The Elimination Procedure for the Competition Number

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Abstract

If D is an acyclic digraph, its competition graph is an undirected graph with the same vertex set and an edge between vertices x and y if there is a vertex a so that (x, a) and (y, a) are both arcs of D. If G is any graph, G together with sufficiently many isolated vertices is a competition graph, and the competition number of G is the smallest number of such isolated vertices. Roberts [1978] gives an elimination procedure for estimating the competition number and Opsut [1982] showed that this procedure could overestimate. In this paper, we modify that elimination procedure and then show that for a large class of graphs it calculates the competition number exactly.

1. Introduction

In this paper, we study the notion of competition graph which was introduced by Cohen [1968] and has been widely studied since. If D is an acyclic digraph (V, A), then its *competition graph* is an undirected graph $G = (V, E)$ with the same vertex set and an edge between vertices x and y if there is a vertex a in V and arcs (x, a) and (y, a) in D. We say that a graph G is a competition graph if it arises as the competition graph of some acyclic digraph. (Sometimes the condition of acyclicity is weakened; see for example the papers by Dutton and Brigham [1983] and Roberts and Steif [1983]. However, we do not weaken the condition here.) Competition graphs arose in connection with an application in ecology and also have

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applications in coding, radio transmission, and modelling of complex economic systems. (See Raychaudhuri and Roberts [1985] for a summary of these applications and Greenberg, Lundgren, and Maybee [1981] for a sample paper on the modelling application.) The vast literature of competition graphs is summarized in the survey paper by Lundgren [1989]. We shall study the notion of competition number which arose in connection with the attempts to characterize competition graphs, and show that a certain elimination procedure computes this number exactly for a large class of graphs, thus going a long way toward settling a question that has been around since 1978.

We will use the notation I_r to denote a graph with r isolated vertices and $G \cup I_r$ for the graph obtained from graph G by adding r isolated vertices. Roberts [1978] observed that if G is any graph, then $G \cup I_r$ is a competition graph of an acyclic digraph for r sufficiently large. He defined the *competition number* of G, $k(G)$, to be the smallest such r, and observed that characterization of competition graphs is equivalent to computation of competition number. The notion of competition number has since been widely studied, as have variants such as niche number and double competition number (see for example Cable, et al. [1989], Fishburn and Gehrlein [1992], Jones, et al. [1987], Lundgren [1989] and Scott [1987]). Opsut [1982] showed that computation of competition number is NP-complete.

Roberts [1978] introduced an elimination procedure for computing a parameter $m(G)$, the elimination number of G, and showed that $k(G) \leq m(G)$ and that for various interesting graphs, $k(G) = m(G)$. Opsut [1982] showed that there were graphs for which these two numbers could be different. In this paper, we modify the elimination procedure so that it computes a parameter $M(G)$ called the modified elimination number, observe that $k(G) \leq M(G) \leq m(G)$ for all graphs G and that for a very large class of graphs, $k(G) = M(G)$. We have not been able to find an example for which $k(G) \neq M(G)$. Both elimination procedures are similar to (but different from) the ones used by Parter [1961], Rose [1970], Golumbic [1978] and others in applying graph theory to study optimal orderings of elimination in Gaussian elimination. Such elimination procedures have been widely studied. Some of the literature about them is summarized in Golumbic [1980]. Neither the original nor our modified elimination procedure for the competition number are very efficient. (Recall that the problem of computing $k(G)$ is NP-complete.) In fact, these elimination procedures require $n!$ different runs. Each of those runs involves a computation that is exponential. Our emphasis here is not so much on the efficiency of the procedure as it is on making progress on the old question of whether an elimination procedure can be used to calculate the competition number.

In what follows, it will be useful to adopt some terminology that is commonly used in the literature of competition graphs and has its origins in the ecological applications of the subject. Specifically, if (x, y) is an arc of digraph D, we call y a *prey* of x and x a *predator* of y. If (x, a) and (y, a) are arcs, we say that a is a *common prey* of x and y. We will also be concerned with *vertex clique coverings* of a graph G; these are collections of cliques that include all the vertices of G. We will use the notation $\theta(G)$ to denote the smallest number of cliques in a vertex clique covering of G. In searching for a vertex clique covering with $\theta(G)$ cliques, we can always limit ourselves to maximal cliques. It will sometimes be necessary to speak of a vertex clique covering of a graph G that is a subgraph of another graph H, and where we are allowed to use cliques of H that may contain vertices not in G. We denote the number of cliques of H in a smallest such vertex clique covering by $\theta_H(G)$. $N_G(v)$ will denote the *open neighborhood* of v in G, the set consisting of all vertices adjacent to v in G. $N_G[v]$ will denote the *closed neighborhood* of G, the set $N_G(v) \cup \{v\}$. We will be interested in $\theta_H(N_G[v])$ for G a subgraph of H. For undefined graph-theoretical terms, the reader is referred to Bondy and Murty [1976].

2. The Elimination Procedure

The following is the modified elimination procedure. After describing it both informally and formally, we shall compare it to the original elimination procedure in Roberts [1978]. The basic idea is to eliminate one vertex of G at a time to build an acyclic digraph whose competition graph is graph G together with some added isolated vertices. We order the vertices of G as $v_1, v_2, \ldots, v_n$. At the j^{th} stage of the procedure, we "eliminate" vertex v_{j+1} by accounting for its adjacencies in G. Beginning the j^{th} stage, we will have built an acyclic digraph $F_j = (V_j, A_j)$ whose vertex set contains the vertex set of G. Digraph F_n will have the property that its competition graph is G together with some isolated vertices. Some edges $\{x, y\}$ of G are *covered* at the j^{th} stage of the procedure in the sense that we add arcs to F_j from x and y to a common prey. We shall keep track of the remaining uncovered edges of G by using a graph G_j which is a spanning subgraph of G. The procedure also keeps track of the subgraph H_j of G induced by vertices that have not been eliminated in previous stages. We pick arcs to add to F_j in the j^{th} stage of the procedure by using maximal cliques in H_j that cover all of the edges from v_{j+1} that are not yet covered, i.e., that remain in graph G_j. For each such clique, we let all of its vertices prey on a common prey; this is chosen from previously eliminated vertices or new vertices added to G where the new vertices play the role of isolated vertices in the competition graph. By this method, we cover all the edges in G_j between v_{j+1} and its neighbors and the edges in those maximal cliques of H_j used. We keep track of the available candidates for common prey by using a set S_j.

The Elimination Procedure

Input: A graph $G = (V, E)$ of n vertices and an ordering of vertices $P = v_1, v_2, \ldots, v_n$.

Output: An acyclic digraph F_n on V together with M new vertices so that $G \cup I_M$ is the competition graph of F_n.

Step 0. Set $G_0 = G$, $S_0 = \emptyset$, $H_0 = G$, and $j = 0$. Let a digraph F_0 have vertex set $V_0 = V$ and arc set $A_0 = \emptyset$.

Step $j1(j \geq 0)$. Let $N_j[v_{j+1}]$ be $N_{G_j}[v_{j+1}]$. Calculate $\theta_{H_j}(N_j[v_{j+1}])$. (By the way that G_j and H_j are defined, $N_j[v_{j+1}]$ is always a subgraph of H_j and therefore $\theta_{H_j}(N_j[v_{j+1}])$ is defined.) Let h_j be this number except when $N_j[v_{j+1}]$ has just one vertex v_{j+1}, in which case take h_j to be 0. If $h_j = 0$, let $S_{j+1} = S_j \cup \{v_{j+1}\}$, $G_{j+1} = G_j$, $H_{j+1} = H_j$, $F_{j+1} = F_j$, $V_{j+1} = V_j$, and $A_{j+1} = A_j$, and go to Step $j3$. If $h_j \neq 0$, find maximal cliques $K_{j_1}, \ldots, K_{j_{h_j}}$ in H_j that form a minimum vertex clique covering of $N_j[v_{j+1}]$. (Note that $v_{j+1} \in K_{j_s}$ for $s = 1, \ldots, h_j$.)

Step $j2$. If $S_j \neq \emptyset$, pick h_j distinct vertices $v_{j_1}, v_{j_2}, \ldots, v_{j_{h_j}}$ ¿from S_j, using lowest indexed vertices of S_j first. If S_j has fewer than h_j elements, then after using all the elements of S_j, add new vertices $v_{j(w+1)}, v_{j(w+2)}, \ldots, v_{jh_j}$, where $w = |S_j|$, that are not in V_j. Add to A_j arcs from the vertices of K_{j_s} to v_{j_s}, $s = 1, \ldots, h_j$, and let A_{j+1} be A_j plus all added arcs and V_{j+1} be V_j plus all new vertices added in this step, and let $F_{j+1} = (V_{j+1}, A_{j+1})$. Set S_{j+1} equal to the set of all vertices of S_j not used as v_{j_s} plus the vertex v_{j+1}, i.e.,

$$S_{j+1} = S_j - \{v_{j_1}, \ldots, v_{j_{h_j}}\} \cup \{v_{j+1}\}.$$

Step $j3$. If $j = n - 1$, output the digraph F_n and stop. If not and $h_j \neq 0$, let G_{j+1} be the graph obtained from G_j by deleting all edges in cliques K_{j_s} (but not their vertices) and let H_{j+1} be obtained from H_j by deleting vertex v_{j+1}. (Thus, H_{j+1} is always an induced subgraph of G.)

Step $j4$. Set $j \longleftarrow j + 1$ and go to Step $j1$.

Remark: F_j, G_j, H_j, and so on depend upon the order P, but our notation suppresses P.

Proposition 1 *The elimination procedure produces acyclic digraphs F_j, $j = 0, \ldots, n$, and the competition graph of F_n is $G \cup I_M$, where $M = M(G, P)$ is the number of vertices added.*

Proof. Since every arc of F_j either goes from a vertex to one with a lower index or to an added vertex, this digraph is clearly acyclic. Moreover, since the added vertices have no outgoing arcs, they are isolated vertices in the competition graph of F_n. If x and y have a common prey in F_n, then they are in a clique K_{j_s} and so are joined by an edge in H_j, which is a subgraph of G. If x and y in $V(G)$ are adjacent, then the edge $\{x, y\}$ remains in the graph G_j until at some step $i1$ it appears in some clique K_{i_s}, at which time we add a common prey for x and y ¿from S_i and so x and y are joined by an edge in the competition graph of F_i, F_{i+1}, ..., F_n. Every such $\{x, y\} = \{v_p, v_q\}$ appears in a clique K_{i_s} no later than step $i1$ where $i + 1 = \min\{p, q\}$, since v_p and v_q are in the closed neighborhood of v_{i+1} and v_{i+1} is in every maximal clique K_{i_s}. Q.E.D.

The modified elimination procedure we have presented differs ¿from that of Roberts [1978]. In the latter, the neighborhood of v_{j+1} in G_j is covered by cliques in G_j and, moreover, we look for cliques that cover not only the vertices of this neighborhood but also its edges. In the modified procedure the neighborhood of v_{j+1} in G_j is covered by cliques in H_j, not G_j, and we search for cliques covering all of the vertices (and hence all of the edges between v_{j+1} and its neighbors in G_j). If $m(P, G)$ is the number of added vertices under the original procedure and $M(P, G)$ is the number of added vertices under the modified procedure, it is easy to see that $M(P, G) \leq m(P, G)$. This follows because $N_{G_j}[v_{j+1}]$ is a subset of $N_{H_j}[v_{j+1}]$ and therefore the number of cliques of H_j required to cover the former is no more than the number of cliques of H_j required to cover the latter.

The number of added isolated vertices depends upon the ordering P. Hence, we define the *modified elimination number* $M(G)$ to be the minimum of $M(P, G)$ over all orders P. The minimum $m(G)$ of $m(P, G)$ is called the *elimination number* by Roberts [1978]. It follows from Proposition 1 and the fact that $M(G, P) \leq m(G, P)$ for any P that $k(G) \leq M(G) \leq m(G)$. Opsut [1982] gives an example (see Figure 1) in which $k(G)$ is less than $m(G)$. It is easy to show that for this graph, $k(G) = M(G)$. This result will follow from Theorem ?? below. It is also easy to see here since $k(G) \geq 1$ whenever G has no isolated vertices, and since $M(G, P) = 1$ if P is the order a, b, c, d, e, f. That $m(G, P) \geq 2$ for any order P is straightforward. (Here, $m(G) = 2$, with the optimum obtained using the order b, a, f, c, d, e.)

3. Sufficient Conditions for $k(G) = M(G)$

The graph of Figure 2 will play an important role in this paper. We shall call it a *kite*. In a kite, the solid edges must appear and the dotted edges cannot appear. The remaining edge is possible. A *kite-free* graph is a graph that does not have a kite as a configuration.

Our main theorem is the following:

Theorem 2 *If G is kite-free, then $k(G) = M(G)$.*

We recall that the computation of the competition number of G is an NP-complete problem. So is the computation of $\theta(G)$. We make no claim that the elimination procedure gives a relatively efficient algorithm for calculating $k(G)$ even for kite-free graphs. Indeed, it requires $n!$ different runs. In each run, we have to compute θ a total of n times. It is not hard to show that although computation of $\theta(G)$ is an NP-complete problem, the greater computational difficulty comes from the need to repeat the entire procedure $n!$ times.

The rest of this paper will be devoted to proving Theorem ??. We already know from Proposition 1 that $k(G) \leq M(G)$. We shall use the notation of the elimination procedure.

Let us say that a kite body is the configuration of Figure 3 and that a graph G is *kite-body-free* if it contains no such configuration.

Lemma 3 *Suppose G is a kite-body-free graph, S is a subset of $V(G)$, H is an induced subgraph of G, and K_1, K_2, ..., K_q is a vertex clique covering of S using maximal cliques of H. Suppose a subset C of S forms a clique in H. Then C is contained in some K_i.*

Proof. Suppose that we have x, y in C such that $x \in K_i$, $y \in K_j$, $x \notin K_j$, $y \notin K_i$ for some $1 \leq i, j \leq q$. Then there is $u \in K_i$ not adjacent to y and $v \in K_j$ not adjacent to x. It follows that u, x, y, v form a kite-body. Since this cannot happen, we conclude that whenever $y \notin K_i$ and $y \in K_j$, then if $x \in K_i$, we must have $x \in K_j$. It follows that $K_i \cap C \subseteq K_j$ and, since $y \notin K_i$ and $y \in K_j$, we have $K_i \cap C$ a proper subset of $K_j \cap C$.

Let i be chosen so that $K_i \cap C$ has the maximum number of vertices. Then C must be a subset of K_i. Otherwise, if y is in $C - K_i$, y must be in some K_j. By our previous argument, $K_i \cap C$ is a proper subset of $K_j \cap C$, which contradicts the above choice of K_i. Q.E.D.

If D is acyclic, we can label the vertices of $V(D)$ with distinct positive integer labels $\pi(x)$ so that if $(x, y) \in A(D)$, then $\pi(y) < \pi(x)$. We call such a labelling of vertices of an acyclic digraph an *acyclic labelling.*

Suppose that G has competition number $k = k(G)$ and F is an acyclic digraph whose corresponding competition graph is $G \cup I_k$. Let π be an acyclic labelling of vertices of F. Let us label the vertices of $V(G)$ as v_1, ..., v_n so that $\pi(v_i) < \pi(v_j)$ if and only if $i < j$. We shall use the ordering P on $V(G)$ defined by the labelling v_1, ..., v_n to perform the elimination procedure and we shall show that the procedure produces a digraph F_n with competition graph $G \cup I_k$.

Let

$\mathcal{D}(P) = \mathcal{D} = \{D : D$ is an acyclic digraph whose competition graph is $G \cup I_k$, that has no outgoing arcs ¿from any vertex outside of $V(G)$, and that has an acyclic labelling that agrees with P on $V(G)\}$.

Certainly $\mathcal{D} \neq \emptyset$ since $F \in \mathcal{D}$. We may restrict ourselves to D's with no outgoing arcs from any added vertex because deleting such arcs does not change $G \cup I_k$. If $D \in \mathcal{D}$, let

$$R_s = R_s(D) = \{x \in V(D) : (v_s, x) \in A(D)\}$$

and

$$Q_x = Q_x(D) = \{y \in V(D) : (y, x) \in A(D)\}.$$

Let

$$\mathcal{V}_0(D) = V(G), \ \mathcal{A}_0(D) = \emptyset, \ \mathcal{W}_0(D) = \emptyset$$

and for $1 \leq j \leq n$, let

$$\mathcal{V}_j(D) = V(G) \cup \{u \in I_k : (v_s, u) \in A(D) \text{ for some } s, 1 \leq s \leq j\},$$

$$\begin{aligned} \mathcal{A}_j(D) &= \cup_{s=1}^{j} \cup_{x \in R_s} \cup_{y \in Q_x} \{(y, x)\} \\ &= \{(y, x) \in A(D) : (v_s, x) \in A(D) \text{ for some } s, 1 \leq s \leq j\} \end{aligned}$$

and

$$\mathcal{W}_j(D) = \{x \in V(D) : j \text{ is the lowest index for a predator of } x \text{ in } D\}.$$

We recall that $n = |V(G)|$; hence $\mathcal{V}_n(D) = V(D)$, $\mathcal{A}_n(D) = A(D)$. Thus, it suffices to show that there is a digraph D in $\mathcal{D}(P) = \mathcal{D}$ so that $(\mathcal{V}_n(D), \mathcal{A}_n(D)) = F_n$. In fact, we show by induction on $j \leq n$ that there is a digraph $D \in \mathcal{D}$ such that

$$(\mathcal{V}_j(D), \mathcal{A}_j(D)) = F_j. \tag{1}$$

(The proof will show that we will not need to change the order P ¿from the one with which we start, and so we obtain the stronger result that the elimination procedure applied to an acyclic order for a digraph D that has competition graph $G \cup I_k$ does indeed give rise to the competition number.) It is clear by definition that for any $D \in \mathcal{D}$, $(\mathcal{V}_0(D), \mathcal{A}_0(D)) = (V(G), \emptyset) = F_0$, so (1) holds when $j = 0$. We assume that (1) holds for some $j < n$ for some $F^* \in \mathcal{D}$ and show the same for $j + 1$. That is, we will set out to construct a digraph D^* in $\mathcal{D}(P)$ so that

$$(\mathcal{V}_{j+1}(D^*), \mathcal{A}_{j+1}(D^*)) = F_{j+1}. \tag{2}$$

Suppose $h_j = 0$. Then $N_j[v_{j+1}]$ has one vertex, v_{j+1}. If there is no arc in F^* from v_{j+1} to an isolated vertex u, or if whenever there is such an arc there is also an arc from v_i to u for some $i < j+1$, then $\mathcal{V}_{j+1}(F^*) = \mathcal{V}_j(F^*)$, $\mathcal{A}_{j+1}(F^*) = \mathcal{A}_j(F^*)$, and $F_{j+1} = F_j$, so (2) holds. If there is an arc in F^* from v_{j+1} to an isolated vertex u but no arc from v_i to u for $i < j+1$, we may drop the arc (v_{j+1}, u) from F^*. This clearly does not change the acyclicity or the acyclic ordering. Since $N_j[v_{j+1}] = \{v_{j+1}\}$, v_{j+1} is not adjacent to any v_s, $s > j+1$, in G, and so the indegree of u is one in F^*; it follows that the competition graph does not change after deleting the arc (v_{j+1}, u). However, in the digraph D^* resulting from dropping such arcs, (2) holds since $\mathcal{V}_{j+1}(D^*) = \mathcal{V}_j(F^*)$, $\mathcal{A}_{j+1}(D^*) = \mathcal{A}_j(F^*)$, and $F_{j+1} = F_j$. Thus, we may assume that $h_j \neq 0$, and we do so for the rest of the proof.

For each $D \in \mathcal{D}$ satisfying (1) and for each $y \in \mathcal{W}_{j+1}(D)$, let

$$C_y(D) = \{x : (x, y) \in A(D)\}.$$

It is easy to see that $C_y(D)$ is a clique in G and since $y \in \mathcal{W}_{j+1}(D)$, there is an arc from v_{j+1} to y in D and hence v_{j+1} belongs to $C_y(D)$.

Let us recall some terminology. In applying the elimination procedure, we encounter edges $\{x, y\}$ in $N_j[v_{j+1}]$ and arcs from x and y to some vertex of S_j or some new vertex added in step j. We then say that edge $\{x, y\}$ is *covered* in step j. All edges $\{x, y\}$ with x, y adjacent to v_{j+1} are covered in a step r with $r \leq j$.

The proof proceeds with a series of lemmas that will be used to replace F^* by other digraphs in $\mathcal{D}$. We first replace F^* by $D \in \mathcal{D}$, then D by $D' \in \mathcal{D}$. We use a procedure called "raising" to replace D' by $D'' \in \mathcal{D}$. From D'', we will construct an explicit $D^{(3)} \in \mathcal{D}$, and we will then again use this raising procedure to change $D^{(3)}$ into $D^{(4)} \in \mathcal{D}$. By continuing with this procedure, we eventually build $D^* \in \mathcal{D}$ satisfying (2).

Lemma 4 *If $D \in \mathcal{D}$ satisfies* (1), *then*

$$\cup_{y \in \mathcal{W}_{j+1}(D)} C_y(D) \supseteq N_j[v_{j+1}].$$

Proof. Suppose that x is in $N_j[v_{j+1}]$ but not in $\cup_{y \in \mathcal{W}_{j+1}(D)} C_y(D)$. Then $x \neq v_{j+1}$. Since x is in $N_j[v_{j+1}]$, the index of x is greater than $j+1$ because all vertices of lower index are isolated in G_j. Since x is adjacent to v_{j+1} in G, x and v_{j+1} are predators of a vertex z in D. If all the other predators of z in D have indices greater than $j+1$, then z is in $\mathcal{W}_{j+1}(D)$. Since x is in $C_z(D)$, we have reached a contradiction. Therefore, one of the incoming arcs to z in D is from a vertex w of index i less than $j+1$. However, since (1) holds, edge $\{x, v_{j+1}\}$ was covered in step r for $r \leq i$ and cannot be in G_j. This is a contradiction. Q.E.D.

Lemma 5 *There exists $D \in \mathcal{D}$ such that D satisfies* (1) *and for each y in $\mathcal{W}_{j+1}(D)$, $C_y(D)$ is included in some K_{j_s}.*

Proof. We take $D = F^*$ and let y be in $\mathcal{W}_{j+1}(D)$. We consider two cases.

Case 1: There is $z \neq v_{j+1}$ in $C_y(D)$ so that $z \in N_j[v_{j+1}]$. Then $z \in K_{j_s}$ for some s. We shall show that $C_y(D) \subseteq K_{j_t}$ for some t. Let G' be the subgraph of G induced by vertices of $N_G(v_{j+1})$, the open neighborhood of v_{j+1} in G. Then since G is kite-free, G' is kite-body-free. Let H be the subgraph of G' induced by vertices in $N_{H_j}(v_{j+1})$ and let $S = C_y(D) - \{v_{j+1}\}$. Now for any $q = 1, \ldots, h_j$, any vertex in $K_{j_q} - \{v_{j+1}\}$ is adjacent to v_{j+1} and $K_{j_q} - \{v_{j+1}\}$ is a clique in G' . It is a maximal clique in H. Moreover, S is a clique in H. It is a clique because every pair of elements in $C_y(D)$ has y as a common prey in D and it is in H because every element in S has index at least $j+1$, by definition of $\mathcal{W}_{j+1}(D)$. We claim that the collection of maximal cliques $K_{j_q} - \{v_{j+1}\}$, $q = 1, \ldots, h_j$, is a vertex clique covering of S. This will allow us to use Lemma 3 (with $C = S$) and conclude that $S \subseteq K_{j_t}$ for some t, as required, and complete the proof in Case 1. Thus, we suppose there is a vertex u in S that is not in any of the maximal cliques $K_{j_q} - \{v_{j+1}\}$, $q = 1, \ldots, h_j$. We will reach a contradiction from this assumption. Since u is not in any K_{j_q} and u is in S which is in H, we conclude that for $q = 1, \ldots, h_j$, there is $z_q \in K_{j_q}$ so that z_q is not adjacent to u in H and therefore in G. Since u is not in any K_{j_q}, u is not in $N_j[v_{j+1}]$, and so edge $\{v_{j+1}, u\}$ is covered in a step i prior to step j. This implies that $\{v_{j+1}, u, v_{i+1}\}$ is included in a maximal clique in H_i, say K_{i_r}, among the maximal cliques used in the elimination procedure at step i. If z is not adjacent to v_{i+1}, then v_{j+1}, z_s, z, u, v_{i+1} form a kite in G, which cannot be. (We get the edge $\{u, z\}$ from $C_y(D)$, the edges $\{u, v_{i+1}\}$, $\{u, v_{j+1}\}$, $\{v_{i+1}, v_{j+1}\}$ from K_{i_r}, and the edges $\{z_s, v_{j+1}\}$, $\{z, v_{j+1}\}$, and $\{z, z_s\}$ from K_{j_s}; we get the non-edges $\{u, z_s\}$ and $\{z, v_{i+1}\}$ by assumption.) Therefore, suppose z is adjacent to v_{i+1}. If z belongs to K_{i_r}, then edge $\{z, v_{j+1}\}$ would have been covered in step i and z could not be in $N_j[v_{j+1}]$. Thus, there is v_k in K_{i_r} such that v_r is not adjacent to z. Then v_{j+1}, z_s, z, u, v_k form a kite in G, which again cannot be. (We get the edges $\{u, v_k\}$ and $\{v_{j+1}, v_k\}$ from clique K_{i_r} and the non-edge $\{z, v_k\}$ by assumption, and the rest of the argument is as in the previous kite.)

Case 2: For every $z \neq v_{j+1}$ in $C_y(D)$, $z \notin N_j[v_{j+1}]$. Consider $z \in C_y(D)$. Recall that $y \in \mathcal{W}_{j+1}(D)$, there is an arc v_{j+1} to y and hence since $z \in C_y(D)$, $\{z, v_{j+1}\}$ is an edge of G. This edge is covered in a step i prior to step j because z is not in $N_j[v_{j+1}]$. Thus, every edge $\{z, v_{j+1}\}$ for $z \in C_y(D)$ is covered in a step prior to step j. Then the new digraph D' obtained from D by deleting arc (v_{j+1}, y) still is acyclic and has the same acyclic order. Also, Equation (1) holds for D' since arc (v_{j+1}, y) was not added in any step prior to step j, by the definition of $\mathcal{W}_{j+1}(D)$. Finally, we note that D' has competition graph $G \cup I_k$ and so is in $\mathcal{D}$. To see why, note that deleting arc (v_{j+1}, y) will result in not covering edges $\{z, v_{j+1}\}$ for $z \in C_y(D)$ at step j. However, these edges are covered in steps prior to step j and so are in the competition graph of D'. By replacing D by D', we get a situation where $\mathcal{W}_{j+1}(D')$ has fewer elements than $\mathcal{W}_{j+1}(D)$. By continuing the process, we end up in a situation where the hypothesis of Case 1 holds or where $\mathcal{W}_{j+1}(D)$ is empty. In the latter case, the lemma holds vacuously. Q.E.D.

If D is any digraph in $\mathcal{D}$, let

$$\psi_j(D) = \max\{\pi(v) : v \in \mathcal{W}_{j+1}(D) \cap S_j\}$$

and let

$$\mathcal{D}(S_j) = \{D \in \mathcal{D} : D \text{ satisfies (1) and there is no } i \leq \psi_j(D) \text{ so that } v_i \in S_j - \mathcal{W}_{j+1}(D)\}$$

Lemma 6 *There is a digraph $D' \in \mathcal{D}(S_j)$ such that for each y in $\mathcal{W}_{j+1}(D')$, $C_y(D')$ is included in some K_{j_s}.*

Proof. By Lemma 5, there is a digraph $D \in \mathcal{D}$ such that D satisfies (1) and for each y in $\mathcal{W}_{j+1}(D)$, $C_y(D)$ is included in some K_{j_s}. Suppose $\mathcal{W}_{j+1}(D) \cap S_j = \{v_{j_1}, \ldots, v_{j_p}\}$, with $j_1 < \ldots < j_p$. Then $j_p = \psi_j(D)$. Suppose that there is an $i \leq j_p$ so that $v_i \in S_j - \mathcal{W}_{j+1}(D)$. Note that $i \neq j_p$ by definition of $\psi_j(D)$. Build D' from D by replacing every arc (x, v_{j_p}) by an arc (x, v_i) and every arc (u, v_i) by an arc (u, v_{j_p}). Obviously, the competition graph does not change. The digraph D' is still acyclic. To see why, note that $j_p \leq j$ because $v_{j_p} \in S_j$. By definition of $\mathcal{W}_{j+1}(D)$, all predators of v_{j_p} in D have indices $\geq j+1 > j \geq j_p > i$. All predators of v_i in D have indices $> j \geq j_p$, for if $(v_r, v_i) \in A(D)$ and $r \leq j$, then $v_i \in R_r(D)$; then since D satisfies (1), we conclude that v_i was used as a prey in the elimination procedure at a step prior to step r and so prior to step j and could not be in S_j, which is a contradiction. The acyclic order for D is still an acyclic order for D'. Hence, D' is in $\mathcal{D}$. Since the only changed arcs in going from D to D' are from vertices of index $\geq j+1$, we note that $(\mathcal{V}_j(D'), \mathcal{A}_j(D')) = (\mathcal{V}_j(D), \mathcal{A}_j(D)) = F_j$ and (1) still holds. Finally, since $\mathcal{W}_{j+1}(D') = \mathcal{W}_{j+1}(D) - \{v_{j_p}\} \cup \{v_i\}$ and $C_{v_i}(D') = C_{v_{j_p}}(D)$, $C_y(D')$ is included in some K_{j_s} for each $y \in \mathcal{W}_{j+1}(D')$. By continuing this process, we end up with D' such that there is no $i \leq j_p$ with $v_i \in S_j - \mathcal{W}_{j+1}(D')$. Q.E.D.

We now describe a method, called *raising*, of transforming one digraph in $\mathcal{D}(S_j)$ into another. We shall use this method several times in the proof.

Given $E \in \mathcal{D}(S_j)$, let $L(E)$ be a list of the vertices of $\mathcal{W}_{j+1}(E)$, starting with vertices in G in order of increasing index, and following by any added vertices not in G, in any order. We define a new digraph $E' = \Delta(E)$ called the *raised digraph* obtained from E. Define E' as follows:

$$V(E') = V(E);$$

$$A(E') = A(E) - \{(x, y) \in A(E) : y \in \mathcal{W}_{j+1}(E)\} + \text{ arcs from all vertices of } K_{j_q} \text{ to the } q^{th} \text{ vertex in list } L(E),\ q = 1, 2, \ldots, h_j.$$

We have enough vertices in $\mathcal{W}_{j+1}(E)$ to do the latter. For, each $C_y(E)$ for y in $\mathcal{W}_{j+1}(E)$ is a clique in H_j and by Lemma 4, these cliques form a vertex clique covering of $N_j[v_{j+1}]$ and so there are at least as many y's as there are cliques K_{j_q} since the K_{j_q} define a minimum vertex clique covering.

The next lemma shows that under appropriate assumptions, raising leaves a digraph in $\mathcal{D}$. Before proving this lemma, we illustrate the procedure with the example in Figure 4. It can easily be checked that F is a digraph which can be obtained from the elimination procedure applied to G by following the vertex ordering $P = v_1, \ldots, v_6$, where a is an added isolated vertex. The digraph E belongs to $\mathcal{D}(P)$. We note that $F_3 = (\mathcal{V}_3(E), \mathcal{A}_3(E))$, $S_3 = \{v_2, v_3\}$, and $\mathcal{W}_4(E) = \{v_2, v_3\}$. Since $E \in \mathcal{D}(S_3)$, $E' = \Delta(E)$ is well-defined. In fact, using raising with $j = 3$, we find that $F_4 = (\mathcal{V}_4(E'), \mathcal{A}_4(E'))$ and $E' = F$ together with arcs (v_5, v_4) and (v_6, v_4). Also, it is easy to check that E' is in $\mathcal{D}(S_4)$, $S_4 = \{v_3, v_4\}$, and $\mathcal{W}_5(E') = \{v_4\}$. Then, using raising with $j = 4$, we have $\Delta(E') = \Delta(\Delta(E)) = F$.

Lemma 7 *If E is in $\mathcal{D}(S_j)$ and for each $y \in \mathcal{W}_{j+1}(E)$, $C_y(E)$ is included in some K_{j_s}, then the raised digraph $E' = \Delta(E)$ is in $\mathcal{D}$, $\mathcal{W}_{j+1}(E') \subseteq \mathcal{W}_{j+1}(E)$, and $|\mathcal{W}_{j+1}(E')| = h_j$. Moreover, $C_y(E')$ is included in some K_{j_s} for each $y \in \mathcal{W}_{j+1}(E')$ and E' belongs to $\mathcal{D}(S_j)$.*

Proof. E' is acyclic since any vertices in $\mathcal{W}_{j+1}(E)$ have indices $\leq j$ if they are in $V(G)$ or no outgoing arcs if they are not in $V(G)$ (by definition of $\mathcal{D}$ and because raising does not add such arcs), and any vertices in K_{j_q} have indices $\geq j + 1$. Also, the same ordering of vertices that was an acyclic ordering for E remains one for E'. The competition graph of E' is still $G \cup I_k$ by the hypothesis of the lemma, since $C_y(E)$ is included in some K_{j_s}. Thus, E' is in $\mathcal{D}$.

Next, we note that $\mathcal{W}_{j+1}(E') \subseteq \mathcal{W}_{j+1}(E)$ since if y is not a prey of v_{j+1} in E, it is still not in E', and if y is a prey of v_i, $i < j + 1$, in E, it is still a prey of v_i in E'.

Next, we show that $|\mathcal{W}_{j+1}(E')| = h_j$. Recall our assumption near the beginning of the proof of the theorem, that $h_j \neq 0$. In E', we have used

h_j vertices from list $L(E)$ and each such vertex is in $\mathcal{W}_{j+1}(E')$. No other vertex is in $\mathcal{W}_{j+1}(E')$ since all vertices not in $\mathcal{W}_{j+1}(E)$ could not be in $\mathcal{W}_{j+1}(E')$.

Clearly, for each $y \in \mathcal{W}_{j+1}(E')$, $C_y(E') = K_{j_q}$ and therefore $C_y(E') \subseteq K_{j_q}$ for some $q \in \{1, \ldots, h_j\}$.

Finally, we show that E' belongs to $\mathcal{D}(S_j)$. We note that $\mathcal{V}_j(E') = \mathcal{V}_j(E)$ because if $u \in I_k$ and $(v_s, u) \in A(E)$ for $s \leq j$, then $(v_s, u) \in A(E')$, and if $u \in I_k$ and $(v_s, u) \notin A(E)$ for $s \leq j$, then (v_s, u) is not added in going to E'. Also, $\mathcal{A}_j(E') = \mathcal{A}_j(E)$. To see why note that if $s \leq j$, then $R_s(E)$ and $R_s(E')$ are the same since each arc added or deleted in going ¿from E to E' goes from a vertex v_k for $k > j$. If $x \in R_s(E) = R_s(E')$, then $x \notin \mathcal{W}_{j+1}(E)$ and so $Q_x(E')$ equals $Q_x(E)$. Thus, (1) holds for E'.

Since $\mathcal{W}_{j+1}(E') \subseteq \mathcal{W}_{j+1}(E)$, it follows that $\mathcal{W}_{j+1}(E') \cap S_j \subseteq \mathcal{W}_{j+1}(E) \cap S_j$. Then, by definition of ψ_j, $\psi_j(E') \leq \psi_j(E)$. Now suppose that $v_i \in S_j - \mathcal{W}_{j+1}(E')$ for some $i \leq \psi_j(E')$. If $v_i \notin \mathcal{W}_{j+1}(E)$, then $\psi_j(E') \leq \psi_j(E)$ implies that $i \leq \psi_j(E)$. Then $v_i \in S_j - \mathcal{W}_{j+1}(E)$ implies that $E \notin \mathcal{D}(S_j)$ contrary to the hypothesis of the lemma. If $v_i \in \mathcal{W}_{j+1}(E)$, then $v_{\psi_j(E')}$ and v_i are in $L(E)$. Hence, since $v_{\psi_j(E')}$ was chosen in the construction of E', v_i should have been chosen before $v_{\psi_j(E')}$ and would be in $\mathcal{W}_{j+1}(E')$, which is a contradiction. Q.E.D.

To complete the proof of Theorem ??, we take a digraph $D' \in \mathcal{D}(S_j)$ such that for each y in $\mathcal{W}_{j+1}(D')$, $C_y(D')$ is included in some K_{j_s}. (This D' exists by Lemma 6.) Then we let $D'' = \Delta(D')$ be the raised digraph obtained from D'. Now we consider the relationship between S_j and $\mathcal{W}_{j+1}(D'')$. Suppose first that $S_j \subseteq \mathcal{W}_{j+1}(D'')$. By Lemma 7, $\mathcal{W}_{j+1}(D'') \subseteq \mathcal{W}_{j+1}(D')$. Thus, $S_j \subseteq \mathcal{W}_{j+1}(D')$ and so the procedure to get D'' from D' (except the process of deleting arcs $(x, y) \in A(D')$ such that $y \in \mathcal{W}_{j+1}(D')$) parallels exactly the elimination procedure in step j since elements of $\mathcal{W}_{j+1}(D')$ are given in list $L(D')$ in increasing order of index if they are in G. It follows that $(\mathcal{V}_{j+1}(D''), \mathcal{A}_{j+1}(D'')) = F_{j+1}$, which is Equation (2) as desired.

Suppose next that $\mathcal{W}_{j+1}(D'') \subseteq S_j$. Let $\mathcal{W}_{j+1}(D'') = \{v_{j_1}, \ldots, v_{j_{h_j}}\}$ with $j_1 < \ldots < j_{h_j}$. By Lemma 7, there is no $i \leq j_{h_j}$ so that $v_i \in S_j - \mathcal{W}_{j+1}(D'')$. But then since $\theta_{H_j}(N_j[v_{j+1}]) = h_j$, and the vertices of $\mathcal{W}_{j+1}(D'')$ are the lowest indexed vertices of S_j, the elimination procedure in step j picks elements of $\mathcal{W}_{j+1}(D'')$ from S_j at the beginning and again we conclude that $(\mathcal{V}_{j+1}(D''), \mathcal{A}_{j+1}(D'')) = F_{j+1}$, as desired.

The last case is where neither $S_j \subseteq \mathcal{W}_{j+1}(D'')$ nor $\mathcal{W}_{j+1}(D'') \subseteq S_j$ hold. Now we choose vertices x and y from $\mathcal{W}_{j+1}(D'') - S_j$ and $S_j - \mathcal{W}_{j+1}(D'')$, respectively. We may choose y to have the lowest index of vertices in $S_j - \mathcal{W}_{j+1}(D'')$. Now y is in $V(G)$ and the index of y is $\leq j$ since y is in

S_j. Moreover, x is not in $V(G)$. For, since $x \in \mathcal{W}_{j+1}(D'') \subseteq \mathcal{W}_{j+1}(D')$, x is a prey of v_{j+1} and of no vertex of index $\leq j$. If x were in $V(G)$, x would have index $\leq j$ and so by the elimination procedure, x would have to be in S_j, which is not the case. We now build a new digraph $D^{(3)}$ as follows. For x and y as chosen above, let

$$V(D^{(3)}) = V(D''),$$

$$\begin{aligned} A(D^{(3)}) &= A(D'') - \{(v,x) : (v,x) \in A(D'')\} - \{(v,y) : (v,y) \in A(D'')\} \\ &\cup \{(v,y) : (v,x) \in A(D'')\} \cup \{(v,x) : (v,y) \in A(D'')\}. \end{aligned}$$

Clearly $D^{(3)}$ has the same competition graph $G \cup I_k$ as D''. Also, $D^{(3)}$ is acyclic. To see why, note that if $(v,x) \in A(D'')$, then v has index $\geq j+1$ by the definition of $\mathcal{W}_{j+1}(D'')$, while y has index $\leq j$ as noted above. Also, since $x \notin V(G)$, note that x has no outgoing arcs in D'', by definition of $\mathcal{D}$ and the raising process, and so replacing the arc (v,y) by the arc (v,x) cannot create a cycle. The acyclic order for D' is still an acyclic order for $D^{(3)}$. Thus, $D^{(3)}$ is in $\mathcal{D}$. Next, we show that $(\mathcal{V}_j(D^{(3)}), \mathcal{A}_j(D^{(3)})) = F_j$. The only possible way that $\mathcal{V}_j(D^{(3)})$ could change from $\mathcal{V}_j(D'')$ would be if we were to include vertex x in the former. That could not happen since y is in S_j and so could not have an incoming arc from a vertex of index j. Thus, $\mathcal{V}_j(D^{(3)}) = \mathcal{V}_j(D'')$. Also, $\mathcal{A}_j(D^{(3)}) = \mathcal{A}_j(D'')$ since neither x nor y has a predator of index $\leq j$ and thus $R_s(D^{(3)})$ is the same as $R_s(D'')$ for $s \leq j$. If for $1 \leq s \leq j$, $z \in R_s(D'') = R_s(D^{(3)})$, then $z \notin \mathcal{W}_{j+1}(D'')$, $z \notin S_j$, and so $Q_z(D^{(3)})$ equals $Q_z(D'')$. We shall show that $D^{(3)} \in \mathcal{D}(S_j)$. If $\pi(y) \leq \psi_j(D'')$, then D'' is not in $\mathcal{D}(S_j)$ since $y \in S_j - \mathcal{W}_{j+1}(D'')$. Hence, $\pi(y) > \psi_j(D'')$. However, by selection of y, $\pi(y) \leq \pi(z)$ for any $z \in S_j - \mathcal{W}_{j+1}(D'')$. Now $\mathcal{W}_{j+1}(D^{(3)}) = \mathcal{W}_{j+1}(D'') - \{x\} \cup \{y\}$. Thus, for every $z \in S_j - \mathcal{W}_{j+1}(D^{(3)})$, we have $z \in S_j - \mathcal{W}_{j+1}(D'') - \{y\}$ (since x is not in $V(G)$ and so could not be z). Hence, $\pi(z) \geq \pi(y)$ and in fact $\pi(z) > \pi(y)$ since $z \neq y$. Then $\pi(z) > \pi(y) > \psi_j(D'')$. But since $\mathcal{W}_{j+1}(D^{(3)}) \cap S_j = (\mathcal{W}_{j+1}(D'') \cup \{y\}) \cap S_j$, $\psi_j(D^{(3)}) = \pi(y)$. Thus, $\pi(z) > \psi_j(D^{(3)})$ and $D^{(3)} \in \mathcal{D}(S_j)$. Moreover, it can be easily checked that for each $v \in \mathcal{W}_{j+1}(D^{(3)})$, $C_v(D^{(3)})$ is included in some K_{j_s}.

Now replace the digraph $D^{(3)}$ by the raised digraph $D^{(4)} = \Delta(D^{(3)})$. By Lemma 7, raising leaves us with a digraph $D^{(4)}$ in $\mathcal{D}$ and one for which $(\mathcal{V}_j(D^{(4)}), \mathcal{A}_j(D^{(4)})) = F_j$. Also by Lemma 7, since $D^{(3)} \in \mathcal{D}(S_j)$, the same is true of $D^{(4)}$. Moreover, raising a second time does not change $\mathcal{W}_{j+1}(D^{(3)})$, i.e., $\mathcal{W}_{j+1}(D^{(4)}) = \mathcal{W}_{j+1}(D^{(3)})$. But since $\mathcal{W}_{j+1}(D^{(3)}) = \mathcal{W}_{j+1}(D'') - \{x\} \cup \{y\}$, $S_j - \mathcal{W}_{j+1}(D^{(4)})$ has fewer elements than $S_j - \mathcal{W}_{j+1}(D'')$ and $\mathcal{W}_{j+1}(D^{(4)}) - S_j$ has fewer elements than $\mathcal{W}_{j+1}(D'') - S_j$. By continuing with the same procedure, we eventually construct a digraph D^* in $\mathcal{D}$ with $(\mathcal{V}_j(D^*), \mathcal{A}_j(D^*)) = F_j$ and either $S_j - \mathcal{W}_{j+1}(D^*) = \emptyset$ or

$\mathcal{W}_{j+1}(D^*) - S_j = \emptyset$. But then either $S_j \subseteq \mathcal{W}_{j+1}(D^*)$ or $\mathcal{W}_{j+1}(D^*) \subseteq S_j$. In either of these cases, we conclude that $(\mathcal{V}_{j+1}(D^*), \mathcal{A}_{j+1}(D^*)) = F_{j+1}$, as desired. This completes the proof of Theorem ??.

References

[1] J.A. Bondy, and U.S.R. Murty, *Graph Theory with Applications*, American Elsevier, New York, 1976.

[2] C. Cable, K.F. Jones, J.R. Lundgren, and S. Seager, "Niche Graphs," *Discrete Applied Math.*, **23** (1989), 231- 241.

[3] J.E. Cohen, "Interval Graphs and Food Webs: A Finding and a Problem," RAND Corporation Document 17696-PR, Santa Monica, CA, 1968.

[4] R.D. Dutton, and R.C. Brigham, "A Characterization of Competition Graphs," *Discrete Applied Math.*, **6** (1983), 315- 317.

[5] P.C. Fishburn, and W.V. Gehrlein, "Niche Numbers," *J. of Graph Theory*, **16** (1992), 131-139.

[6] M.C. Golumbic, "A Note on Perfect Gaussian Elimination," *J. Math. Anal. Appl.*, **64** (1978), 455-457.

[7] M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York, 1980.

[8] H.J. Greenberg, J.R. Lundgren, and J.S. Maybee, "Graph-theoretic Foundations of Computer-Assisted Analysis," in H.J. Greenberg, and J.S. Maybee (eds.), *Computer-Assisted Analysis and Model Simplification*, Academic Press, New York, 1981, pp. 481-495.

[9] K.F. Jones, J.R. Lundgren, F.S. Roberts, and S. Seager, "Some Remarks on the Double Competition Number of a Graph," *Congressus Numerantium*, **60** (1987), 17-24.

[10] J.R. Lundgren, "Food Webs, Competition Graphs, Competition-Common Enemy Graphs, and Niche Graphs," in F.S. Roberts (ed.), *Applications of Combinatorics and Graph Theory in the Biological and Social Sciences*, Vol. 17 of IMA Volumes in Mathematics and its Applications, Springer-Verlag, New York, 1989, 221-243.

[11] R.J. Opsut, "On the Computation of the Competition Number of a Graph," *SIAM J. Alg. & Discr. Meth.*, **3** (1982), 420- 428.

[12] S. Parter, "The Use of Linear Graphs in Gauss Elimination," *SIAM Review*, **3** (1961), 119-130.

[13] A. Raychaudhuri, and F.S. Roberts, "Generalized Competition Graphs and their Applications," in P. Brucker and R. Pauly (eds.), *Methods of Operations Research*, **49**, Anton Hain, Konigstein, West Germany, 1985, 295-311.

[14] F.S. Roberts, "Food Webs, Competition Graphs, and the Boxicity of Ecological Phase Space," in Y. Alavi and D. Lick (eds.), *Theory and Applications of Graphs*, Springer-Verlag, New York, 1978, 477-490.

[15] F.S. Roberts, and J.E. Steif, "A Characterization of Competition Graphs of Arbitrary Digraphs," *Discrete Applied Math.*, **6** (1983), 323-326.

[16] D.J. Rose, "Triangulated Graphs and the Elimination Process," *J. Math. Anal. & Appl.*, **32** (1970), 597-609.

[17] D. Scott, "The Competition-Common Enemy Graph of a Digraph," *Discrete Appl. Math.*, **17** (1987), 269-280.

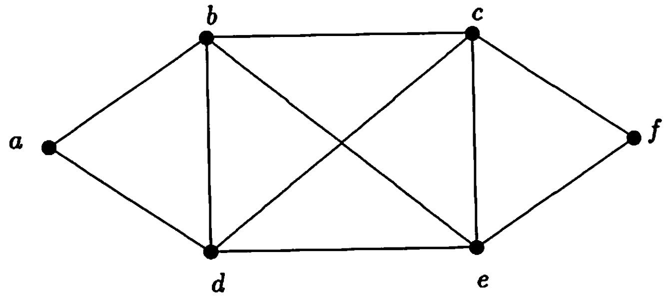

Figure 1: Opsut's example of a graph where $k(G) < m(G)$.

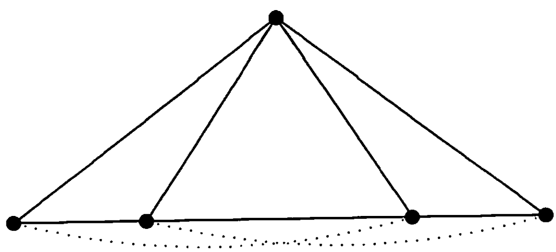

Figure 2: A kite. All solid lines are in the graph and all dotted lines are not.

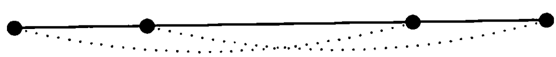

Figure 3: A kite-body. All solid lines are in the graph and all dotted lines are not.

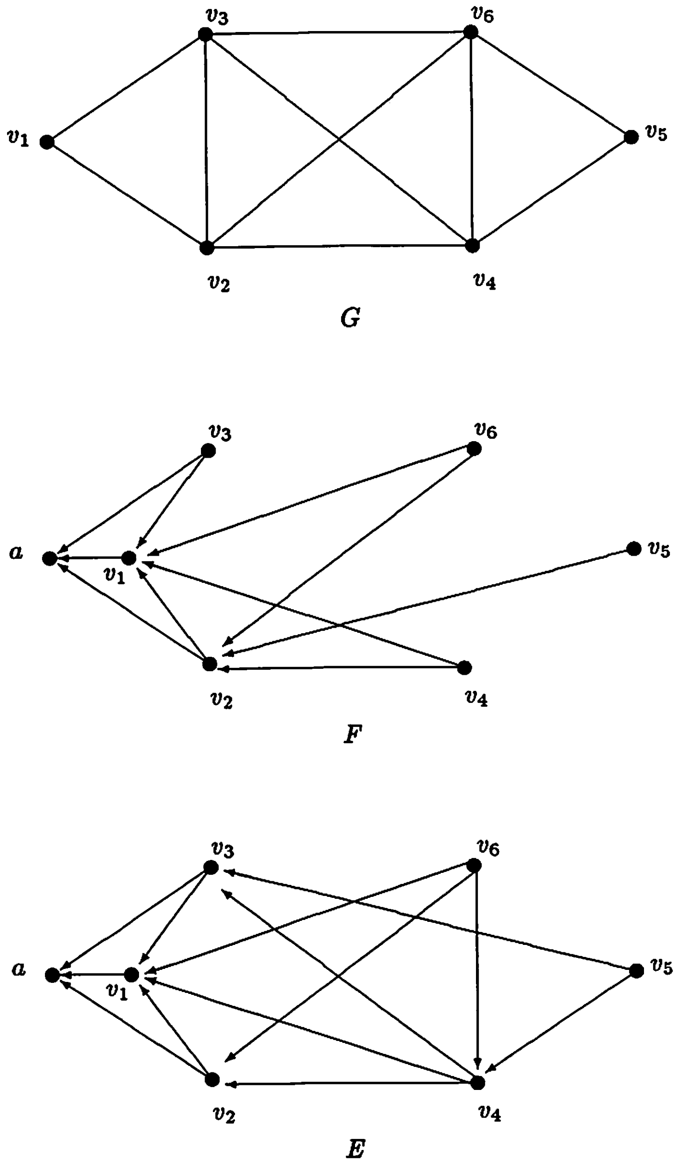

Figure 4: An illustration for the raising procedure. The competition graphs of F and E are both $G \cup I_1$ and $\Delta(\Delta(E)) = F$.