-2\}$.

In particular we get that $v_k v_{k-2}$ and $v_{k-2} v_{k-1}$ have the same colour, so $v_{k-1} v_k$ has a different colour from that. But then we obtain that $v_1 v_2 \dots v_{k-1} v_k$ is a properly coloured Hamiltonian path, contradicting the assumption that no such path existed. □

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