# STIFF GENUS OF GROUPS

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ABSTRACT.
A new concept of genus for finite groups, called stiff genus, is developed. Cases of stiff embeddings in orientable or nonorientable surfaces are dealt with. Computations of stiff genus of several classes of abelian and non-abelian groups are presented. A comparative analysis between the stiff genus and the Tucker symmetric genus is also undertaken.

## 0. INTRODUCTION

In [3] C. Gagliardi introduced a particular class of 2-cell embeddings, called (strongly-) regular, for $n$-coloured graphs. By means of this type of embeddings, he defined a new topological invariant for closed PL-manifolds of arbitrary dimension [4] – the regular genus – which is a generalization of the classical genus of 2-manifolds and of the Heegaard genus of 3-manifolds. The concepts of regular embedding and regular genus was later extended to $n$-coloured digraphs by Attilia Ceré [1].

Here the research is carried on, by applying regular embeddings to Cayley colour graphs of groups, with the aim of transferring to finite groups the notion of regular genus. The term "stiff" will be used instead of "regular", since the latter seems to be too widely used under other meanings.

As well known, an arc-coloured digraph $C(G,X)$, called Cayley digraph or Cayley colour graph, can be associated to each set $X$ of generators of a group $G$ (see [5] or [9]). We shall consider the orientable (resp. nonorientable) stiff genus of Cayley digraphs, as defined in [1]. The notion of orientable (resp. nonorientable) stiff genus of a group $G$ will be given, in a natural way, as the minimun orientable (resp. nonorientable) stiff genus of a Cayley digraph of $G$. The stiff genus of several classes of abelian and

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non-abelian groups (elementary abelian groups, dihedral groups, dicyclic groups, non-abelian groups of order $p^3$) will be computed in Section 5. The reader will note that the techniques used in the proofs are directly imported from the works on the "classical" group genus.

Moreover, a comparative analysis between stiff genus and symmetric genus will be presented in the final section.

## 1. Definitions and notations

Throughout this paper we shall only consider finite groups and finite digraphs. Moreover, digraphs will be always connected, except when explicitly stated. The cardinality of a set $X$ will be indicated by $\#X$.

Let $\Gamma = (V, A, \alpha, \beta)$ be a digraph, where $V(\Gamma) = V$ and $A(\Gamma) = A$ are respectively the vertex set and the arc set of $\Gamma$, and $\alpha, \beta : A \to V$ are the incidence maps (i.e., $\alpha(a)$ is the tail and $\beta(a)$ is the head of the arc $a$). We shall request $\alpha(a) \neq \beta(a)$, for all $a \in A$ (no loops). For each $v \in V$, let us define $A_v^+ = \{a \in A \mid \alpha(a) = v\}$, $A_v^- = \{a \in A \mid \beta(a) = v\}$ and $A_v = A_v^+ \cup A_v^-$. As usual, $|\Gamma|$ denotes the topological representation of $\Gamma$.

A *coloured digraph* is a digraph $\Gamma$ equipped with an arc colouration $\gamma : A(\Gamma) \to C$ such that

$$\alpha(a) = \alpha(b) \text{ or } \beta(a) = \beta(b) \quad \Longrightarrow \quad \gamma(a) \neq \gamma(b),$$

for all $a, b \in A(\Gamma)$. Moreover, $\Gamma$ is said to be *n-coloured* if $\#A_v^+ = \#A_v^- = \#C = n$, for all $v \in V(\Gamma)$. The sets $C$ and $C^\pm = C \times \{+1, -1\}$ are said respectively the *colour set* and the *signed colour set* of $\Gamma$. We shall write $c^+$ for $(c, +1)$ and $c^-$ for $(c, -1)$.

A cyclic permutation $\varepsilon$ of $C^\pm$ is called *balanced* if $\varepsilon^n(c^+) = c^-$, for every $c \in C$. For instance, if $C = \{a, b, c\}$, then the cyclic permutation $\varepsilon = (a^+\ c^-\ b^-\ a^-\ c^+\ b^+)$ is balanced. The set of balanced cyclic permutations of $C^\pm$ will be denoted by $\mathcal{B}(C^\pm)$ and it has cardinality $\#\mathcal{B}(C^\pm) = (n-1)!2^{n-1}$.

## 2. Stiff embeddings and stiff genus of digraphs

In what follows, the term "embedding" will always mean "2-cell embedding" [5].

Two embeddings $\iota, \iota' : |\Gamma| \to F$ of a digraph $\Gamma$ in a closed surface $F$ are said to be *equivalent* if there exists an homeomorphism $\Theta : F \to F$ such that: $\Theta \circ \iota = \iota'$. If $F$ is orientable and $\Theta$ preserves orientation, then $\iota$ and $\iota'$ are called *strongly equivalent*.

Let $\Gamma$ be an $n$-coloured digraph with arc colouration $\gamma$ and colour set $C$. For each $v \in V(\Gamma)$, let $\gamma_v : A_v \to C^{\pm}$ be the bijection defined by

$$\gamma_v(a) = \begin{cases} \gamma(a)^+ & \text{if } v = \alpha(a) \\ \gamma(a)^- & \text{if } v = \beta(a) \end{cases}.$$

If $\varphi_v$ is a cyclic permutation of $A_v$, then the map $\varepsilon_v = \gamma_v \circ \varphi_v \circ \gamma_v^{-1}$ is a cyclic permutation of $C^{\pm}$. Therefore, an arc rotation system [8] $\Phi = \{\varphi_v \mid v \in V(\Gamma)\}$ of $\Gamma$ can be given by a set of cyclic permutation of $C^{\pm}$: $\Phi = \{\varepsilon_v \mid v \in V(\Gamma)\}$. This allows us to compare rotations of different vertices of $\Gamma$, for selecting a particular class of "stiff" embeddings. If $\varepsilon$ is a cyclic permutation of $C^{\pm}$, let us denote by $\Phi_\varepsilon$ the "constant" arc rotation system having $\varepsilon_v = \varepsilon$, for all $v \in V(\Gamma)$.

An arc rotation system $\Phi = \{\varepsilon_v \mid v \in V(\Gamma)\}$ of an $n$-coloured digraph $\Gamma$ is called *stiff* if there exists a cyclic permutation $\varepsilon$ of the signed colour set $C^{\pm}$, such that (see Figure 1):

(i) $\varepsilon$ is balanced;

(ii) for all $v \in V(\Gamma)$, $\varepsilon_v = \varepsilon$ or $\varepsilon_v = \varepsilon^{-1}$;

(iii) if $v, w \in V(\Gamma)$ are adjacent vertices, then $\varepsilon_v = \varepsilon_w^{-1}$.

An embedding of $\Gamma$ in an orientable surface is called *stiff* if it arises from a stiff arc rotation system.

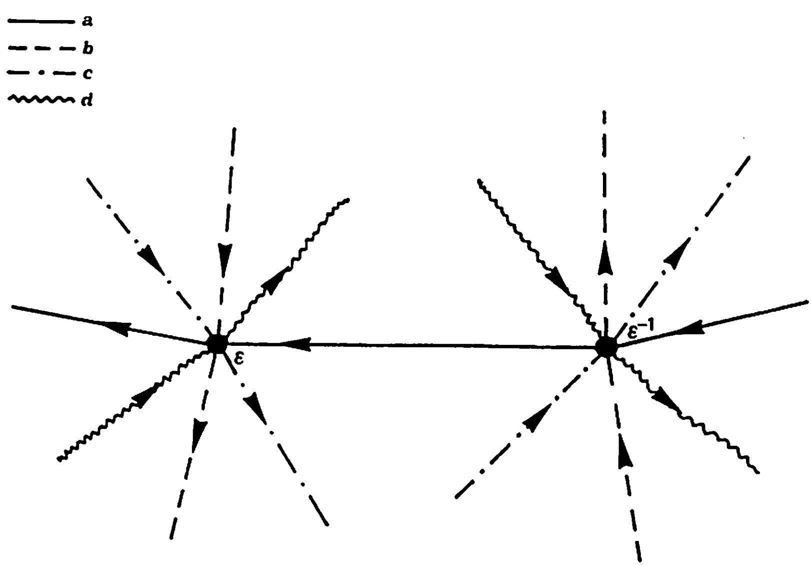


FIGURE 1. — $\varepsilon = (a\, d^-\, b\, c\, a^-\, d\, b^-\, c^-)$ —

The case $n = 1$ is trivial, since a 1-coloured digraph $\Gamma$ is simply a circuit: every orientable embedding of $\Gamma$ is stiff and occurs in a sphere.

Let us suppose $n \geq 2$, then $\varepsilon \neq \varepsilon^{-1}$. Thus, an $n$-coloured digraph admitting an orientable stiff embedding is bipartite, since it can not have odd cycles, by condition (iii). Moreover, each balanced cyclic permutation $\varepsilon \in \mathcal{B}(C^{\pm})$ defines a stiff arc rotation system on a bipartite $n$-coloured digraph $\Gamma$, by assigning $\varepsilon_v = \varepsilon$ to a fixed vertex $v \in V(\Gamma)$, and extending by connectedness, in accordance with condition (iii), the system to all vertices. Let us denote by $\iota_\varepsilon$ the embedding arising from this rotation system. It is easy to see that $\iota_{\varepsilon^{-1}}$ is equivalent to $\iota_\varepsilon$, by an orientation reversing homeomorphism of the surface. Moreover, a different choice of the fixed vertex $v$ leads to an equivalent stiff embedding.

*REMARK 2.1.* The boundary of every region of the embedding is constituted by arcs having exactly two distinct colours. However, this property does not characterize stiff embeddigs, unless $n = 2$. An example of such a "two-coloured" embedding which is not stiff can already be found for the Cayley digraph of $\mathbf{Z}_2^4$, associated to the canonical presentation.

To each $n$-coloured bipartite digraph $\Gamma$, with partition of vertices $\{V', V''\}$ and colour set $C$, we can associate a $2n$-coloured graph [3] $\widehat{\Gamma}$, having $C^{\pm}$ as colour set, with edge colouration $\widehat{\gamma} : E(\widehat{\Gamma}) = A(\Gamma) \to C^{\pm}$ defined by $\widehat{\gamma}(a) = \begin{cases} \gamma(a)^+ & \text{if } \alpha(a) \in V' \\ \gamma(a)^- & \text{if } \alpha(a) \in V'' \end{cases}$. Besides, the orientable stiff embedding $\iota_\varepsilon : |\Gamma| \to S$ corresponds to a regular embedding of $\widehat{\Gamma}$ in the same surface $S$, arising from the same permutation $\varepsilon$. This will allow us to utilize some results about regular embeddings of coloured graphs, included in [2] and [3].

In order to investigate nonorientable stiff embeddings, we make use of the standard tool of embedding schemes [7], which is a more general technique involving both orientable and nonorientable cases. An embedding scheme of a (di)graph $\Gamma$ is a pair $(\Phi, \lambda)$, where $\Phi$ is an arc rotation system of $\Gamma$ and $\lambda$ is a voltage-map from the set of arcs of $\Gamma$ to the cyclic group $\mathbf{Z}_2 = \{0, 1\}$.

Let $\Gamma = (V, A, \alpha, \beta)$ be a digraph and let $(\Phi, \lambda)$ be an embedding scheme of $\Gamma$, where $\lambda$ is a voltage map such that $\Gamma$ contains a cycle having an odd number of arcs $a$ with $\lambda(a) = 1$. Let $\Gamma^\lambda = (V \times \mathbf{Z}_2, A \times \mathbf{Z}_2, \alpha^\lambda, \beta^\lambda)$ be the derived digraph associated to $\lambda$. Notice that the incidence maps are given by $\alpha^\lambda(a, s) = (\alpha(a), s)$ and $\beta^\lambda(a, s) = (\beta(a), s + \lambda(a))$, for all $(a, s) \in A \times \mathbf{Z}_2$. Each embedding of $\Gamma$ in a nonorientable surface $S$ arises from a embedding scheme of this type, through the embedding of $\Gamma^\lambda$ on the orientable double covering $\tilde{S}$ of $S$, induced by the lifted arc rotation system

$\Phi^\lambda$. The genus $\bar{g} = 2 - \chi(S)$ of $S$ is related to the genus $\tilde{g} = 1 - \chi(\tilde{S})/2$ of $\tilde{S}$ by the formula

$$\bar{g} = 1 + \tilde{g}, \qquad (1)$$

since $\chi(\tilde{S}) = 2\chi(S)$. If $\Gamma$ is an $n$-coloured digraph, then $\Gamma^\lambda$ becomes an $n$-coloured digraph, with the same colour set, through the *lifted arc colouration* $\gamma^\lambda : A \times \mathbb{Z}_2 \to C$ defined by $\gamma^\lambda(a,s) = \gamma(a)$, for all $(a,s) \in A \times \mathbb{Z}_2$. If $\Phi^\lambda = \{\varepsilon_{(v,s)} \mid (v,s) \in V \times \mathbb{Z}_2\}$ is the lifting of $\Phi = \{\varepsilon_v \mid v \in V\}$ to the derived digraph, then $\varepsilon_{(v,0)} = \varepsilon_v$ and $\varepsilon_{(v,1)} = \varepsilon_v^{-1}$, for all $v \in V$.

An embedding scheme $(\Phi, \lambda)$ of $\Gamma$ is called *stiff* if and only if the arc rotation system $\Phi^\lambda$ is stiff for $\Gamma^\lambda$. The embedding of $\Gamma$ in a nonorientable surface is said to be *stiff* if it arises from a stiff embedding scheme.

If $n = 1$, the result is again trivial. In fact, every nonorientable embedding of a 1-coloured digraph is stiff and takes place in a projective plane, since the derived digraph is a circuit and stiffly embeds only in a sphere.

If $n \geq 2$, we can check, analogously to the proof of Corollary 24 of [3], that each nonorientable stiff embedding of a $n$-coloured digraph $\Gamma$ is equivalent to an embedding $\bar{\iota}_\varepsilon$ arising from the embedding scheme $(\Phi, \lambda)$, where $\lambda \equiv 1$ and $\Phi$ is the constant arc rotation system $\Phi_\varepsilon$, for a suitable $\varepsilon \in \mathcal{B}(C^\pm)$. Therefore, an $n$-coloured digraph admits a nonorientable stiff embeddigs if and only if it is non-bipartite.

**Definition 2.1.** *The orientable (resp. nonorientable) stiff genus $\sigma(\Gamma)$ (resp. $\bar{\sigma}(\Gamma)$) of an $n$-coloured digraph $\Gamma$ is the smallest integer $k$ such that $\Gamma$ admits a stiff embedding in an orientable (resp. a nonorientable) surface of genus $k$. If $\Gamma$ does not admit any orientable (resp. nonorientable) stiff embedding, then $\sigma(\Gamma) = +\infty$ (resp. $\bar{\sigma}(\Gamma) = +\infty$).*

The next lemma is a consequence of the previous considerations.

**Lemma 2.2.** *Let $\Gamma$ be an $n$-coloured digraph.*

*(a) If $n = 1$, then $\sigma(\Gamma) = 0$ and $\bar{\sigma}(\Gamma) = 1$.*

*(b) If $n \geq 2$ and $\Gamma$ is bipartite, then $\sigma(\Gamma) < +\infty$ and $\bar{\sigma}(\Gamma) = +\infty$.*

*(c) If $n \geq 2$ and $\Gamma$ is non-bipartite, then $\sigma(\Gamma) = +\infty$ and $\bar{\sigma}(\Gamma) < +\infty$.* □

**Lemma 2.3.** *Let $n \geq 2$. If $\Gamma$ is a non-bipartite $n$-coloured digraph and $\Gamma^1$ is the derived digraph obtained by the constant voltage map $\lambda \equiv 1$, then $\bar{\sigma}(\Gamma) = 1 + \sigma(\Gamma^1)$.*

*Proof.* Every nonorientable stiff embedding $\bar{\iota}_\varepsilon$ of $\Gamma$ is induced by an orientable stiff embedding $\iota_\varepsilon$ of $\Gamma^1$. Viceversa, let us consider an orientable stiff embedding $\iota_\varepsilon$ of $\Gamma^1$, which arises from a stiff arc rotation system $\Lambda = \{\varepsilon_{(v,u)} \mid (v,u) \in V(\Gamma)\times \mathbb{Z}_2\}$, where $\varepsilon_{(v,0)} = \varepsilon = \varepsilon^{-1}_{(v,0)}$, for all $v \in V(\Gamma)$. Since $\Lambda$ is the lifting of the constant arc rotation system $\Phi_\varepsilon$ on $\Gamma$, $\iota_\varepsilon$ induces a nonorientable stiff embeddings $\bar{\iota}_\varepsilon$ of $\Gamma$. Therefore, the statement follows from (1). □

## 3. Stiff embeddings of Cayley digraphs

If $G$ is a group, $X \subseteq G$ and $g \in G$, then $o(g)$ denotes the order of $g$ and $< X >$ denotes the subgroup of $G$ generated by $X$.

Let $\mathcal{X}(G) = \{X \subseteq G \mid < X > = G,\ e \notin X\}$ be the class of all generating sets for $G$, not containing the identity element. The Cayley digraph of $G$, associated to $X \in \mathcal{X}(G)$, is the $\#X$-coloured digraph $\mathcal{C}(G,X)$, which has $G$ as vertex set, $G \times X$ as edge set, the incidence maps $\alpha, \beta : G \times X \to G$ and the arc colouration $\gamma : G \times X \to X$ respectively given by $\alpha(g,x) = g$, $\beta(g,x) = gx$ and $\gamma(g,x) = x$, for all $(g,x) \in G \times X$.

Moreover, if $X^{-1} = \{g \in G \mid g^{-1} \in X\}$, we can take the set $X^{\pm} = X \cup X^{-1}$ (disjoint union) as signed colour set for $\mathcal{C}(G,X)$.

Now, we define the following subclasses of $\mathcal{X}(G)$:

$$\mathcal{O}(G) = \{X \in \mathcal{X}(G) \mid \mathcal{C}(G,X) \text{ is bipartite}\},$$

$$\mathcal{N}(G) = \{X \in \mathcal{X}(G) \mid \mathcal{C}(G,X) \text{ is non-bipartite}\}.$$

If $\#X \geq 2$, then $\mathcal{C}(G,X)$ stiffly embeds in an orientable surface (resp. in a nonorientable surface) if and only if $X \in \mathcal{O}(G)$ (resp. if and only if $X \in \mathcal{N}(G)$ ).

For each $\varepsilon \in \mathcal{B}(X^{\pm})$, we denote by $\iota_\varepsilon$ the orientable stiff embedding of a bipartite Cayley digraph $\mathcal{C}(G,X)$, induced by the choice $\varepsilon_e = \varepsilon$.

In the following we need the next two lemmas.

**Lemma 3.1.** *Let $G$ be a group. Then, $X \in \mathcal{O}(G)$ if and only if $G$ has a subgroup $A$ of index two such that $A \cap X = \emptyset$.*

*Proof.* $\Longrightarrow$ Let $\mathcal{C}(G,X)$ be bipartite, with partition of vertices $\{A,B\}$ such that $e \in A$. Each $a \in A$ (resp. $a \in B$) is the product of an even (resp. an odd) number of generators or their inverses. Thus, $A \cap X = \emptyset$. Since $\#A = \#B$ and moreover $x, y \in A \Rightarrow xy \in A$, then $A$ is a subgroup of index two of $G$. $\Longleftarrow$ If $g$ and $h$ are incident, then $h = gx$, for some $x \in X^{\pm}$, and moreover $g \in A \Leftrightarrow h \notin A$. Therefore, $\mathcal{C}(G,X)$ is bipartite with vertex set partition $\{A, G - A\}$. □

**Lemma 3.2.** *Let $G$ be a group and $X \in \mathcal{N}(G)$. If $\widetilde{X} = \{(g,1) \in G \oplus \mathbf{Z}_2 \mid g \in X\}$, then $\widetilde{X} \in \mathcal{O}(G \oplus \mathbf{Z}_2)$ and*

$$\bar{\sigma}(\mathcal{C}(G,X)) = 1 + \sigma(\mathcal{C}(G \oplus \mathbf{Z}_2, \widetilde{X})).$$

*Proof.* Let $\Gamma = \mathcal{C}(G,X)$. We show that $\widetilde{X} \in \mathcal{O}(G \oplus \mathbf{Z}_2)$. Since $\Gamma$ is non-bipartite, there exists an odd number of elements $h_1, h_2, \ldots, h_r \in X^{\pm}$ such that $\prod h_i = e$. Therefore, $\prod (h_i, 1) = (e,1) \in <\widetilde{X}>$. For each $g \in G$, there exist $k_1, k_2, \ldots, k_s \in X^{\pm}$ such that $\prod k_i = g$. Thus, for each $u \in \mathbf{Z}_2$, we have either $(g,u) = \prod (k_i, 1)$ or $(g,u) = (e,1) \prod (k_i,1)$, and therefore $(g,u) \in <\widetilde{X}>$. The set $A = \{(g,0) \mid g \in G\}$ is a subgroup of index two of $G \oplus \mathbf{Z}_2$ and $A \cap \widetilde{X} = \emptyset$. Since $(e,0) \notin \widetilde{X}$, we have $\widetilde{X} \in \mathcal{O}(G \oplus \mathbf{Z}_2)$ by Lemma 3.1. Now we are going to prove that the Cayley digraph $\Omega = \mathcal{C}(G \oplus \mathbf{Z}_2, \widetilde{X})$ and the derived digraph $\Gamma^1$, obtained by the choice $\lambda \equiv 1$, are isomorphic coloured digraphs. Let us consider the three functions $\phi : V(\Gamma^1) \to V(\Omega)$, $\psi : A(\Gamma^1) \to A(\Omega)$, $\omega : X \to \widetilde{X}$ defined by $\phi(g,u) = (g,u)$, $\psi((g,x),s) = ((g,s),(x,1))$ and $\omega(x) = (x,1)$. These maps are bijections between the sets of vertices, arcs and colours of the two coloured digraphs. Moreover, the maps preserve the incidence structures and the arc colourations, i.e. $\alpha' \circ \psi = \phi \circ \alpha^{\lambda}$, $\beta' \circ \psi = \phi \circ \beta^{\lambda}$ and $\gamma' \circ \psi = \omega \circ \gamma^{\lambda}$. Therefore, $\Gamma^1 \cong \Omega$ and the statement follows from Lemma 2.3. □

We close this section with two propositions containing an explicit connection between the genus of the surfaces where a Cayley digraph $\mathcal{C}(G,X)$ stiffly embeds and the order of certain elements of $G$. These results will be often used in Section 5 for obtaining the genus of some important classes of groups.

**Proposition 3.3.** *Let $G$ be a group, $X \in \mathcal{O}(G)$ such that $\#X = n \geq 2$, and $\varepsilon \in \mathcal{B}(X^{\pm})$. Then, the stiff embedding $\iota_\varepsilon$ of $\mathcal{C}(G,X)$ occurs in an orientable surface of genus*

$$g_\varepsilon = 1 + \frac{\#G}{2}(n - 1 - \sum_{x \in X} \frac{1}{l_x}), \tag{2}$$

*where $l_x = \mathrm{o}(x\varepsilon^{-1}x^{-1})$.*

*In particular, if $n = 2$ ($X = \{a,b\}$), $g_\varepsilon$ is independent of $\varepsilon$ and is equal to the orientable stiff genus of $\mathcal{C}(G,X)$:*

$$\sigma(\mathcal{C}(G,X)) = g_\varepsilon = 1 + \frac{\#G}{2}(1 - \frac{1}{\mathrm{o}(ab)} - \frac{1}{\mathrm{o}(ab^{-1})}). \tag{2'}$$

*Proof.* Let $y = \varepsilon^{-1}(x^{-1})$ and let $\{A, B\}$ be the partition of $V(C(G,X))$, with $e \in A$. Since $\varepsilon$ is balanced, $\varepsilon(y^{-1}) = x$. Therefore, the boundary of each region of $\iota_\varepsilon$ subsequently meets the $2\mathrm{o}(xy)$ vertices $g, gx, gxy, gxyx, \dots, g(xy)^{m-1}x$, for suitable $g \in A$ and $x \in X^\pm$. Once fixed $x \in X^\pm$, each vertex lies exactly in one of these circuits. Hence, the number of this type of regions is $R_x = \#G/(2\mathrm{o}(xy^{-1}))$. Since $\mathrm{o}(x^{-1}y^{-1}) = \mathrm{o}(yx) = \mathrm{o}(xy)$, we get $R_{x^{-1}} = R_x$, for every $x \in X$. Therefore, the number of regions of the embedding is $R = \sum_{x \in X^\pm} R_x = 2\sum_{x\in X} R_x$. The genus $g_\varepsilon$ follows from Euler formula. If $n = 2$, then $\mathcal{B}(X^\pm) = \{\varepsilon, \varepsilon'\}$, where $\varepsilon = (a\ b\ a^{-1}\ b^{-1})$ and $\varepsilon' = (a\ b^{-1}\ a^{-1}\ b)$. Since $\mathrm{o}(ba^{-1}) = \mathrm{o}(ab^{-1})$ and $\mathrm{o}(ba) = \mathrm{o}(ab)$, we get $g_\varepsilon = g_{\varepsilon'} = 1 + \#G(1 - 1/\mathrm{o}(ab) - 1/\mathrm{o}(ab^{-1}))/2 = \sigma(C(G,X))$. □

**Proposition 3.4.** *Let $G$ be a group, $X \in \mathcal{N}(G)$ such that $\#X = n \geq 2$, and $\varepsilon \in \mathcal{B}(X^\pm)$. If $l_x$ is defined as in Proposition 3.3, then the stiff embedding $\bar{\iota}_\varepsilon$ of $C(G,X)$ occurs in a nonorientable surface of genus:*

$$\bar{g}_\varepsilon = 2 + \#G(n - 1 - \sum_{x \in X} \frac{1}{l_x}). \tag{3}$$

*In particular, if $n = 2$ ($X = \{a, b\}$), $\bar{g}_\varepsilon$ is independent of $\varepsilon$ and is equal to the nonorientable stiff genus of $C(G,X)$:*

$$\bar{\sigma}(C(G,X)) = \bar{g}_\varepsilon = 2 + \#G(1 - \frac{1}{\mathrm{o}(ab)} - \frac{1}{\mathrm{o}(ab^{-1})}). \tag{3'}$$

*Proof.* Let us consider the Cayley digraph $C(G \oplus \mathbb{Z}_2, \widetilde{X})$, where $\widetilde{X}$ is defined as in Lemma 3.2, and identify $\varepsilon$ with $\tilde{\varepsilon} = \omega \circ \varepsilon \circ \omega^{-1} \in \mathcal{B}(\widetilde{X}^\pm)$, being $\omega$ the map defined in the proof of Lemma 3.2. Since $\mathrm{o}((x,1)^u(y,1)^v) = \mathrm{o}(x^u y^v)$, for all $u, v \in \{+1, -1\}$, we can apply (2) and (2′), using the elements of $X$ in the computation of the orders, and the results follow from (1). □

## 4. Stiff genus of groups

Let us start to give the notion of stiff genus of groups.

**Definition 4.1.** *The orientable stiff genus $\rho(G)$ (resp. nonorientable stiff genus $\bar{\rho}(G)$) of a group $G$ is the minimum of the set $\{\sigma(C(G,X)) \mid X \in \mathcal{X}(G)\}$ (resp. of the set $\{\bar{\sigma}(C(G,X)) \mid X \in \mathcal{X}(G)\}$).*

For cyclic groups, the computation is immediate.

**Proposition 4.2.** *Let $\mathbf{Z}_m$ be the cyclic group of order $m > 1$, then:*

$$\rho(\mathbf{Z}_m) = 0, \qquad \bar{\rho}(\mathbf{Z}_m) = 1$$

*Proof.* Apply Lemma 2.2.a to $C(\mathbf{Z}_m, \{1\})$. □

In order to compute the orientable (resp. the nonorientable) stiff genus of a non-cyclic group, the attention can be restricted to bipartite (resp. non-bipartite) Cayley digraphs.

**Proposition 4.3.** *(a) Let $G$ be a non-trivial group, then $\bar{\rho}(G) < +\infty$.*

*(b) Let $G$ be a non-cyclic group, then $\rho(G) < +\infty$ if and only if $G$ has a subgroup of index two.*

*(c) Let $G$ be a non-cyclic group, then $\bar{\rho}(G) \geq 1 + \rho(G \oplus \mathbf{Z}_2)$.*

*Proof.* (a) If $G$ is cyclic, the result follows from Proposition 4.2. Otherwise, let us choose $X = G - \{e\}$. Evidently, $X \in \mathcal{X}(G)$ and there is no subgroup of index two disjoint from $X$. By Lemma 3.1, $C(G, X)$ is non-bipartite and stiffly embeds in a nonorientable surface.

(b) $\Longrightarrow$ If $\rho(G) < +\infty$, there exists a Cayley digraph $C(G, X)$ stiffly embeddable in an orientable surface. Therefore, $C(G, X)$ is bipartite and, by Lemma 3.1, there exists a subgroup of $G$ of index two. $\Longleftarrow$ Let us define $X = G - A$, where $A$ is a subgroup of index two of $G$. Since $e \notin X$ and $X \cup \{e\} \subseteq \langle X \rangle$, then $\#G/2 < \# \langle X \rangle$ and therefore $\langle X \rangle = G$. By Lemma 3.1, $C(G, X)$ is bipartite and admits an orientable stiff embedding.

(c) Follows from Lemma 3.2. □

From point (b) of Proposition 4.3, the following results can be easily obtained.

**Corollary 4.4.** *Let $G$ be a non-cyclic group.*

*(a) If $\#G$ is odd, then $\rho(G) = +\infty$.*

*(b) If $G$ is abelian, then $\#G$ is odd if and only if $\rho(G) = +\infty$.*

*(c) If $G$ is simple, then $\rho(G) = +\infty$.*

*Proof.* (a) Trivial.

(b) Every abelian group of even order has a subgroup of index two.

(c) A simple group has no subgroups of index two. □

*REMARK 4.1.* Proposition 4.3.c can not be extended to cyclic groups. In fact $\bar{\rho}(\mathbf{Z}_2) = 1$ and, as we shall see in Section 5, $\rho(\mathbf{Z}_2 \oplus \mathbf{Z}_2) = 1$. The converse of Corollary 4.4.a is false for non-abelian groups. For example, the alternating group $A_4$ has order 12, but it has no subgroups of index two and therefore $\rho(A_4) = +\infty$.

A generating set $X$ for a group $G$ is called *irredundant* if every proper subgroup of $X$ does not generate $G$. Otherwise, $X$ is called *redundant*. Irredundant generating sets for a group play an important role in the computation of the stiff genus.

**Lemma 4.5.** *Let $X, Y \in \mathcal{X}(G)$ such that $X \subseteq Y$.*
*(a) $\sigma(\mathcal{C}(G,X)) \leq \sigma(\mathcal{C}(G,Y))$.*
*(b) If $X \in \mathcal{N}(G)$, then $\bar{\sigma}(\mathcal{C}(G,X)) \leq \bar{\sigma}(\mathcal{C}(G,Y))$.*

*Proof.* Obviously, if $\mathcal{C}(G,X)$ is non-bipartite then so is $\mathcal{C}(G,Y)$. The statements follow from Lemma 2.2.a, if $\#X = 1$, and from Lemma 4.1 of [2], if $\#X \geq 2$. □

**Corollary 4.6.** *Let $G$ be a group, then there exists $X \in \mathcal{X}(G)$ irredundant such that $\sigma(\mathcal{C}(G,X)) = \rho(G)$.*

*Proof.* $\rho(G) = \sigma(\mathcal{C}(G,Y))$, for some $Y \in \mathcal{X}(G)$. If $X$ is any irredundant generating set such that $X \subseteq Y$, then $\sigma(\mathcal{C}(G,X)) = \rho(G)$, by Lemma 4.5.a. □

As a consequence of Corollary 4.6, in order to compute the orientable stiff genus of a group, we are allowed to just consider irredundant generating sets. In particular, the following condition can be assumed:

$$x, x^{-1} \in X \implies x^{-1} = x. \tag{4}$$

Corollary 4.6 is false for the nonorientable case. In fact, if $G = \mathbf{Z}_2 \oplus \mathbf{Z}_2$, then $\mathcal{N}(G)$ contains only the redundant generator set $X = G - \{(0,0)\}$. However, by Lemma 4.5.b we may consider, in the computation of the nonorientable stiff genus of a non-cyclic group, only generating sets that are minimal with regard to inclusion in $\mathcal{N}(G)$. Since if $X \in \mathcal{N}(G)$ and $x^{-1} \neq x \in X$, then $X - \{x^{-1}\} \in \mathcal{N}(G)$, condition (4) can be assumed also in nonorientable cases.

**Proposition 4.7.** *Let $X \in \mathcal{X}(G)$ such that: (i) $\#X \geq 2$, (ii) $x, x^{-1} \in X \Rightarrow x^{-1} = x$.*
*(a) If $X \in \mathcal{O}(G)$, then $\sigma(\mathcal{C}(G,X)) \geq 1 + \#G(\#X - 2)/4$.*
*(b) If $X \in \mathcal{N}(G)$, then $\bar{\sigma}(\mathcal{C}(G,X)) \geq 2 + \#G(\#X - 2)/2$.*

*Proof.* (a) By condition (ii), we can apply (2) with $l_x \geq 2$, for all $x \in X$.

(b) If $\widetilde{X}$ is the set defined in Lemma 3.2, then it verifies (i) and (ii). By point (a) and Lemma 3.2 we have $\bar{\sigma}(\mathcal{C}(G,X)) = 1 + \sigma(\mathcal{C}(G \oplus \mathbf{Z}_2, \widetilde{X})) \geq 2 + \#G(\#X - 2)/2$. □

**Corollary 4.8.** *If G is a non-cyclic group, then:*

$$\rho(G) \geq 1, \qquad \bar{\rho}(G) \geq 2. \qquad \square$$

## 5. Examples of computation

Proposition 4.2 states that $\rho(\mathbf{Z}_m) = 0$ and $\bar{\rho}(\mathbf{Z}_m) = 1$, for every cyclic group of order $m > 1$. Now, we are going to compute the stiff genus of several classical families of non-cyclic groups. As explained before, we can consider only generator sets $X \in \mathcal{X}(G)$ verifying condition (4). So, applying (2), (2′), (3) and (3′), we shall always assume $o(ab^{\pm 1}) \geq 2$ and $l_x \geq 2$, for all $x \in X$.

**Elementary abelian groups.**

Let us consider the elementary abelian $p$-group $\mathbf{Z}_p^m$, with $m \geq 2$. We can regard $\mathbf{Z}_p^m$ as a vector space of dimension $m$ over $\mathbf{Z}_p$. Since $< X > = L(X)$, for every $X \subseteq \mathbf{Z}_p^m$, the set $X$ is an irredundant subset of generators for the group $\mathbf{Z}_p^m$ if and only if $X$ is a basis for the vector space $\mathbf{Z}_p^m$. Note that $o(g) = p$, when $g \neq e$.

**Proposition 5.1.** *Let $p$ be a prime number and $m \geq 2$, then:*

$$\rho(\mathbf{Z}_p^m) = \begin{cases} 1 + (m-2)2^{m-2} & \text{if } p = 2 \\ +\infty & \text{if } p \geq 3 \end{cases},$$

$$\bar{\rho}(\mathbf{Z}_p^m) = \begin{cases} 2 + (m-1)2^{m-1} & \text{if } p = 2 \\ 2 + p^{m-1}(mp - m - p) & \text{if } p \geq 3 \end{cases}.$$

*Proof.* (a) – orientable case – Let us suppose $p = 2$. Each generator set of $\mathbf{Z}_2^m$ has at least $m$ elements. Therefore, $\rho(\mathbf{Z}_2^m) \geq 1+(m-2)2^{m-2}$ by Proposition 4.7.a. Now, let us take the canonical basis $B$ as a generator set of $\mathbf{Z}_2^m$. From Lemma 3.1, applied with the subgroup $A = \{(a_1, a_2, \ldots, a_m) \in \mathbf{Z}_2^m \mid a_1 + a_2 + \cdots + a_m \text{ is even}\}$, follows that $C(\mathbf{Z}_2^m, B)$ is bipartite. Using (2) with $l_x = 2$, for all $x \in X$, we obtain $g_\varepsilon = 1 + (m-2)2^{m-2}$, for every $\varepsilon \in \mathcal{B}(X^{\pm})$. Finally, if $p \geq 3$, $\#\mathbf{Z}_p^m$ is odd and the result is a straightforward consequence of Corollary 4.4.a.

(b) – nonorientable case – Let us suppose $p = 2$. From Proposition 4.3.c we get $\bar{\rho}(\mathbf{Z}_2^m) \geq 1 + \rho(\mathbf{Z}_2^{m+1}) = 2 + (m-1)2^{m-1}$. Now, let us take $X = B \cup \{e\}$, where $B = \{e_i \mid 1 \leq i \leq m\}$ is the canonical basis of $\mathbf{Z}_2^m$ and $e = e_1 + e_2$. Clearly, $X$ is redundant and $\Gamma = C(\mathbf{Z}_2^m, X)$ is non-bipartite because $e + e_1 + e_2 = 0$. We can apply (3) with $l_x = 2$, for all $x \in X$. Since $\bar{g}_\varepsilon = 2 + (m-1)2^{m-1}$, for every $\varepsilon \in \mathcal{B}(X^{\pm})$, the statement follows. Now, suppose $p \geq 3$. Since $\mathcal{X}(\mathbf{Z}_p^m) = \mathcal{N}(\mathbf{Z}_p^m)$, we can consider only irredundant subsets of generators (i.e. bases). Let $X$ be any of them, we can use (3) with $l_x = p$, for all $x \in X$. Since $\bar{g}_\varepsilon = 2 + p^{m-1}\big((p-1)(m-1) - 1\big)$, for every $\varepsilon \in \mathcal{B}(X^{\pm})$, the proof is completed. □

In Figure 2 is presented a stiff embedding on the torus of the "canonical" Cayley digraph of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.

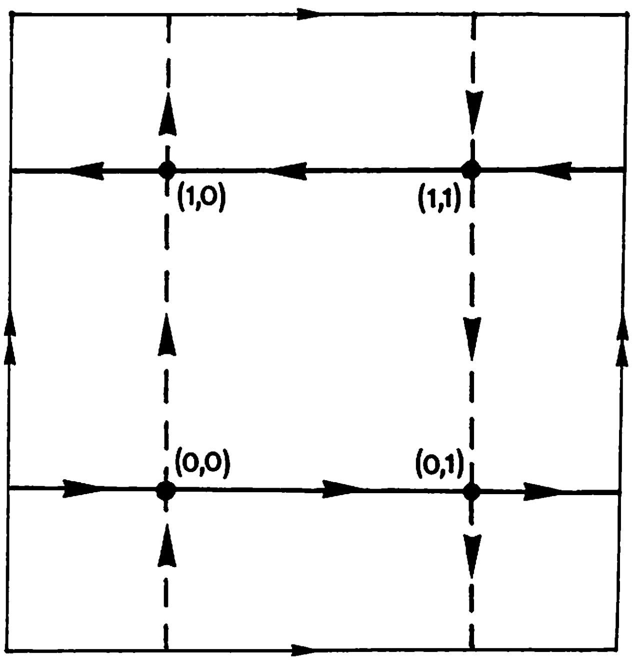


FIGURE 2. $\mathcal{C}(\mathbf{Z}_2 \oplus \mathbf{Z}_2, \{e_1, e_2\})$ on the torus

**Dihedral groups.**

Let $D_{2m}$ ($m \geq 3$) be the group of all symmetries of a regular polygon with $m$ sides, called dihedral group, which admits the presentation

$$D_{2m} = < x, y \mid x^2 = y^2 = (xy)^m = e >$$

and has cardinality $\#D_{2m} = 2m$. From these relations we obtain $y = y^{-1} = (xy)^{m-1}x$. Therefore, each $g \in D_{2m}$ can be written in a unique way in the form $g = (xy)^k x^h$, where $0 \leq k \leq m-1$ and $0 \leq h \leq 1$. If $m$ is odd, the set $A_1 = \{(xy)^k \mid 0 \leq k \leq m-1\}$ is the only subgroup of index two of $D_{2m}$. Instead, if $m$ is even, then there exist other two subgroups of index two: $A_2 = \{(xy)^{2i}x^h \mid 0 \leq i \leq m/2-1, 0 \leq h \leq 1\}$ and $A_3 = \{(xy)^k x^h \mid k+h \text{ is even}\}$. Note that $g^2 = e$ for all $g \notin A_1$.

**Proposition 5.2.** *Let $D_{2m}$ be the dihedral group of order $2m$ ($m \geq 3$). Then*

$$\rho(D_{2m}) = \begin{cases} 1 & \text{if } m \text{ is even} \\ m-1 & \text{if } m \text{ is odd} \end{cases},$$

$$\bar{\rho}(D_{2m}) = \begin{cases} 2 & \text{if } m \text{ is odd} \\ 2+m & \text{if } m \text{ is even and } 4 \not| m \end{cases}.$$

*Moreover, if $m$ is even, then $\bar{\rho}(D_{2m}) \geq 2+m$.*

*Proof.* (a) – orientable case – Let us suppose $m$ even. Choose $X = \{xy, x\}$ as a generator set of $D_{2m}$. Then $X \in \mathcal{O}(D_{2m})$ because $X \cap A_3 = \emptyset$. If $a = xy$ and $b = x$, then $ab^{\pm 1} \notin A_1$ and therefore $o(ab^{\pm 1}) = 2$. From (2′) we obtain $\sigma(\mathcal{C}(G, X)) = 1$.

Now, let us suppose $m$ odd. Since $A_1$ is the only subgroup of index two of $D_{2m}$, necessarily $A_1 \cap X = \emptyset$, for all $X \in \mathcal{O}(D_{2m})$. If $\#X = 2$, then $X = \{a, b\}$, whit $a = (xy)^k x$ and $b = (xy)^h x$ such that $\gcd(|k-h|, m) = 1$. If $k > h$, then $ab^{\pm 1} = (xy)^{k-h}$ and therefore $o(ab^{\pm 1}) = m$. From (2′), we obtain $\sigma(\mathcal{C}(D_{2m}, X)) = m-1$. Let us consider the case $\#X \geq 3$. If $x, y \in X$, with $x \neq y$, then $g = xy^{\pm 1} \in A_1$. Since $A_1$ is a cyclic group of order $m$, necessarily $o(g) \mid m$ and therefore $o(g) \geq 3$. Using (2) with $l_x \geq 3$, for all $x \in X$, we obtain $g_\varepsilon \geq 1+m$, for every $\varepsilon \in \mathcal{B}(X^{\pm})$. Thus, $\rho(D_{2m}) = m-1$

(b) – nonorientable case – Let us suppose $m$ odd. In this case $X = \{xy, x\} \in \mathcal{N}(D_{2m})$, since $o(xy) = m$. Applying (3′) we get $\bar{\sigma}(\mathcal{C}(D_{2m}, X)) = 2$ and the result follows from Corollary 4.8. Conversely, suppose $n$ even. It is easy to check that if $X \in \mathcal{N}(D_{2m})$, then $\#X \geq 3$. By Proposition 4.7.b, we get $\bar{\sigma}(\mathcal{C}(D_{2m}, X)) \geq 2 + m(\#X - 2) \geq 2+m$ and therefore $\bar{\rho}(D_{2m}) \geq 2+m$. If $4 \not| m$, let us select $X = \{xy, x, z\}$ where $z = (xy)^{m/2}x$. Since $zx(xy)^{m/2} = e$, there exists an odd cycle in $\mathcal{C}(D_{2m}, X)$ and therefore $X \in \mathcal{N}(D_{2m})$. We get $o(xyx^{\pm 1}) = o(xyz^{\pm 1}) = o(xz^{\pm 1}) = 2$. Hence, applying (3) with $l_{xy} = l_x = l_z = 2$, we obtain $\bar{g}_\varepsilon = 2+m$, for every $\varepsilon \in \mathcal{B}(X^{\pm})$, and the statement is proved. □

Two embeddings of Cayley digraphs of $D_6$ in the Klein bottle and in an orientable surface $T_2$ of genus two are respectively shown in Figure 3 and Figure 4.

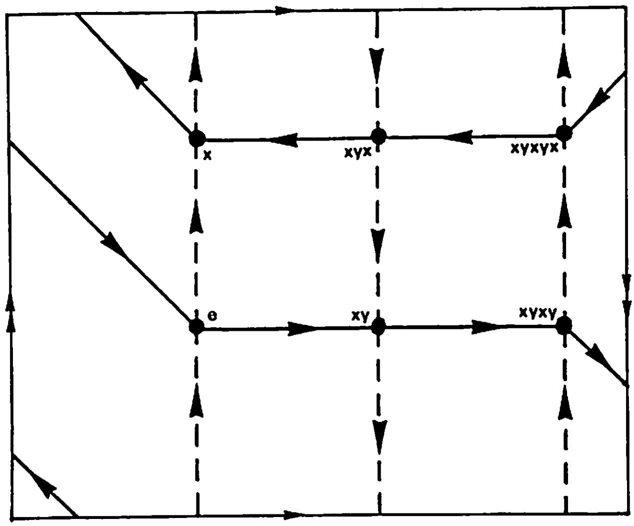


FIGURE 3. $C(D_6, \{x, xy\})$ on the Klein bottle

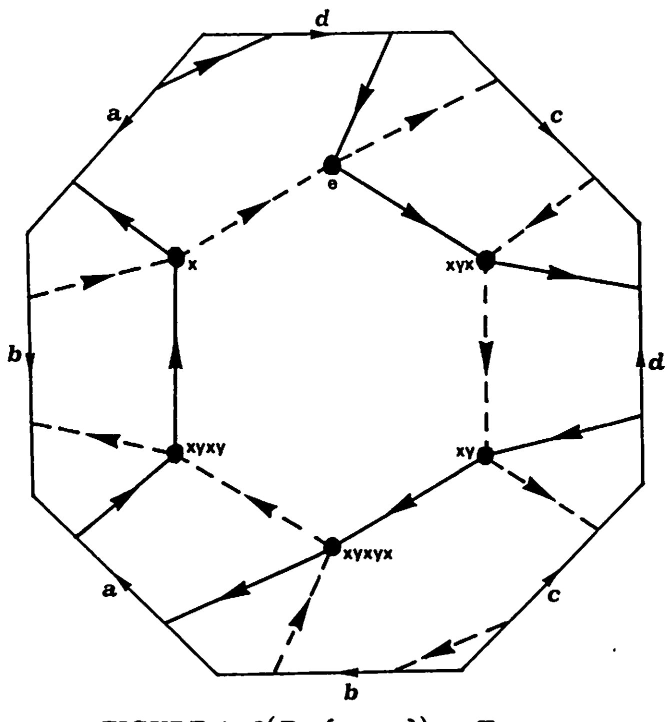


FIGURE 4. $C(D_6, \{x, xyx\})$ on $T_2$

**Dicyclic groups.**

Let $Q_m$ ($m \geq 2$) be the dicyclic group of order $4m$. The group admits the presentation

$$Q_m = < x, y \mid x^{2m} = x^m y^{-2} = y^{-1} x y x = e >$$

and, in particular, $Q_2$ is the quaternion group. From these relations we get $yx^k = x^{-k}y$ and $y^{-1} = x^m y$. Thus, each $g \in Q_m$ can be written in a unique way in the form $g = x^k y^h$, where $0 \leq k \leq 2m-1$ and $0 \leq h \leq 1$. Let $A_1, A_2, A_3 \subseteq Q_m$ be the subsets: $A_1 = \{x^k \mid 0 \leq k \leq 2m-1\}$, $A_2 = \{x^k y^h \mid h+k \text{ is even}\}$ and $A_3 = \{x^{2i} y^h \mid 0 \leq i \leq m-1, 0 \leq h \leq 1\}$. If $m$ is even, they are the only subgroups of index two of $Q_m$. Conversely, if $m$ is odd, $A_1$ is the only subgroup of index two, whereas $A_2$ and $A_3$ are not subgroups. Finally, note that $g^4 = e$, for all $g \notin A_1$.

**Proposition 5.3.** *Let $Q_m$ be the dicyclic group of order $4m$ ($m \geq 2$), then:*

$$\rho(Q_m) = \begin{cases} m+1 & \text{if } m \text{ is even} \\ 2m-2 & \text{if } m \text{ is odd} \end{cases},$$

$$\bar{\rho}(Q_m) = \begin{cases} 2m+2 & \text{if } m \text{ is odd} \\ 5m+2 & \text{if } m \text{ is even and } 4 \nmid m \end{cases}.$$

*Moreover, if $m$ is even, then $\bar{\rho}(Q_m) \geq 5m+2$.*

*Proof.* (a) – orientable case – Let us suppose $m$ even. Define the family of generating sets $\mathcal{S} = \{X \in \mathcal{O}(Q_m) \mid \#X = 2\}$. If $X \in \mathcal{S}$, then necessarily $\#X \cap A_1 \leq 1$. Furthermore, let $\mathcal{S}_1 = \{X \in \mathcal{S} \mid \#X \cap A_1 = 1\}$ and $\mathcal{S}_0 = \mathcal{S} - \mathcal{S}_1$. Then $\mathcal{S}_1, \mathcal{S}_0 \neq \emptyset$, since $\{y, xy\} \in \mathcal{S}_0$ and $\{x, y\} \in \mathcal{S}_1$ (note that $\{x, y\} \cap A_2 = \emptyset$). For each $X = \{a, b\} \in \mathcal{S}_1$ we have $ab^{\pm 1} \notin A_1$ and therefore $\mathrm{o}(ab^{\pm 1}) = 4$. From (2′) we obtain $\sigma(\mathcal{C}(Q_m, X)) = 1 + m$. Conversely, if $X \in \mathcal{S}_0$, then $X = \{a, b\}$, with $a = x^k y$ and $b = x^j y$ such that $\gcd(|k-j|, m) = 1$. Since $\mathrm{o}(ab^{\pm 1}) = 2m$, from (2′) we obtain $\sigma(\mathcal{C}(Q_m, X)) = 2m - 1 \geq 1 + m$. Finally, if $X \in \mathcal{O}(Q_m) - \mathcal{S}$, from Proposition 4.7.a follows $\sigma(\mathcal{C}(Q_m, X)) \geq 1 + \#Q_m(\#X - 2)/4 \geq 1 + m$. Therefore, $\rho(Q_m) = m + 1$.

Conversely, suppose $m$ odd. Since $A_1$ is the only subgroup of index two of $Q_m$, we have $\mathcal{S}_1 = \emptyset$. Let $X = \{a, b\} \in \mathcal{S}_0$, then $\mathrm{o}(ab^u) = m$ if and only if $\mathrm{o}(ab^{-u}) = 2m$, for every $u \in \{+1, -1\}$. From (2′), we obtain $\sigma(\mathcal{C}(Q_m, X)) = 2m - 2$. Now, let us take $X \in \mathcal{O}(Q_m) - \mathcal{S}$. For every $a, b \in X$ ($a \neq b^{\pm 1}$) we have $ab^{\pm 1} \in A_1$, and moreover $\mathrm{o}(ab^{\pm 1}) \neq 2$. Using (2), with $l_x \geq 3$ for all $x$, we get $g_\varepsilon \geq 1 + \#Q_m(\#X - 1 - \#X/3)/2 = 1 + 2m(2\#X/3 - 1) \geq 1 + 2m \geq 2m - 2$, for every $\varepsilon \in \mathcal{B}(X^{\pm})$.

(b) – nonorientable case – Suppose $m$ odd. If $X \in \mathcal{N}(Q_m)$ and $\#X = 2$, then $X = \{a, b\}$ with $a \in A_1$ and $b \notin A_1$. We have $ab^{\pm 1} \notin A_1$ and therefore $o(ab^{\pm 1}) = 4$. Using (3′) we obtain $\bar{\sigma}(C(Q_m, X)) = 2m + 2$. Otherwise, if $\#X \geq 3$ we obtain, by Proposition 4.7.b, $\bar{\sigma}(C(Q_m, X)) \geq 2+2m(\#X-2) \geq 2m + 2$.

Conversely, suppose $m$ even. If $X \in \mathcal{N}(Q_m)$, then $\#X \geq 3$ and moreover there exist $a, b \in X$ such that $a \in A_1$ and $b \notin A_1$. Since $ab^{\pm 1} \notin A_1$, at least four of the different types of circuits of every stiff embedding of $C(Q_m \oplus \mathbb{Z}_2, \tilde{X})$ must have length 8. First let us suppose $\#X = 3$. Then, necessarily the remaining two types of circuits have at least length 8 (otherwise $X \notin \mathcal{N}(Q_m)$). Hence, using (3) with $l_x \geq 4$ for all $x \in X$, we obtain $\bar{g}_\varepsilon \geq 2 + 5m$, for every $\varepsilon \in \mathcal{B}(X^\pm)$. On the other hand, if $\#X \geq 4$, we can apply again (3) with $l_x = 4$ two times, and with $l_x \geq 2$ the remaining $\#X - 2$ times. We obtain $\bar{g}_\varepsilon \geq 2 + 4m(\#X - 1 - 1/2 - (\#X - 2)/2) = 2+2m(\#X-1) \geq 2+6m \geq 2+5m$, for every $\varepsilon \in \mathcal{B}(X^\pm)$. If $4 \not| m$, then take $X = \{x, y, z\}$, where $z = x^{m/2-1}$. Since $zx^{3m/2+1} = e$, there exists an odd cycle in $C(Q_m, X)$ and therefore $X \in \mathcal{N}(Q_m)$. Choosing the permutation $\varepsilon = (y\ x^{-1}\ z\ y^{-1}\ x\ z^{-1})$, we can apply (3) with $l_x = l_y = l_z = 4$, in order to obtain $\bar{g}_\varepsilon = 5m + 2$. This complete the proof. □

**Non-abelian groups of order $p^3$.**

Given a prime odd number $p$, then there exist two non-isomorphic non-abelian groups of order $p^3$, one with exponent $p$ and presentation

$$G_p = < x, y \mid x^p = y^p = e,\ x^{-1}[x, y]x = [x, y] = y^{-1}[x, y]y >,$$

the other with exponent $p^2$ and presentation

$$G'_p = < x, y \mid (x^p)^p = y^p = e,\ y^{-1}xy = x^{1+p} >$$

(see [6], p. 141).

**Proposition 5.4.** *Let $p$ be a prime number ($p \geq 3$). If $G_p$ and $G'_p$ are the non-abelian groups of order $p^3$ and exponent respectively $p$ and $p^2$, then:*

$$\rho(G_p) = +\infty, \qquad \rho(G'_p) = +\infty,$$

$$\bar{\rho}(G_p) = 2 + p^2(p - 2), \qquad \bar{\rho}(G'_p) = 2 + p(p^2 - p - 1).$$

*Proof.* (a) – orientable case – Follows from Corollary 4.4.a, since the two groups have odd order.

(b) – nonorientable case – $G_p$ has exponent $p$, therefore $o(g) = p$, for all $g \in G_p - \{e\}$. For every $X \in \mathcal{X}(G_p) = \mathcal{N}(G_p)$, we can apply (3) with $l_x = p$, for all $x \in X$. For every $\varepsilon \in \mathcal{B}(X^{\pm})$, we get $\bar{g}_\varepsilon = 2 + (\#X(p-1) - p)p^2$ and when $\#X = 2$ this value achieves the minimum $\bar{\rho}(G_p) = 2 + (p-2)p^2$.

The elements of $G'_p$ can be uniquely write in the form $y^h x^k$, where $0 \le h \le p-1$ and $0 \le k \le p^2 - 1$. The set $H = \{y^h x^{lp} \mid 0 \le h, l \le p-1\}$ is a subgroup of index $p$ of $G'_p$ and $o(g) = p$, for all $g \in H - \{e\}$. Furthermore, $o(g) = p^2$, for all $g \in G'_p - H$, and $\mathrm{Frat}(G'_p) = \{x^{lp} \mid 0 \le l \le p-1\}$. Since $[G'_p : \mathrm{Frat}(G'_p)] = p^2$, every set of generators of $G'_p$ always contains a subset of generators with two elements, by the Burnside Basis Theorem (see [6], 5.3.2). Thus, we can only consider the case $\#X = 2$ ($X = \{a, b\}$). If $\#X \cap H = 1$, then $ab^{\pm 1} \notin H$ and therefore $o(ab^{\pm 1}) = p^2$. Using (3′), we obtain $\bar{\sigma}(C(G'_p, X)) = 2 + p(p^2 - 2)$. Suppose now $X \cap H = \emptyset$. In this case, if $ab^{\pm 1} \in H$, then $ab^{\mp 1} \notin H$. Therefore, either $o(ab) = p^2$ or $o(ab) = p^2$. Using again (3′) we get $\bar{\sigma}(C(G'_p, X)) \ge 2 + p(p^2 - p - 1)$. Finally, if we choose $X = \{a, b\}$, with $a = x$ and $b = yx$, we have $o(ab^{-1}) = p$ and $o(ab) = p^2$. By (3′) we obtain $\bar{\sigma}(C(G'_p, X)) = 2 + p(p^2 - p - 1)$. This completes the proof. □

## 6. Stiff genus compared with symmetric genus

An *automorphism* of a coloured digraph $\Gamma$ is a pair $(\phi, \psi)$, where $\phi$ is a bijection on $A(\Gamma)$ and $\psi$ is a bijection on $V(\Gamma)$ such that: $\alpha \circ \phi = \psi \circ \alpha$, $\beta \circ \phi = \psi \circ \beta$ and $\gamma \circ \phi = \gamma$.

Let $G$ be a group and $C(G, X)$ be a Cayley digraph of $G$. The group $G$ acts on $C(G, X)$ (and on $|C(G, X)|$), through the family of automorhisms $\{(\phi_g, \psi_g) \mid g \in G\}$ defined by $\phi_g(a) = ga$ and $\psi_g(a, x) = (ga, x)$, for all $a \in G$ and for all $(a, x) \in G \times X$. The action is free and vertex transitive and it is called the *natural action* of $G$ on $C(G, X)$. As well known (see, e.g. [9]), $G$ is isomorphic to $\mathrm{Aut}(C(G, X))$, for every choice of $X$.

Now we shall recall some definitions from [5] about the symmetric genus.

An embedding $\iota : |\Gamma| \to S$ of the Cayley digraph $\Gamma = C(G, X)$ in an orientable surface $S$ is said to be *symmetric* and only if the natural action of $G$ on $\Gamma$ extends to $S$. That is to say, if there exists an action $\mathcal{A}$ of $G$ on $S$ such that $\iota(|\Gamma|)$ is invariant for it and moreover the restriction of $\mathcal{A}$ to $\iota(|\Gamma|)$ corresponds to the natural action of $G$ on $\Gamma$. Furthermore, if every $g \in G$ preserves the orientation (resp. there exists a $g \in G$ which reverses the orientation) of $S$, then the embedding is called *strongly symmetric* (resp. is called *weakly symmetric*).

So, the property of an embedding to be symmetric is topological and it is equivalent to some combinatorial conditions on the arc rotation system defining the embedding:

**Lemma 6.1.** *(a) ([5], p. 266) An embedding of a Cayley digraph is strongly symmetric if and only if the rotation $\varepsilon_v$ at each vertex $v$ is the same.*

*(b) ([5], p. 282) An embedding of a Cayley digraph $C(G,X)$ is weakly symmetric if and only if there exists a cyclic permutation $\varepsilon$ of $X^{\pm}$ and a subgroup $A$ of index two in $G$ such that $\varepsilon_v = \varepsilon_w^{-1} = \varepsilon$, for all $v \in A$ and for all $w \in G - A$.* □

Comparing Lemma 6.1.b with the definition of orientable stiff embedding given in Section 2 and with Lemma 3.1, we note that, when $\#X \geq 2$, the conditions imposed to orientable embeddings to be stiff are stronger than the conditions of Lemma 6.1.b. Therefore, every stiff embedding is weakly symmetric.

Instead, if $\#X = 1$, the two notions are coincident: an embedding is stiff if and only if is symmetric (weakly or strongly). More precisely, if $\#G$ is even, then every stiff embedding is both strongly symmetric and weakly symmetric. Conversely, if $\#G$ is odd then the stiff embedding is just strongly symmetric.

The *symmetric genus* $\nu(G)$ (resp. *strongly symmetric genus* $\nu_0(G)$) of a group $G$ is the minimum genus of an orientable surface where some Cayley digraph of $G$ can be embedded in a symmetric way (resp. in a strongly symmetric way). Of course, if $\gamma(G)$ denotes the "classical" genus of $G$, we have $\gamma(G) \leq \nu(G) \leq \nu_0(G)$.

From the above considerations follows immediately that $\nu(G) \leq \rho(G)$, for every group $G$. Moreover, there are groups for which the two genera are different: for instance, $\nu(D_{2m}) = 0$ (see [5], p. 287) and $\rho(D_{2m}) \geq 1$. Clearly, for cyclic groups the two genera coincide: $\nu(\mathbf{Z}_m) = \rho(\mathbf{Z}_m) = 0$.

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## References

[1] Ceré A. *Regular embeddings of coloured digraphs,* Geom. Dedicata, 29 (1989), 97-107. *Errata corrige,* Geom. Dedicata, 33 (1990), 239.
[2] Chiavacci R. and Pareschi G. *Some bounds for the regular genus of PL-manifolds,* Discrete Math. 2 (1990), 165-180.
[3] Gagliardi C. *Regular embeddings of edge-coloured graphs,* Geom. Dedicata 11 (1981), 397-414.
[4] Gagliardi C. *Extending the concept of genus to dimension n,* Proc. Amer. Math. Soc. 81 (1981), 473-481.
[5] Gross J.L. and Tucker T.W. *Topological graph theory,* Wiley, New York, (1987).
[6] Robinson D.J.S. *A course in the theory of groups,* Springer-Verlag, New York, (1982).
[7] Stahl S. *Generalized embedding schemes,* J. Graph Theory 2 (1978), 41-52.
[8] Stahl S. *The embedding of a graph - a survey,* J. Graph Theory 2 (1978), 275-298.
[9] White A.T. *Graphs, groups and surfaces,* North Holland, (1973).



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