INDEPENDENCE IN DIRECT-PRODUCT GRAPHS

Pranava K Jha
Dept of Computer Engineering
Delhi Institute of Technology: Delhi
Kashmere Gate, Delhi 110 006, INDIA
e-mail: pkj@dit.ernet.in

Sandi Klavžar *
Department of Mathematics, PEF
University of Maribor
Koroška cesta 160, 62000 Maribor, SLOVENIA
e-mail: sandi.klavzar@uni-lj.si

Abstract

Let $\alpha(G)$ denote the independence number of a graph G and let $G \times H$ be the direct product of graphs G and H. Set $\underline{\alpha}(G \times H) = \max\{\alpha(G) \cdot |H|, \alpha(H) \cdot |G|\}$. If G is a path or a cycle and H is a path or a cycle then $\alpha(G \times H) = \underline{\alpha}(G \times H)$. Moreover, this equality holds also in the case when G is a bipartite graph with a perfect matching and H is a traceable graph. However, for any graph G with at least one edge and for any $i \in \mathbb{N}$ there is a graph H such that $\alpha(G \times H) > \underline{\alpha}(G \times H) + i$.

1 Introduction

Problems of determining the independence number and the matching number of a graph are also important because of applications of these invariants in many areas, notably, (*i*) selection by PRAM, (*ii*) VLSI layout, (*iii*) wire coloring, and (*iv*) processor scheduling. While independence number problem is NP-hard [5], matching is solvable in polynomial time [17]. Recently it was shown that independence number is not even approximable within a

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factor of n^c for any $c > 0$ unless P=NP [2, 3]. For a product graph, solving these problems via factor graphs is economical, since problem size is much smaller in the factors than in the product. This natural view forms the basis of several studies, for example, [7, 13, 19].

The present paper addresses the twin problems with respect to the direct product. The *direct product* $G \times H$ of graphs G and H is a graph with $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{\{(u,x),(v,y)\} \mid \{u,v\} \in E(G)$ and $\{x,y\} \in E(H)\}$. This product (which is also known as *Kronecker product*, *tensor product*, *categorical product* and *graph conjunction*) is the most natural graph product. It is commutative and associative in a natural way. However, dealing with this product is also most difficult in many respects among standard products. For instance, a Cartesian product or a strong product of two graphs is connected if and only if both factors are connected, and this fact is easily provable. On the other hand, it is not completely straightforward to see that $G \times H$ is connected if and only if both G and H are connected and at least one of them is non-bipartite, cf. [20]. Furthermore, if both G and H are connected and bipartite, then $G \times H$ consists of two connected components.

The direct product has several applications, for instance it may be used as a model for concurrency in multiprocessor systems [15]. Some other applications are listed in [12].

By a graph is meant a finite, simple, undirected graph. Unless indicated otherwise, graphs are also connected and have at least two vertices. Let $|G|$ stand for $|V(G)|$. For $X \subseteq V(G)$, $\langle X \rangle$ denotes the subgraph induced by X. By $\chi(G)$, $\alpha(G)$ and $\tau(G)$ we will denote the *chromatic number*, the *independence number* and the *matching number* of G, respectively. A graph has a *perfect matching* if $\tau(G) = |G|/2$. If G is a bipartite graph with $V(G) = V_0 + V_1$ and $|V_0| \leq |V_1|$ then a *complete matching from V_0 to V_1* is a matching which includes every vertex of V_0.

The main open problem concerning the direct product is the Hedetniemi's conjecture. Let $\overline{\chi}(G \times H) = \min\{\chi(G), \chi(H)\}$. It is easily seen that $\chi(G \times H) \leq \overline{\chi}(G \times H)$ holds for any graphs G and H. In 1966, Hedetniemi [9] conjectured that for all graphs G and H, $\chi(G \times H) = \overline{\chi}(G \times H)$. For surveys on the conjecture we refer to [4, 14]. The chromatic number of a graph G and its independence number are closely related via the inequality $\chi(G) \geq |G|/\alpha(G)$. It is easy to see (and well-known [13, 18]) that

$$\alpha(G \times H) \geq \max\{\alpha(G) \cdot |H|, \alpha(H) \cdot |G|\} =: \underline{\alpha}(G \times H).$$

The main topic of this paper is the study of relation between $\alpha(G \times H)$ and $\underline{\alpha}(G \times H)$. For instance, is it the case (analogous to the Hedetniemi's conjecture) that $\alpha(G \times H) = \underline{\alpha}(G \times H)$ for any two graphs G and H? In particular, does $\alpha(G \times C_n) = \underline{\alpha}(G \times C_n)$ hold for any graph G?

Independence numbers of direct products have also been studied earlier. Greenwell and Lovász [6] proved that the independence number of the direct product of k copies of K_n is equal to n^{k-1}. For some other results we refer to [13].

In the next section we prove that for any graph G with at least one edge and for any $i \in \mathbb{N}$ there is a graph H such that $\alpha(G \times H_i) > \underline{\alpha}(G \times H_i) + i$. On the other hand we show that $\alpha(G \times H) = \underline{\alpha}(G \times H)$ when G is a bipartite graph with a perfect matching and H is a Hamiltonian graph. In Section 3 we compute the independence numbers of the direct products of paths and cycles. As a by-product we also give the matching numbers of these graphs. We conclude the paper with some remarks concerning dense graphs with high independence number related to the direct graph construction.

2 $\alpha(G \times H)$ versus $\underline{\alpha}(G \times H)$

As we will show later, for many graphs G and H, $\alpha(G \times H)$ indeed equals $\underline{\alpha}(G \times H)$. However, we can answer both questions from the introduction negatively with the next theorem.

Theorem 2.1 *For any graph G with at least one edge and for any $i \in \mathbb{N}$, there is a graph H_i such that*

$$\alpha(G \times H_i) > \underline{\alpha}(G \times H_i) + i.$$

Proof. Let G be an arbitrary graph with at least one edge. Let $|G| = n$ and $\alpha(G) = k$. Clearly $n \geq 2$ and $k \leq n - 1$.

Define a graph $H(p,q)$ as follows. Let $V(H(p,q)) = P \cup Q$, where P induces a complete graph on $p \geq 2$ vertices and Q induces an independent set on $q \geq 1$ vertices. In addition, every vertex of Q is adjacent to a fixed vertex, say w, of P. Clearly, $|H(p,q)| = p + q$ and $\alpha(H(p,q)) = q + 1$. Hence

$$\underline{\alpha}(G \times H(p,q)) = \max\{k(p+q),\ (q+1)n\}.$$

Let u be an arbitrary vertex of G and let X be the subset of $V(G \times H(p,q))$ defined by

$$X = (V(G) \times Q) \cup (\{u\} \times (P \setminus \{w\})).$$

Clearly, $|X| = nq + (p-1)$. Furthermore, it is straightforward to see that X is an independent set of $G \times H(p,q)$. Thus $\alpha(G \times H(p,q)) \geq nq + (p-1)$. For $i \in \mathbb{N}$ we now define

$$H_i = H(p_i, q_i) = H(n + i + 2,\ n(n+i)).$$

Then we have

$$\begin{aligned} nq_i + (p_i - 1) &= nq_i + n + i + 1 \\ &> (q_i + 1)n + i \end{aligned}$$

and also

$$\begin{aligned} nq_i + (p_i - 1) &= nq_i - q_i + n(n+i) + p_i - 1 - (2i+3) + (2i+3) \\ &= q_i(n-1) + (n+i+2)(n-1) + (2i+3) \\ &= (n-1)(p_i + q_i) + (2i+3) \\ &\geq k(p_i + q_i) + (2i+3) \\ &> k(p_i + q_i) + i. \end{aligned}$$

It follows that

$$\alpha(G \times H_i) > \underline{\alpha}(G \times H_i) + i$$

and the theorem is proved. □

Theorem 2.1 can be reinterpreted by saying that there is no graph G which is *universal* in the sense that $\alpha(G \times H) = \underline{\alpha}(G \times H)$ holds for any graph H.

A graph is called *traceable* if it contains a Hamiltonian path. Clearly, paths and cycles are traceable. We are going to show that α is equal to $\underline{\alpha}$ for the direct product of a bipartite graph with a perfect matching and a traceable graph. But first we need several lemmas.

Lemma 2.2 *(i) For any graphs G and H, $\tau(G \times H) \geq 2 \cdot \tau(G) \cdot \tau(H)$.*
(ii) $\alpha(K_2 \times C_n) = n$.

Proof. (i) If M is a matching of G and M' is a matching of H then $M \times M'$ is a matching of $G \times H$.

(ii) $C_{2i} \times K_2$ consists of two disjoint copies of C_{2i} while $C_{2i+1} \times K_2$ is isomorphic to C_{4i+2}. □

The bound from Lemma 2.2 (i) can be arbitrarily smaller than the matching number of $G \times H$. Consider, for instance, the graph $K_{1,m} \times K_{1,n}$. It consists of two connected components $K_{1,mn}$ and $K_{m,n}$. Thus $\tau(K_{1,m} \times K_{1,n}) = 1 + \min\{m, n\}$ whereas the bound from Lemma 2.2 (i) gives only 2.

Lemma 2.3 *If G is a bipartite graph with a perfect matching and H is a Hamiltonian graph, then*

$$\alpha(G \times H) = \underline{\alpha}(G \times H) = |G| \cdot |H| / 2.$$

Proof. Let S be an independent set of $G \times H$ and let e be an edge of G. Since H contains a Hamiltonian cycle, Lemma 2.2 (ii) implies

$$|S \cap (e \times H)| \leq |H|.$$

As G has a perfect matching it follows that $\alpha(G \times H) \leq |G| \cdot |H|/2$.

Conversely, since G is bipartite and has a perfect matching, $\alpha(G) = |G|/2$. But then $\alpha(G \times H) \geq |G| \cdot |H|/2$. □

In fact, Lemma 2.3 can be slightly generalized by assuming that G is a graph with a perfect matching and with $\alpha(G) = |G|/2$.

Lemma 2.4 *If G is a bipartite graph with a perfect matching and H is a traceable graph, then*

$$\alpha(G \times H) \leq |G| \cdot \left\lceil \frac{|H|}{2} \right\rceil.$$

Proof. Since G has a perfect matching, $\tau(G) = |G|/2$ and as H is traceable, $\tau(H) = \lfloor |H|/2 \rfloor$. Thus by Lemma 2.2 (i),

$$\tau(G \times H) \geq 2 \cdot \frac{|G|}{2} \cdot \left\lfloor \frac{|H|}{2} \right\rfloor = |G| \cdot \left\lfloor \frac{|H|}{2} \right\rfloor.$$

Because G is bipartite, $G \times H$ is bipartite as well. Since for bipartite graphs $\tau + \alpha$ equals the number of vertices (cf. [8]) we have

$$\alpha(G \times H) = |G| \cdot |H| - \tau(G \times H).$$

By the above we obtain

$$\begin{aligned} \alpha(G \times H) &\leq |G| \cdot |H| - |G| \cdot \left\lfloor \tfrac{|H|}{2} \right\rfloor \\ &= |G| \cdot (|H| - \left\lfloor \tfrac{|H|}{2} \right\rfloor) \\ &= |G| \cdot \left\lceil \tfrac{|H|}{2} \right\rceil. \end{aligned}$$

□

Theorem 2.5 *Let G be a bipartite graph with a perfect matching and let H be a traceable graph. Then*

$$\alpha(G \times H) = \underline{\alpha}(G \times H) = |G| \cdot |H|/2.$$

Proof. If $|H|$ is even, then the bound of Lemma 2.4 coincides with the $\underline{\alpha}$.

Let $|H| = 2i+1$. If $\alpha(H) = i+1$, then $|G| \cdot \alpha(H)$ equals the bound of Lemma 2.4 and the proof is done also for this case.

The last case to consider is when $|H| = 2i+1$ and $\alpha(H) \leq i$. Let $v_1 v_2 \ldots v_{2i+1}$ be a Hamiltonian path of H. Then, at least two vertices with odd indices must be adjacent, for otherwise these vertices would form an independent set of size $i+1$. Let v_{2j+1} and v_{2k+1} be adjacent vertices, $j < k$. Consider now the subgraphs H_1, H_2 and H_3 of H which are induced by the vertices $v_1, \ldots, v_{2j}$; $v_{2j+1}, \ldots, v_{2k+1}$ and $v_{2k+2}, \ldots, v_{2i+1}$, respectively. Note that H_1 and H_3 contain paths on $2j$ and $2i-2k$ vertices, respectively, and that H_2 contains a (Hamiltonian) cycle on $2k-2j+1$ vertices. Clearly,

$$\alpha(G \times H) \leq \alpha(G \times H_1) + \alpha(G \times H_2) + \alpha(G \times H_3),$$

which is in turn by Lemmas 2.3 and 2.4 at most

$$|G|\cdot j + |G|\cdot(2k-2j+1)/2 + |G|\cdot(i-k) = |G|\cdot(i+1/2) = |G|\cdot|H|/2$$

and the theorem is proved. □

We will apply Theorem 2.5 in the next section for the case of paths and cycles. But before we close this section let us give the following connection of our study with the Hedetniemi's conjecture.

Proposition 2.6 *Let G and H be graphs such that $\chi(G)\alpha(G) = |G|$, $\chi(H)\alpha(H) = |H|$ and $\alpha(G \times H) = \underline{\alpha}(G \times H)$. Then $\chi(G \times H) = \overline{\chi}(G \times H)$.*

Proof. It suffices to show that $\chi(G \times H) \geq \overline{\chi}(G \times H)$. We may without loss of generality assume

$$\chi(G) = \frac{|G|}{\alpha(G)} \geq \frac{|H|}{\alpha(H)} = \chi(H).$$

Then we have

$$\begin{aligned}
\chi(G \times H) &\geq \frac{|G \times H|}{\alpha(G \times H)} = \frac{|G||H|}{|G|\alpha(H)} \\
&= \frac{|H|}{\alpha(H)} = \chi(H) \\
&= \min\{\chi(G), \chi(H)\} = \overline{\chi}(G \times H).
\end{aligned}$$

We now give an example which illustrates Proposition 2.6. For $m \geq 1$ let G_m denote the direct product of n copies of the complete graph K_n. Let $k, s \geq 1$. Then by the result of Greenwell and Lovász from [6] we have $\alpha(G_k) = n^{k-1}$. Since $\chi(G_k) = n$ we have $\chi(G_k)\alpha(G_k) = n^k = |G_k|$. The same holds for G_s as well. Furthermore, using the result of Greenwell and Lovász again we obtain

$$\alpha(G_k \times G_s) = n^{k+s-1} = \alpha(G_k)|G_s| = \underline{\alpha}(G_k \times G_s).$$

Proposition 2.6 now implies that $\chi(G_{k+s}) = \chi(G_k \times G_s) = \overline{\chi}(G_k \times G_s) = n$.

3 Products of paths and cycles

In this section we will obtain the independence numbers for the direct product of paths and cycles. As a byproduct we will also list the matching numbers of these graphs.

For the path P_m and the cycle C_n, let $V(P_k) = V(C_k) = \{0, \cdots, k-1\}$. As we already mentioned, the graph $P_m \times P_n$ consists of two connected components. In addition, vertices (p,q) and (r,s) of $P_m \times P_n$ belong to the same component if and only if $p+q$ and $r+s$ are of the same parity. A component of $P_m \times P_n$ will be called an *even component* (resp. *odd component*) if vertices (p,q) of that component are such that $p+q$ is even (resp. odd). Note that the even component has $\lceil mn/2 \rceil$ vertices while the odd component has $\lfloor mn/2 \rfloor$ vertices. Further, each component has $(m-1)\cdot(n-1)$ edges. The graph $P_9 \times P_5$ appears in Fig. 1.

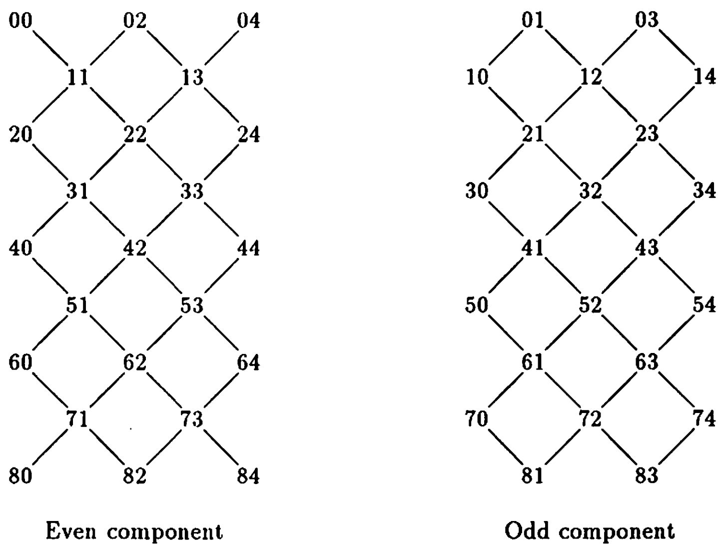

Figure 1: Graph $P_9 \times P_5$

First we present two results which might be of independent interest.

Proposition 3.1 *Each component of* $P_{2i+1} \times P_{2j+1}$ *has a complete matching from the smaller partite set to the bigger one.*

Proof. Let $i \geq j \geq 1$. Let E_1 be a (maximum) matching of P_{2i+1} with i edges and let E_2 be a (maximum) matching of P_{2j+1} with j edges. Then it is easy to see that the matching $E_1 \times E_2$ (which is of size $2ij$) of the graph $P_{2i+1} \times P_{2j+1}$ is evenly divided between the two components. Thus the matching number of each component of $P_{2i+1} \times P_{2j+1}$ is at least ij, which coincides with the size of the smaller partite set of the even component, and hence the result follows for this component.

For the odd component, the partite sets are $\{0,2,\ldots,2i\} \times \{1,3,\ldots,2j-1\}$ and $\{1,3,\ldots,2i-1\} \times \{0,2,\ldots,2j\}$, which are of cardinalities $(i+1)\cdot j$

and $i \cdot (j+1)$, respectively. Partition the vertex set of this component into the following subsets: $V_1, V_3, \ldots, V_{2(i+j)-1}$, where $V_{2k-1} = \{(p,q) \mid p+q = 2k-1\}$, $1 \le k \le i+j$. Clearly, this is a well-defined partition. Further, the reader may verify that for $1 \le 2k-1 \le 2j-1$, $\langle V_{2k-1} \rangle$ and $\langle V_{2(i+j-k)+1} \rangle$ are both isomorphic to P_{2k} and that for $2j+1 \le 2k-1 \le 2i-1$, $\langle V_{2k-1} \rangle$ is isomorphic to P_{2j+1}. It follows that this component contains a matching of size $2 \cdot (1+2+\cdots+j) + (i-j) \cdot j = (j+1) \cdot j + (i-j) \cdot j = (i+1) \cdot j$, which coincides with the size of the smaller partite set of this component. Hence the result. □

Proposition 3.2 *If m and n be both odd and $m \ge n$, then $C_m \times C_n$ admits of a vertex decomposition into n m-cycles.*

Proof. Consider the following vertex subset of $C_m \times C_n$:

$$\{(0,b_0), (1,b_1), \ldots, (m-1, b_{m-1})\},$$

where $b_i = i$ for $0 \le i \le n-1$, and $b_i = (i+1) \bmod 2$ for $n \le i \le m-1$. It is clear that these m vertices induce a cycle of length m, say σ_0. For $i \in \{1, \ldots, n-1\}$, consider the vertex subsets

$$\{(0, b_0+i), (1, b_1+i), \ldots, (m-1, b_{m-1}+i)\},$$

where the sum $b_j + i$ is taken modulo n. It is easy to check that these sets of vertices induce cycles of length m, say σ_i, in $C_m \times C_n$. Finally, the cycles $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ thus constructed form a partition of the vertex set of $C_m \times C_n$. □

Theorem 3.3 *If G is a path or a cycle and H is a path or a cycle, then* $\alpha(G \times H) = \underline{\alpha}(G \times H)$.

Proof. By Theorem 2.5, the result is true for all the cases except when both G and H have odd number of vertices.

Consider first the case $P_{2i+1} \times P_{2j+1}$ and assume without loss of generality that $i \ge j$. By Proposition 3.1, the independence number of a component of $P_{2i+1} \times P_{2j+1}$ equals to the size of a larger partite set. Hence $\alpha(P_{2i+1} \times P_{2j+1}) = (i+1) \cdot (j+1) + i \cdot (j+1) = (2i+1) \cdot (j+1)$ which is in turn equal to $\underline{\alpha}(P_{2i+1} \times P_{2j+1})$.

For the case $C_{2i+1} \times P_{2j+1}$ we recall from [10] that this graph admits a decomposition into two cycles of length $2 \cdot (2i+1) \cdot j$. Thus, $C_{2i+1} \times P_{2j+1}$ contains a matching of size $(2i+1) \cdot j$. (We note, however, that considering subproducts of the type $C_{2i+1} \times K_2$, we can also see that directly.) It follows that $\alpha(C_{2i+1} \times P_{2j+1}) \le (2i+1) \cdot (2j+1) - (2i+1) \cdot j = (2j+1) \cdot (j+1)$. This is equal to $\underline{\alpha}(C_{2i+1} \times P_{2j+1})$.

The last case $C_{2i+1} \times C_{2j+1}$ follows immediately from Proposition 3.2. □

Note that all the products from Theorem 3.3 are bipartite with the exception of $C_{2i+1} \times C_{2j+1}$. Thus, since for bipartite graphs $\tau + \alpha$ equals the number of vertices, Theorem 3.3 also gives all matching numbers but one for these graphs. The remaining matching number $\tau(C_{2i+1} \times C_{2j+1})$ is equal to $((2i+1)(2j+1)-1)/2$. This follows from the fact that this graph is Hamiltonian, see [11]. Results of this section are collected in Table 1.

m	n	G	H	$\alpha(G \times H)$	$\tau(G \times H)$
odd	odd	C_m	C_n, $m \geq n$	$(m-1) \cdot n/2$	$(mn-1)/2$
odd	even	C_m	C_n or P_n	$mn/2$	$mn/2$
odd	odd	C_m	P_n	$m \cdot (n+1)/2$	$m \cdot (n-1)/2$
even	even	C_m or P_m	C_n or P_n	$mn/2$	$mn/2$
even	odd	C_m or P_m	P_n	$m \cdot (n+1)/2$	$m \cdot (n-1)/2$
odd	odd	P_m	P_n, $m \geq n$	$m \cdot (n+1)/2$	$m \cdot (n-1)/2$

Table 1: α and τ in products of paths and cycles

Note that each of $C_{2i} \times C_{2j}$, $C_{2i} \times P_{2j}$ and $P_{2i} \times P_{2j}$ has the same independence number. Analogous statement holds for $C_{2i+1} \times C_{2j}$ and $C_{2i+1} \times P_{2j}$ (resp. $C_{2i} \times P_{2j+1}$ and $P_{2i} \times P_{2j+1}$).

4 Concluding remarks

Albertson, Chan and Haas [1] established a relationship between the independence number and the odd girth of a graph. Their main result states that if the odd girth of a graph G is at least 7 and the smallest degree of G is greater than $|G|/4$ then $\alpha(G)/|G|$ is at least $3/7$. The result is interesting because a dense graph is suspected to have a low independence number. Results of this paper indicate that the direct product construction can be used to obtain examples of graphs in this direction. To make this more precise we give the following example. Let $G = K_{2i+1} \times C_{2i+1}$, and let $n = |V(G)|$. It is easy to see that $|E(G)| = n \cdot (\sqrt{n} - 1)$, $\alpha(G) \geq (n - \sqrt{n})/2$, $\tau(G) = (n-1)/2$ (since this graph is Hamiltonian), the odd girth of G is $\sqrt{n}$, and the even girth of G is 4. Thus we have a graph with (i) density of order $\sqrt{n}$, (ii) independence number of order $n/2$, (iii) high odd girth, and (iv) low even girth. Importance of graphs having high density and high girth appears in [16].

Finally, we want to add that Proposition 3.1 is true not only in the case of two odd paths but also (with natural changes of the statement) for the graphs $C_m \times C_{2j}$, $C_m \times P_n$ and $P_m \times P_n$. This can be seen from Table 1.

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