On Matchings in Graphs

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ABSTRACT

A matching in a graph G is a set of independent edges and a maximal matching is a matching that is not properly contained in any other matching in G. A maximum matching is a matching of maximum cardinality. The number of edges in a maximum matching is denoted by $\beta_1(G)$; while the number of edges in a maximal matching of minimum cardinality is denoted by $\beta_1^-(G)$. Several results concerning these parameters are established including a Nordhaus-Gaddum result for $\beta_1^-(G)$. Finally, in order to compare the maximum matchings in a graph G, a metric on the set of maximum matchings of G is defined and studied. Using this metric, we define a new graph $M(G)$, called the matching graph of G. Several graphs are shown to be matching graphs; however, it is shown that not all graphs are matching graphs.

1. Maximum and Maximal Matchings

A *matching* in a graph G is a set of independent (pairwise nonadjacent) edges of G. The *edge independence number* $\beta_1 = \beta_1(G)$ of G is the maximum size of a matching in G, that is, β_1 is the maximum positive integer h such that hK_2 is a subgraph of G. A matching of size β_1 is thus referred to as a *maximum matching*. Obviously, for every graph G of order n, $\beta_1 \le \lfloor n/2 \rfloor$. A *maximal matching* in G is a matching that is not properly contained in any other matching in G. Let $\beta_1^- = \beta_1^-(G)$ denote the minimum size among the maximal matchings of G. (Of course, the maximum size among the maximal matchings of G is β_1.)

For the path P_6 shown in Figure 1, $\beta_1 = 3$, where $\{e_1, e_3, e_5\}$ is the unique maximum matching. On the other hand, $\beta_1^- = 2$, where $\{e_1, e_4\}$,

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$\{e_2, e_4\}$, and $\{e_2, e_5\}$ are the three maximal matchings of minimum size, i. e., the minimum maximal matchings.

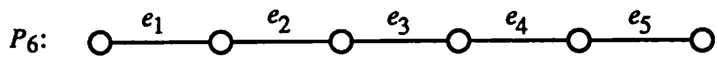


Figure 1

The following result of Lesk [5] establishes bounds for the edge independence number β_1 of a graph G in terms of β_1^-. The bounds are analogous to those for the diameter of a graph in terms of its radius.

Theorem A For every nonempty graph, $\beta_1^- \leq \beta_1 \leq 2\beta_1^-$.

It is not difficult to observe that β_1 and β_1^- can attain any positive integer values subject to the restrictions given in Theorem A. In particular, let a and b be integers with $a \leq b \leq 2a$, and define $G = (b-a)P_4 \cup (2a-b)K_2$. Then G is a graph of order $2b$ with $\beta_1 = b$ and $\beta_1^- = a$. Since the order of every graph having a matching of size b is at least $2b$, the graph G has minimum order with the prescribed properties. However, G is disconnected, in fact, has a components. As we shall see next, the minimum order of a *connected* graph G having $\beta_1 = b$ and $\beta_1^- = a$, where $a \leq b \leq 2a$, is also $2b$.

Theorem 1 For positive integers a and b with $a \leq b \leq 2a$, the minimum order of a connected graph with $\beta_1 = b$ and $\beta_1^- = a$ is $2b$.

Proof The proof is constructive. We consider two cases.

Case 1 *Suppose that* $a \geq \lceil 2b/3 \rceil$. Let G be the graph obtained by identifying one vertex of the complete graph $K_{6a-4b+2}$ with one end-vertex of the path $P_{6(b-a)-1}$, the path on $6(b-a)-1$ vertices. Then $\beta_1 = (3a-2b+1) + 3(b-a) - 1 = b$, where $3a-2b+1$ counts the number of edges in a maximum matching of the complete subgraph of G and $3(b-a)-1$ counts every other edge of $P_{6(b-a)-1}$; and $\beta_1^- = (3a-2b+1) + [2(b-a)-1] = a$, where $3a-2b+1$ again counts the number of edges in a maximum matching of the complete subgraph of G and $2(b-a)-1$ counts every third edge of $P_{6(b-a)-1}$.

Case 2 *Suppose that* $a < \lceil 2b/3 \rceil$. Then it follows that $a \le (2b-1)/3$. Let P denote the path $P_{6a-2b+2}$: $v_1, v_2, \dots, v_{6a-2b+2}$, and let G be the graph obtained from P by adding one pendant edge at each of the vertices v_1, $v_2, \dots, v_{4b-6a-2}$. (Since $a \le (2b-1)/3$, we have $4b - 6a - 2 \ge 0$.) Then $\beta_1 = (4b - 6a - 2) + (6a - 3b + 2) = b$ and $\beta_1^- = (2b - 3a - 1) + (4a - 2b + 1) = a$. ❑

A *cut-vertex* of a connected graph is a vertex whose removal results in a disconnected graph. A graph is 2-*connected* if it has no cut-vertices. Perhaps surprisingly, there also exists a 2-connected graph of order $2b$ having $\beta_1 = b$ and $\beta_1^- = a$ for every pair a, b of integers with $a \ge 1$, $b \ge 2$, and $a \le b \le 2a$..

Theorem 2 For integers $a \ge 1$ and $b \ge 2$ with $a \le b \le 2a$, the minimum order of a 2-connected graph with $\beta_1 = b$ and $\beta_1^- = a$ is $2b$.

Proof The proof is constructive. We consider six cases. The first four cases deal with the four possible specific values of b (in terms of a): (1) $b = a$, (2) $b = 2a$, (3) $b = 3a/2$, (4) $b = (3a+1)/2$, where $a \ge 3$.

Case 1 *Suppose that* $b = a$. Then the complete graph K_{2b} is a 2-connected graph with $\beta_1 = \beta_1^- = b$.

Case 2 *Suppose that* $b = 2a$. First, if $a = 1$, then $K_4 - e$ has the desired properties. So assume that $a \ge 2$. Consider the graph G obtained from the cycle C_b: $u_1, u_2, \dots, u_b, u_1$ by adding b new vertices $v_1, v_2, \dots, v_b$ and the edges v_iu_i and v_iu_{i+1} for $i = 1, 2, \dots, b$, where $i + 1$ is expressed modulo b. Then G has $\beta_1 = b$ and $\beta_1^- = a$.

Case 3 *Suppose that* $b = 3a/2$. Then the cycle C_{2b} has $\beta_1 = b$ and $\beta_1^- = a$.

Case 4 *Suppose that* $b = (3a+1)/2$, *where* $a \ge 3$. Let G be the graph obtained from the cycle C_{2b-2}: $u_1, u_2, \dots, u_{2b-2}, u_1$ by adding two new vertices x and y and the edges xu_1, xu_2, yu_3, yu_4. Then G has $\beta_1 = b$ and $\beta_1^- = a$.

We are now left with the two cases (5) $a + 1 \le b \le (3a-1)/2$ and (6) $(3a+2)/2 \le b \le 2a - 1$.

Case 5 *Suppose that* $a + 1 \le b \le (3a - 1)/2$. Let G be the graph obtained from the complete graph K_{6a-4b} and the path $P_{6b-6a+2}$ by identifying the end-vertices of the path to two distinct vertices of the complete graph. Since $a \ge (2b + 1)/3$, we have $6a - 4b \ge 2$ and since $a \le b - 1$, it follows that $6b - 6a + 2 \ge 8$. Then G is a 2-connected graph of order $2b$ with $\beta_1(G) = (3a - 2b) + (3b - 3a) = b$ and $\beta_1^-(G) = (3a - 2b) + (2b - 2a) = a$.

Case 6 *Suppose that* $(3a + 2)/2 \le b \le 2a - 1$. We begin with the cycle $C_{4b-6a}: u_1, u_2, \ldots, u_{4b-6a}, u_1$. Since $b \ge (3a + 2)/2$, it follows that $4b - 6a \ge 4$. Now let G' be the graph obtained from C_{4b-6a} by adding $4b - 6a$ new vertices $v_1, v_2, \ldots, v_{4b-6a}$ and the edges v_iu_i and v_iu_{i+1} for $i = 1, 2, \ldots, 4b - 6a$, where $i + 1$ is expressed modulo $4b - 6a$. Finally, G is obtained by identifying one end-vertex of the path $P_{12a-6b+2}$ to u_1 and the other to u_2. Since $b \le 2a - 1$, we have $12a - 6b + 2 \ge 9$. Then G is a 2-connected graph of order $2b$ with $\beta_1(G) = (4b - 6a) + (6a - 3b) = b$ and $\beta_1^-(G) = (2b - 3a) + (4a - 2b) = a$. ❑

Before leaving this section, we present an intermediate value theorem for maximal matchings, sometimes called an interpolation theorem as in Harary and Plantholt [4]. First, the following notation and terminology will be useful. Let M be a matching of a graph G. A *weak vertex* of G is not incident with any edge of M. An *alternating path* of G has alternate edges in M and not in M. The following result is due to Berge [1].

Theorem B A matching M in a graph G is maximum if and only if there exists no alternating path between two distinct weak vertices of G.

This aids in establishing an interpolation theorem for maximal matchings.

Theorem 3 If G is a graph and k is an integer with $\beta_1^- \le k \le \beta_1$, then G has a maximal matching of size k.

Proof It suffices to show that if there is a maximal matching of size m in G, where $\beta_1^- \le m < \beta_1$, then there is a maximal matching of size $m + 1$ in G. Let M be a maximal matching of size m, where $\beta_1^- \le m < \beta_1$. By Theorem A, since M is not a maximum matching, there exists an alternating path P in G between two distinct weak vertices of G. Let $S \subset E(G)$ be the symmetric difference of M and $E(P)$ which then consists of the edges of M that are not in P and the edges of P that are not in M, that is,

$$S = [M - E(P)] \cup [E(P) - M].$$

Observe that S is a matching with $|S| = m + 1$. Also since M is a maximal matching, $V(G) - V(\langle S\rangle)$ is an independent set of vertices. Hence S is a maximal matching. ❑

2. A Nordhaus-Gaddum Result for β_1^-

Ever since Nordhaus and Gaddum [6] presented bounds for the sum of the chromatic number of a graph G and the chromatic number of the complement of G, many others have investigated analogous results for various parameters. In particular, for edge independence numbers, it was shown in [2] that for any graph G of order $n \geq 3$,

$$\left\lfloor \frac{n}{2} \right\rfloor \leq \beta_1(G) + \beta_1(\bar{G}) \leq 2\left\lfloor \frac{n}{2} \right\rfloor,$$

and further, that for any integers a and b with $0 \leq a, b \leq \lfloor n/2 \rfloor$ and $a + b \geq \lfloor n/2 \rfloor$, there exists a graph G having order n, $\beta_1(G) = a$, and $\beta_1(\bar{G}) = b$. The second portion of this clearly shows that the presented bounds are sharp. We show that in general the same bounds hold for β_1^-; however, we shall see that the upper bound can be improved when we restrict ourselves to graphs of order n, where $n \equiv 2 \pmod 4$.

Theorem 4 For every graph G of order $n \geq 3$,

$$\left\lfloor \frac{n}{2} \right\rfloor \leq \beta_1^-(G) + \beta_1^-(\bar{G}) \leq 2\left\lfloor \frac{n}{2} \right\rfloor.$$

Proof Suppose that G has order n and that $\beta_1^-(G) = a$. Then G must be a subgraph of $K_{2a} + \bar{K}_{n-2a}$. Hence, it follows that $\bar{G}$ contains $\overline{K_{2a} + \bar{K}_{n-2a}}$ as a subgraph. But

$$\overline{K_{2a} + \bar{K}_{n-2a}} = \bar{K}_{2a} \cup K_{n-2a},$$

so that $\bar{K}_{2a} \cup K_{n-2a}$ is a subgraph of $\bar{G}$. This implies that

$$\beta_1^-(\bar{G}) \geq \lfloor \frac{n-2a}{2} \rfloor,$$

and so

$$\beta_1^-(G) + \beta_1^-(\bar{G}) \ \geq \ a + \lfloor \frac{n-2a}{2} \rfloor = \lfloor \frac{n}{2} \rfloor.$$

Next, observe that, by the result in [2],

$$\beta_1^-(G) + \beta_1^-(\bar{G}) \ \leq \ \beta_1(G) + \beta_1(\bar{G}) \ \leq \ 2\lfloor \frac{n}{2} \rfloor. \quad \square$$

The lower bound presented in Theorem 4 is sharp since $G = K_n$ has $\beta_1^-(G) = \lfloor n/2 \rfloor$ and $\beta_1^-(\bar{G}) = 0$. The upper bound is sharp also except when $n \equiv 2 \pmod 4$. We consider two cases. First, suppose that $n \equiv 0 \pmod 4$ and write $n = 4k$, where $k \geq 1$. Then the graph $G \cong K_{2k,2k}$ has $\beta_1^-(G) = \beta_1^-(\bar{G}) = 2k = n/2$, showing that the upper bound is sharp in this case. Next, let n be odd, say that $n = 2k + 1$, where $k \geq 1$. Then the graph $G \cong K_{k,k+1}$ has $\beta_1^-(G) + \beta_1^-(\bar{G}) = 2k = 2\lfloor n/2 \rfloor$, and, again, the bound is sharp. In the remaining case, when $n \equiv 2 \pmod 4$, we shall see that the upper bound can be lowered by one and that this new bound is sharp. We begin with a useful lemma.

Lemma 5 If G is a graph of even order $n \geq 4$ with $\beta_1^-(G) = n/2$, then the end-vertices of every path of length 3 are adjacent.

Proof Let M be a maximal matching with $|M| = \beta_1^-(G) = n/2$, and consider a path $P \cong P_4$, say that $P: v_1, v_2, v_3, v_4$. We show that $v_1 v_4 \in E(G)$. We will consider three cases, but first, for $i = 1, 2, 3$, let $e_i = v_i v_{i+1}$. Note that at most two of the edges e_1, e_2, e_3 belong to M and if, in fact, two of these edges belong to M, they must be e_1 and e_3.

Case 1 *Suppose that* $e_1, e_3 \in M$ *and* $e_2 \notin M$. If $v_1 v_4 \notin E(G)$, then

$$M' = M - \{e_1, e_3\} \cup (e_2)$$

is a maximal matching with $|M'| = n/2 - 1$, contradicting the definition of M. So $v_1 v_4 \in E(G)$ in this case.

Case 2 *Suppose that exactly one of* $e_1, e_2,$ *and* e_3 *is in* M. We consider two subcases.

Subcase 2.1 *Suppose that* $e_2 \in M$ *and* $e_1, e_3 \notin M$. Each of the edges $e_1 = v_1v_2$ and $e_3 = v_3v_4$ is adjacent to an edge of M other than e_2, say that e_1 is adjacent to $v_1x \in M$ and that e_3 is adjacent to $v_4y \in M$. See Figure 2, and note that the vertical edges in Figure 2 are the edges that belong to M. By Case 1, the edges v_3x and v_2y must be in G. Now if $v_1v_4 \notin E(G)$, then

$$M' = M - \{e_2, v_1x, v_4y\} \cup \{v_3x, v_2y\}$$

is a maximal matching with $|M'| = n/2 - 1$, producing a contradiction.

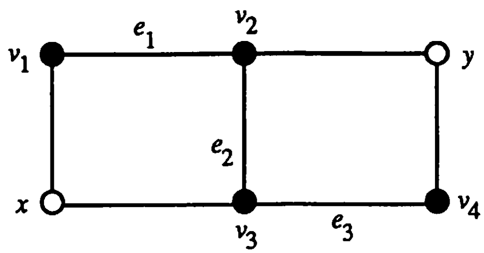


Figure 2

Subcase 2.2 *Suppose that either* e_1 *or* e_3 *is in* M, *but not both.* Without loss of generality, assume that $e_1 \in M$ and $e_2, e_3 \notin M$. As before, each of the vertices v_3 and v_4 are incident to edges of M, say that $v_3x, v_4y \in M$. Figure 3 illustrates this, where again, the vertical edges are the edges belonging to M. Again, by Case 1, $xy \in E(G)$. Now if $v_1v_4 \notin E(G)$, then

$$M' = M - \{e_1, v_3x, v_4y\} \cup \{e_2, xy\}$$

is a maximal matching with $|M'| = n/2 - 1$, again contradicting $\beta_1^-(G) = n/2$.

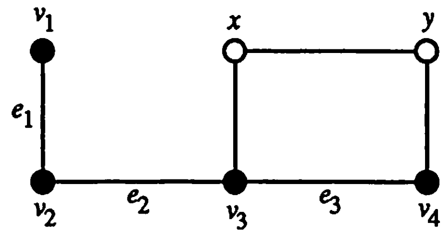


Figure 3

Case 3 *Assume that* $e_1, e_2, e_3 \notin M$. Then for each i with $1 \le i \le 4$, there exists a vertex $u_i \in V(G)$ such that $u_i v_i \in M$. Now, by Case 1, the edges $u_1u_2, u_3u_4 \in E(G)$. See Figure 4. If $v_1v_4 \notin E(G)$, then

$$M' = M - \{u_i v_i \mid 1 \le i \le 4\} \cup \{e_2, u_1u_2, u_3u_4\}$$

is a maximal matching with $|M'| < \beta_1^-(G)$, a contradiction. ❑

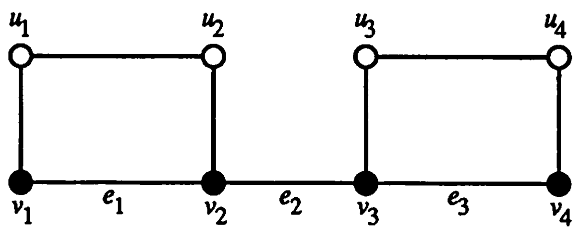


Figure 4

Let G be a graph of even order n having $\beta_1^-(G) = n/2$, and let $M = \{e_i \mid 1 \le i \le n/2\}$ be a maximal matching. Using Lemma 5, it is possible to partition the edges of $E(G) - M$ into pairs of edges as follows. Let $e \in E(G) - M$. Then since M is a maximal matching, e is adjacent to two edges e_i and e_j $(i \ne j)$ of M. Now $\langle\{e, e_i, e_j\}\rangle \cong P_4$ and hence, by Lemma 5, the end-vertices of this path must be adjacent, that is, the edge e' such that $\langle\{e, e', e_i, e_j\}\rangle \cong C_4$ must be in G. So e and e' are paired together, and we can do the same for any edge $e \in E(G) - M$. Thus $|E(G) - M| = |E(G)| - n/2$ is even and so the parity of $|E(G)|$ is the same as the parity of $n/2$.

With this, we are ready to show that the upper bound of Theorem 4 can be improved for many graphs of even order.

Theorem 6 If G is a graph of order n, where $n \equiv 2 \pmod 4$, and $\beta_1^-(G) = n/2$, then $\beta_1^-(\bar{G}) < n/2$.

Proof Let G be a graph of order $n \equiv 2 \pmod 4$ having $\beta_1^-(G) = n/2$. Further, assume, to the contrary, that $\beta_1^-(\bar{G}) = n/2$. Then since $n/2$ is odd, both $|E(G)|$ and $|E(\bar{G})|$ are odd. However, we also know that

$$|E(G)| + |E(\bar{G})| = \binom{n}{2},$$

which is odd, producing the desired contradiction. ❑

Hence we obtain an immediate consequence.

Corollary 7 If G is a graph of order $n \equiv 2 \pmod 4$, then

$$\frac{n}{2} \leq \beta_1^-(G) + \beta_1^-(\bar{G}) \leq n - 1.$$

In order to see that the upper bound in Corollary 7 is sharp, observe that the graph $G \cong K_{2k+1,2k+1}$ has $\beta_1^-(G) = 2k + 1$ and $\beta_1^-(\bar{G}) = 2k$.

3. (Maximum) Matching Graphs

Usually a graph has several maximum matchings, which can share some common edges or be disjoint. In this section, we discuss one possible way of studying the maximum matchings of a graph and the relationships between them. Of course, if two maximum matchings consist of the same edges, then they are identical. Otherwise, they differ by at least one edge. Let M and M' be two maximum matchings in a graph G, and suppose further that M and M' differ by exactly one edge, say that $M - M' = \{e\}$ and $M' - M = \{e'\}$. Note that e and e' must be adjacent, for otherwise $M \cap M' \cup \{e, e'\}$ is a matching larger than the maximum, producing a contradiction. Hence, when two maximum matchings differ by exactly one edge, we say that they are *adjacent matchings*. With this definition in mind, it makes sense to say that two maximum matchings M and M' in a graph G are *connected* if there exists a sequence

$$M = M_0, M_1, M_2, \ldots, M_k = M',$$

where each M_i $(0 \leq i \leq k)$ is a maximum matching and such that every two consecutive matchings M_i, M_{i+1} $(0 \leq i \leq k-1)$ are adjacent. The minimum such k is then defined to be the *distance* $d(M, M')$ between M and M'. If G is a graph in which every two maximum matchings are connected, then this distance is a metric on the set of all maximum matchings of G.

In this context, the maximum matchings of a graph can themselves be represented by a graph, namely, the *(maximum) matching graph* $M(G)$ of a graph G is that graph whose vertices are the maximum matchings of G and such that two vertices M and M' are adjacent in $M(G)$ if and only if M and M' are adjacent matchings in G. Certainly, then, the distance between two maximum matchings of a graph G is simply the ordinary distance between the corresponding vertices of $M(G)$. Since each maximum matching of $K_{1,n}$ consists of one edge and every pair is adjacent, $M(K_{1,n}) = K_n$. As a second example, consider the 5-cycle G with edges labeled as shown in Figure 5. The maximum matchings of G are $M_1 = \{1, 3\}$, $M_2 = \{1, 4\}$, $M_3 = \{2, 4\}$, $M_4 = \{2, 5\}$, and $M_5 = \{3, 5\}$. Furthermore, $M(G) = C_5$ with the appropriate adjacencies shown in Figure 5.

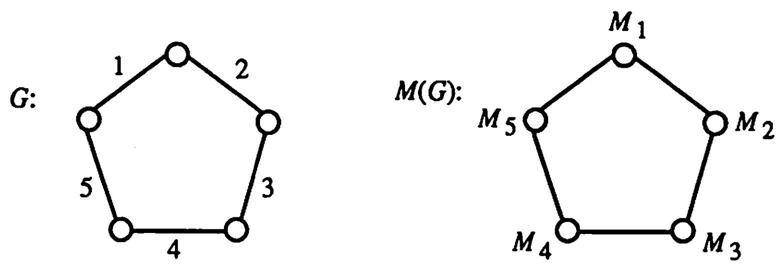


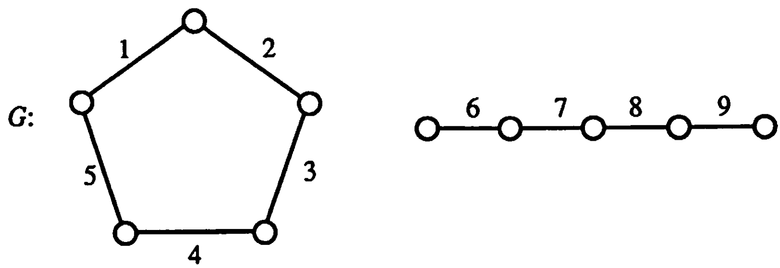
Figure 5

In fact, the matching graph of any odd cycle C_{2n+1}, where $n \geq 2$, is a $(2n + 1)$-cycle.

Theorem 8 Let n be a positive integer. Then $M(C_{2n+1}) = C_{2n+1}$.

Proof Let M be a maximum matching of C_{2n+1}. So M contains n edges. There are exactly two adjacent edges of C_{2n+1} that are not in M. Hence each maximum matching M is adjacent to exactly two matchings implying that $M(C_{2n+1})$ is 2-regular. In fact, it is now easy to see that $M(C_{2n+1}) = C_{2n+1}$. □

As another straightforward example, one can check that $M(P_{2n+1}) = P_{n+1}$ for $n \geq 1$. Clearly $M(C_{2n}) = 2K_1$ for $n \geq 2$ and $M(P_{2n}) = K_1$ for $n \geq 1$. The matching graph of a disconnected graph has a nice relationship with the cartesian product. The *cartesian product* of two graphs G_1 and G_2 is that graph with vertex set $V(G_1) \times V(G_2)$ and such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either (1) $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or (2) $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. Prior to presenting the result, we illustrate it with the following example. Consider $G = C_5 \cup P_5$ with edges labeled as in Figure 6. Then a maximum matching of G consists of a maximum matching of C_5 together with a maximum matching of P_5. When we fix a maximum matching M of C_5 and consider all maximum matchings of P_5 containing M, we see that a copy of $M(P_5) = P_3$ is obtained as a subgraph of $M(G)$. Similarly, when a maximum matching of P_5 is fixed, a copy of $M(C_5) = C_5$ is produced. Figure 6 shows $M(G)$ where each vertex is labeled with the edges belonging to the corresponding maximum matching of G. Note that $M(G) = M(C_5) \times M(P_5)$.



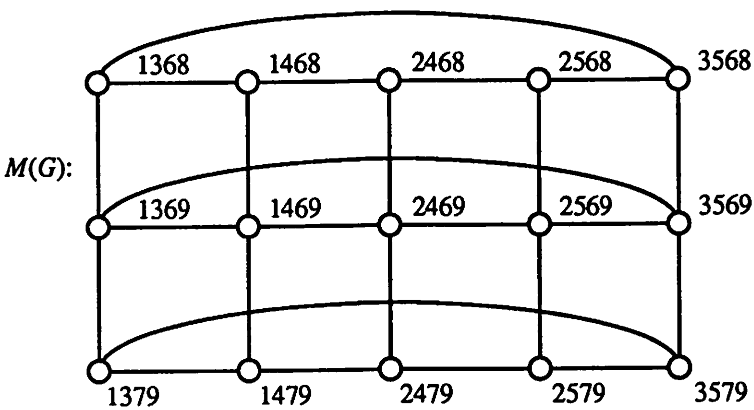


Figure 6

In general, we have the next result.

Theorem 9 If a graph G consists of two nonempty components G_1 and G_2, then

$$M(G) = M(G_1) \times M(G_2).$$

Proof Since a maximum matching of G consists of a maximum matching of G_1 together with a maximum matching of G_2, we have

$$V(M(G)) = V(M(G_1) \times M(G_2)).$$

Next, we show that $E(M(G)) = E(M(G_1) \times M(G_2))$. First, let $(M_1, M_2)(N_1, N_2)$ be an edge of $M(G_1) \times M(G_2)$. This means that M_1 and N_1 are maximum matchings of G_1, while M_2 and N_2 are maximum matchings of G_2, and thus, that $M_1 \cup M_2$ and $N_1 \cup N_2$ are maximum matchings of G. We show that, in fact, $M_1 \cup M_2$ and $N_1 \cup N_2$ are adjacent in G. Since $(M_1, M_2)(N_1, N_2) \in E(M(G_1) \times M(G_2))$, we may assume, without loss of generality, that $M_1 = N_1$ and M_2 and N_2 are adjacent in $M(G_2)$. Hence $M_1 \cup M_2$ and $N_1 \cup N_2$ are adjacent maximum matchings in G. Thus $E(M(G_1) \times M(G_2)) \subseteq E(M(G))$. Now let MN be an edge of $M(G)$. Then $M = M_1 \cup M_2$ and $N = N_1 \cup N_2$, where M_i, N_i are maximum matchings of G_i for each $i = 1, 2$. Since M and N are adjacent, they differ by exactly one edge, say that $M - N = \{e\}$ and $N - M = \{f\}$. so $e \in M_1 \cup M_2$ but $e \notin N_1 \cup N_2$. Now e is either an edge of G_1 or of G_2. Assume, without loss of generality, that $e \in E(G_1)$. Then $e \in M_1$ and $e \notin N_1$. Now since M_1 and N_1 are maximum matchings of G_1, they contain the same number of edges. Hence there exists an edge $g \neq e$ with $g \in N_1$ and $g \notin M_1$. Now if $g \neq f$, then $N - M$ contains both f and g, which is a contraction. So $g = f$, $M_1 - N_1 = \{e\}$, and $N_1 - M_1 = \{f\}$. Thus M_1 and N_1 are adjacent in $M(G_1)$. Moreover, $M_2 = N_2$. This implies that MN corresponds to an edge, namely $(M_1, M_2)(N_1, N_2)$, of $E(M(G_1) \times M(G_2))$. Thus $E(M(G)) \subseteq E(M(G_1) \times M(G_2))$, and so $M(G) = M(G_1) \times M(G_2)$. □

The next result follows immediately.

Corollary 10 If $G_1, G_2, \ldots, G_k$ are the components of G, then

$$M(G) = M(G_1) \times M(G_2) \times \ldots \times M(G_k).$$

A *perfect matching* of a graph of order n is a matching containing $n/2$ edges. If G is a graph of order n containing a perfect matching, then $M(G)$ is empty, that is, $E(M(G)) = \emptyset$, and the order of $M(G)$ is the number of perfect matchings in G. For example, $M(P_{2n}) = K_1$ and $M(C_{2n}) = \bar{K}_2$ for $n \geq 2$. Consequently, $M(G)$ is empty for every hamiltonian graph of even order. A graph H is a *matching graph* if there exists a graph G such that $M(G) = H$. A natural question now is which graphs are matching graphs? We have already observed that every complete graph, every path, and every odd cycle of order $n \geq 3$ is a matching graph. With regard to even cycles, it is straightforward to check that $M(2P_3) = C_4$ and $M(K_{2,3}) = C_6$. That C_{2n}, where $n \geq 4$, is a matching graph is shown in [3], where matching graphs are explored in greater detail.

It can further be shown that every star $K_{1,n}$ is a matching graph. The graph G of Figure 7 has $M(G) = K_{1,n}$, where $\{e_1', e_2', \dots, e_n'\}$ is the maximum matching corresponding to the central vertex of $M(G)$.

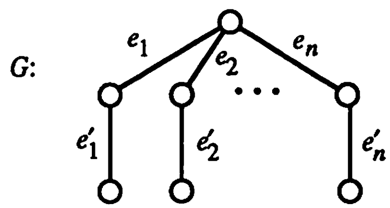


Figure 7

Since $M(K_{1,2}) = K_2$ and the n-cube Q_n $(n \geq 1)$ is the repeated cartesian product of K_2, it follows by Corollary 10 that Q_n is a matching graph. In fact, Q_n is the matching graph of the union of n copies of $K_{1,2}$.

With this, one might expect that every graph is a matching graph; however, this is not the case.

Theorem 11 No graph containing $K_4 - e$ as an induced subgraph is a matching graph.

Proof Suppose that the theorem is false. Then there exists a matching graph H containing $K_4 - e$ as an induced subgraph. Therefore, there exists a graph G such that $H = M(G)$. Consequently, G contains maximum matchings M_1, M_2, M_3, M_4 such that $H' = \langle\{M_1, M_2, M_3, M_4\}\rangle = K_4 - e$, where we may assume that H' is labeled as shown in Figure 8.

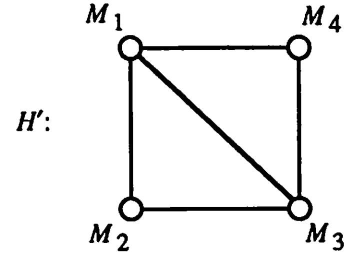


Figure 8

Since M_1 is adjacent to M_2 in G, each of $M_1 - M_2$ and $M_2 - M_1$ consists of exactly one edge of G, say $M_1 - M_2 = \{e_1\}$ and $M_2 - M_1 = \{e_2\}$. Hence $M_i = (M_1 \cap M_2) \cup \{e_i\}$ for $i = 1, 2$. Now since M_1 and M_2 are adjacent, e_1 and e_2 are adjacent. Next, we consider the matching M_3. Since e_1 and e_2 are adjacent in G, at most one of these edges belongs to M_3. We consider two cases.

Case 1 *Exactly one of* e_1 *and* e_2 *belongs to* M_3. Without loss of generality, assume that $e_1 \in M_3$ and $e_2 \notin M_3$. Since M_2 and M_3 are adjacent matchings, each of $M_2 - M_3$ and $M_3 - M_2$ consists of exactly one edge. Moreover, since $e_2 \in M_2$ but $e_2 \notin M_3$, it follows that $M_2 - M_3 = \{e_2\}$. By hypothesis, $e_1 \in M_3$. However, $e_1 \notin M_2$; so $M_3 - M_2 = \{e_1\}$. This implies that $M_3 = (M_2 \cap M_3) \cup \{e_1\}$ and $M_2 = (M_2 \cap M_3) \cup \{e_2\}$. But $M_2 = (M_1 \cap M_2) \cup \{e_2\}$; thus $M_2 \cap M_3 = M_1 \cap M_2$. So $M_1 = (M_2 \cap M_3) \cup \{e_1\} = M_3$, which is a contradiction.

Case 2 *Neither* e_1 *nor* e_2 *belongs to* M_3. Since M_1 and M_3 are adjacent matchings, there exists an edge e_3 in G such that $M_3 - M_1 = \{e_3\}$. On the other hand, $e_1 \in M_1$ but $e_1 \notin M_3$; so $M_1 - M_3 = \{e_1\}$. Consequently, e_1 and e_3 are adjacent edges in G. Also, M_2 and M_3 are adjacent matchings. Since $e_2 \in M_2$ and $e_2 \notin M_3$, it follows that $M_2 - M_3 = \{e_2\}$. If $e_3 \in M_2$, then necessarily $e_3 \in M_1$, which contradicts the fact that $e_1 \in M_1$ and e_1 and e_3 are adjacent. Consequently, $e_3 \notin M_2$. Therefore, $M_3 - M_2 = \{e_3\}$. Thus $M_3 = M_2 \cap M_3 \cup \{e_3\}$. Since $M_2 = M_1 \cap M_2 \cup \{e_2\}$ and M_2 and M_3 have all but one edge in common, $M_2 \cap M_3 = M_1 \cap M_2$ so that $M_3 = M_1 \cap M_2 \cup \{e_3\}$.

Hence we have shown that if a matching graph contains three mutually adjacent matchings M_1, M_2, M_3, then these matchings are precisely $M_i = M_1 \cap M_2 \cup \{e_i\}$, where e_1, e_2, e_3 are distinct edges with $e_i \notin M_1 \cap M_2$ for $i = 1, 2, 3$. Thus since M_1, M_3, and M_4 are mutually adjacent matchings, it now follows that $M_4 = M_1 \cap M_2 \cup \{e_4\}$, where $e_4 \in M_4 - M_1$. But since M_2 and M_4 are matchings that differ by exactly one edge, they are adjacent, producing the desired contradiction. ❑

There are many problems yet to be studied regarding matching graphs. Clearly, it would be of interest to characterize those graphs that are matching graphs. Assuming that this is a difficult task, one may wish to determine those graphs whose matching graph has a specified property. For example, which graphs have a connected or a hamiltonian matching graph? It has been previously noted that if G is a graph with j (≥ 1) perfect matchings, then $M(G) = jK_1$. Also, let $G = H_1 \cup H_2$, where H_1 has j (≥ 1) perfect matchings. By Theorem 9, $M(G) = jM(H_2)$, and thus if $M(H_2)$ is connected, then the matching graph of G consists of j copies of some connected graph. It is not known whether every disconnected matching graph consists only of isomorphic components.

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