Closure, Path-Factors and Path Coverings in Claw-Free Graphs

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Abstract

In this paper, we study path-factors and path coverings of a claw-free graph and those of its closure. For a claw-free graph G and its closure $\mathrm{cl}(G)$, we prove (1) G has a path-factor with k components if and only if $\mathrm{cl}(G)$ has a path-factor with k components, and (2) $V(G)$ is covered by k paths in G if and only if $V(\mathrm{cl}(G))$ is covered by k paths in $\mathrm{cl}(G)$.

1. Introduction

In this paper, we deal with finite undirected graphs $G = (V(G), E(G))$ without loops and multiple edges. A graph is called **claw-free** if it contains no induced subgraph isomorphic to $K_{1,3}$. A graph is defined to be **hamiltonian** if it contains a hamiltonian cycle, while **traceable** if it contains a hamiltonian path. Let $H_1, \dots, H_k$ be subgraphs of a graph G. Then G is said to be **covered** by $H_1, \dots, H_k$ if $V(G) = V(H_1) \cup \cdots \cup V(H_k)$. A **path-factor** of a graph G is a set of vertex-disjoint paths covering G.

2. Ryjáček Closure

Recently, Ryjáček [3] has introduced a new closure for claw-free graphs. A vertex x of a graph G is said to be **locally connected** if the neighborhood

$N_G(x)$ of x in G induces a connected graph. For a locally connected vertex x of a graph G, we consider the operation of adding edges between every pair of nonadjacent vertices in $N_G(x)$ so that $N_G(x)$ induces a complete graph in the resulting graph. This operation is called **local completion** of G at x. Now we consider a series of local completions. For a graph G, let $G = G_0, G_1, \ldots, G_{r-1}, G_r = H$ be a sequence of graphs in which G_i is obtained from G_{i-1} by a local completion ($1 \le i \le r$). If we cannot obtain a new graph from $G_r = H$ by any local completion, i.e. if $N_H(x)$ induces a complete graph in H for every locally connected vertex x in H, we call H a **closure** of G and denote it by cl(G). Ryjáček has proved the following theorem.

Theorem A ([3]). *Let G be a claw-free graph. Then*
(1) a graph obtained from G by local completion is also claw-free, and
(2) cl(G) is uniquely determined.

3. Hamiltonicity and Traceability

It has been proved that the closure preserves hamiltonicity and traceability.

Theorem B ([3]). *Let G be a claw-free graph. Then G is hamiltonian if and only if cl(G) is hamiltonian.*

Theorem C ([1]). *Let G be a claw-free graph. Then G is traceable if and only if cl(G) is traceable.*

Ryjáček, Saito and Schelp proved the following two theorems as generalizations of Theorem B.

Theorem D ([4]). *Let G be a claw-free graph. If cl(G) has a 2-factor with k components, then G has a 2-factor with at most k components.*

Theorem E ([4]). *Let G be a claw-free graph. Then G is covered by k cycles if and only if cl(G) is covered by k cycles.*

4. Main Results

We prove the following two theorems as generalizations of Theorem C.

Theorem 1. *Let G be a claw-free graph. Then G has a path-factor with k components if and only if $cl(G)$ has a path-factor with k components.*

Theorem 2. *Let G be a claw-free graph. Then G is covered by k paths if and only if $cl(G)$ is covered by k paths.*

Before proving the above theorems, we introduce some terminology and notation which are used in the subsequent arguments. For a graph G and $T \subset V(G)$, the subgraph of G induced by T is denoted by $G[T]$. When we consider a path or a cycle, we always assign orientation. Let $P = x_0x_1 \cdots x_m$ be a path. We call x_0 and x_m the **starting vertex** and the **terminal vertex** of P, respectively. The set of internal vertices of P is denoted by $\mathrm{int}(P)$, namely $\mathrm{int}(P) = \{x_1, x_2, \ldots, x_{m-1}\}$. The **length** of P is the number of edges in P, and is denoted by $l(P)$. We define $x_i^{+(P)} = x_{i+1}$ and $x_i^{-(P)} = x_{i-1}$. Furthermore, we define $x_i^{++(P)} = x_{i+2}$. When it is obvious which path is considered in the context, we sometimes write x_i^+ and x_i^- instead of $x_i^{+(P)}$ and $x_i^{-(P)}$, respectively. For $x_i, x_j \in V(P)$ with $i \leq j$, we denote the subpath $x_ix_{i+1} \cdots x_j$ by $x_i \overrightarrow{P} x_j$. The same path traversed in the opposite direction is denoted by $x_j \overleftarrow{P} x_i$. We also use the same notation for a cycle. For graph theoretic terminology and notation not defined in this paper, we refer the reader to [2].

Before closing this section, we present the following lemma which is used in the proof of the main theorems.

Lemma 3. *Let G be a claw-free graph and let x be a locally connected vertex of G. Let $T_1, T_2 \subset V(G)$ with $T_1 \cap T_2 = \{x\}$. Suppose $G[T_1]$ is traceable and $G[T_2]$ is either hamiltonian or isomorphic to K_2 but $G[T_1 \cup T_2]$ is not traceable. Choose a path P_1 and a cycle or an edge C_2 with $V(P_1) \cup V(C_2) = T_1 \cup T_2$ and $V(P_1) \cap V(C_2) = \{x\}$, and a path P_0 in $G[N_G(x)]$ with starting vertex in $\{x^{+(P_1)}, x^{-(P_1)}\}$ and terminal vertex in $\{x^{+(C_2)}, x^{-(C_2)}\}$ (if C_2 is a cycle) or $V(C_2) - \{x\}$ (if C_2 is an edge) so that P_0 is as short as* *possible. Then* $2 \leq l(P_0) \leq 3$ *and* $int(P_0) \cap (V(P_1) \cup V(C_2)) = \emptyset$.

Proof. Note first that each hamiltonian path Q_1 in $G[T_1]$ and each hamiltonian cycle D_2 in $G[T_2]$ (if $G[T_2]$ is hamiltonian) or an edge D_2 in $G[T_2]$ (if $G[T_2]$ is isomorphic to K_2) satisfy $V(Q_1) \cup V(D_2) = T_1 \cup T_2$ and $V(Q_1) \cap V(D_2) = \{x\}$. Let s_1 and t_1 be the starting and terminal vertices of Q_1, respectively. If $x = s_1$, then $G[T_1 \cup T_2]$ is traceable. This contradicts the assumption. Hence we have $x \neq s_1$. By symmetry, we have $x \neq t_1$. These imply that both $x^{-(Q_1)}$ and $x^{+(Q_1)}$ exist on Q_1. Furthermore, since x is a locally connected vertex of G, there exists a path in $G[N_G(x)]$ with starting vertex in $\{x^{+(Q_1)}, x^{-(Q_1)}\}$ and terminal vertex in $\{x^{+(D_2)}, x^{-(D_2)}\}$ (if D_2 is a cycle) or $V(D_2) - \{x\}$ (if D_2 is an edge). Therefore, we can make a choice of (P_1, C_2, P_0).

Let s and t be the starting and terminal vertices of P_1, respectively. By the same reason as in the above, both $x^{-(P_1)}$ and $x^{+(P_1)}$ exist on P_1. Let $u_1 = x^{-(P_1)}$ and $v_1 = x^{+(P_1)}$. If C_2 is a cycle, let $u_2 = x^{+(C_2)}$ and $v_2 = x^{-(C_2)}$. If C_2 is an edge, let $C_2 = u_2x$. (An edge u_2x is also denoted by $u_2 \overline{C_2}\, x$ in the subsequent arguments.) In either case, we may assume that the starting and terminal vertices of P_0 are u_1 and u_2, respectively.

If $u_1u_2 \in E(G)$, then $P = s \overline{P_1}\, u_1u_2 \overline{C_2}\, x \overline{P_1}\, t$ is a path in G with $V(P) = V(P_1) \cup V(C_2) = T_1 \cup T_2$. This contradicts the assumption that $G[T_1 \cup T_2]$ is not traceable. Hence we have $u_1u_2 \notin E(G)$. Similarly, we have $u_1v_2, v_1u_2, v_1v_2 \notin E(G)$ if C_2 is a cycle, $v_1u_2 \notin E(G)$ if C_2 is an edge. Since x and $\{u_1, v_1, u_2\}$ do not form a claw in G, $u_1v_1 \in E(G)$. Similarly, $u_2v_2 \in E(G)$ if C_2 is a cycle. (See Figure 1.)

First, we prove $2 \leq l(P_0) \leq 3$. By the choice of (P_1, C_2, P_0), P_0 is an induced path. Hence, if $l(P_0) \geq 4$, $\{u_1, u_1^{++(P_0)}, u_2\}$ is an independent set. Since $V(P_0) \subset N_G(x)$ and G is claw-free, this is a contradiction. Thus, $l(P_0) \leq 3$. Since $u_1u_2 \notin E(G)$, $l(P_0) \geq 2$.

Next, we prove $\text{int}(P_0) \cap (V(P_1) \cup V(C_2)) = \emptyset$. Let $w = u_1^{+(P_0)}$ and $y = u_2^{-(P_0)}$. Note that, if $l(P_0) = 2$, $w = y$. We show the following two facts.

Claim 1. $w \notin V(P_1) \cup V(C_2)$

Proof. Assume, to the contrary, $w \in V(P_1) \cup V(C_2)$. Since $w \in V(P_0) \subset$

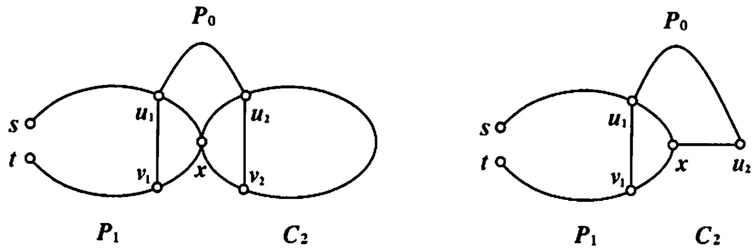

Figure 1:

$N_G(x)$ and $u_1 u_2 \notin E(G)$, $w \notin \{x, u_2\}$.

Case 1.1. $w \in s\overrightarrow{P_1}\, u_1^-$

First, suppose $w \neq s$. Thus, $w \in s^+ \overrightarrow{P_1}\, u_1^-$. Let $w^- = w^{-(P_1)}$ and $w^+ = w^{+(P_1)}$. Since w and $\{x, w^-, w^+\}$ do not form a claw in G, $\{w^- w^+, w^- x, x w^+\} \cap E(G) \neq \emptyset$. If $w^- w^+ \in E(G)$, let

$$P_1' = s\overrightarrow{P_1}\, w^- w^+ \overrightarrow{P_1}\, u_1 w x \overrightarrow{P_1}\, t, \quad C_2' = C_2 \text{ and } P_0' = w\overrightarrow{P_0}\, u_2.$$

If $w^- x \in E(G)$, let

$$P_1' = s\overrightarrow{P_1}\, w^- x w \overrightarrow{P_1}\, u_1 v_1 \overrightarrow{P_1}\, t, \quad C_2' = C_2 \text{ and } P_0' = w\overrightarrow{P_0}\, u_2.$$

If $x w^+ \in E(G)$, let

$$P_1' = s\overrightarrow{P_1}\, w x w^+ \overrightarrow{P_1}\, u_1 v_1 \overrightarrow{P_1}\, t, \quad C_2' = C_2 \text{ and } P_0' = w\overrightarrow{P_0}\, u_2.$$

Then in each case, we have $V(P_1') \cup V(C_2') = V(P_1) \cup V(C_2) = T_1 \cup T_2$ and $V(P_1') \cap V(C_2') = \{x\}$. Furthermore, $w \in \{x^{+(P_1')}, x^{-(P_1')}\}$ and $l(P_0') < l(P_0)$. This contradicts the choice of (P_1, C_2, P_0).

Next, suppose $w = s$. Let

$$P = u_2\overrightarrow{C_2}\, x w \overrightarrow{P_1}\, u_1 v_1 \overrightarrow{P_1}\, t.$$

Then P is a path in G with $V(P) = V(P_1) \cup V(C_2) = T_1 \cup T_2$. This contradicts the assumption.

Case 1.2. $w \in v_1\overrightarrow{P_1}\, t$

By the choice of P_0, $w \neq v_1$. Thus, $w \in v_1^+ \overrightarrow{P_1}\, t$. By similar arguments as in Case 1.1, we have a contradiction.

Furthermore, if C_2 is a cycle, we also consider the following case.

Case 1.3. $w \in u_2^+ \overrightarrow{C_2}\, v_2$

Since $u_1 v_2 \notin E(G)$, $w \neq v_2$. Thus, $w \in u_2^+ \overrightarrow{C_2}\, v_2^-$. Let $w' = w^{-(C_2)}$ and $w'' = w^{+(C_2)}$. Since w and $\{x, w', w''\}$ do not form a claw in G, $\{w'w'', w'x, w''x\} \cap E(G) \neq \emptyset$. If $w'w'' \in E(G)$, let

$$P_1' = s\overrightarrow{P_1}\, u_1 w x \overrightarrow{P_1}\, t,\quad C_2' = x\overrightarrow{C_2}\, w'w'' \overrightarrow{C_2}\, x \text{ and } P_0' = w\overrightarrow{P_0}\, u_2.$$

Then $V(P_1') \cup V(C_2') = V(P_1) \cup V(C_2) = T_1 \cup T_2$, $V(P_1') \cap V(C_2') = \{x\}$, $w = x^{-(P_1')}$ and $l(P_0') < l(P_0)$. This contradicts the choice of (P_1, C_2, P_0). If $w'x \in E(G)$, let

$$P = s\overrightarrow{P_1}\, u_1 w \overrightarrow{C_2}\, v_2 u_2 \overleftarrow{C_2}\, w' x \overrightarrow{P_1}\, t.$$

If $w''x \in E(G)$, let

$$P = s\overrightarrow{P_1}\, u_1 w \overleftarrow{C_2}\, u_2 v_2 \overrightarrow{C_2}\, w'' x \overrightarrow{P_1}\, t.$$

Then in either case, P is a path in G with $V(P) = V(P_1) \cup V(C_2) = T_1 \cup T_2$. This contradicts the assumption. □

Claim 2. $y \notin V(P_1) \cup V(C_2)$

Proof. Assume, to the contrary, $y \in V(P_1) \cup V(C_2)$. Since $y \in V(P_0) \subset N_G(x)$ and $u_1 u_2, v_1 u_2 \notin E(G)$, $y \notin \{x, u_1, v_1\}$.

Case 2.1. $y \in s\overrightarrow{P_1}\, u_1^-$

First, suppose $y \neq s$. Thus, $y \in s^+ \overrightarrow{P_1}\, u_1^-$. Let $y^- = y^{-(P_1)}$ and $y^+ = y^{+(P_1)}$. Since y and $\{x, y^-, y^+\}$ do not form a claw in G, $\{y^-y^+, y^-x, xy^+\} \cap E(G) \neq \emptyset$. If $y^-y^+ \in E(G)$, let

$$P_1' = s\overrightarrow{P_1}\, y^- y^+ \overrightarrow{P_1}\, x \overrightarrow{P_1}\, t,\quad C_2' = xyu_2 \overrightarrow{C_2}\, x \text{ and } P_0' = u_1 \overrightarrow{P_0}\, y.$$

Then $V(P_1') \cup V(C_2') = V(P_1) \cup V(C_2) = T_1 \cup T_2$, $V(P_1') \cap V(C_2') = \{x\}$, $y = x^{+(C_2')}$ and $l(P_0') < l(P_0)$. This contradicts the choice of (P_1, C_2, P_0).

If $y^-x \in E(G)$, let

$$P = s\overleftarrow{P_1}\, y^- x\, \overleftarrow{C_2}\, u_2 y\, \overrightarrow{P_1}\, u_1 v_1\, \overrightarrow{P_1}\, t.$$

If $xy^+ \in E(G)$, let

$$P = s\overleftarrow{P_1}\, y u_2\, \overleftarrow{C_2}\, x y^+\, \overrightarrow{P_1}\, u_1 v_1\, \overrightarrow{P_1}\, t.$$

Then in either case, P is a path in G with $V(P) = V(P_1) \cup V(C_2) = T_1 \cup T_2$. This contradicts the assumption.

Next, suppose $y = s$. Let

$$P = u_2\, \overleftarrow{C_2}\, x y\, \overrightarrow{P_1}\, u_1 v_1\, \overrightarrow{P_1}\, t.$$

Then P is a path in G with $V(P) = V(P_1) \cup V(C_2) = T_1 \cup T_2$. This contradicts the assumption.

Case 2.2. $y \in v_1^+ \overrightarrow{P_1} t$

By similar arguments as in Case 2.1, we have a contradiction.

Furthermore, if C_2 is a cycle, we also consider the following case.

Case 2.3. $y \in u_2^+ \overrightarrow{P_1} v_2$

By the choice of P_0, $y \neq v_2$. Thus, $y \in u_2^+ \overrightarrow{C_2} v_2^-$. Let $y' = y^{-(C_2)}$ and $y'' = y^{+(C_2)}$. Since y and $\{x, y', y''\}$ do not form a claw in G, $\{y'y'', y'x, y''x\} \cap E(G) \neq \emptyset$. If $y'y'' \in E(G)$, let

$$P_1' = P_1, \quad C_2' = x y u_2\, \overleftarrow{C_2}\, y' y''\, \overrightarrow{C_2}\, x \text{ and } P_0' = u_1\, \overrightarrow{P_0}\, y.$$

If $y'x \in E(G)$, let

$$P_1' = P_1, \quad C_2' = x y\, \overrightarrow{C_2}\, v_2 u_2\, \overleftarrow{C_2}\, y' x \text{ and } P_0' = u_1\, \overrightarrow{P_0}\, y.$$

If $y''x \in E(G)$, let

$$P_1' = P_1, \quad C_2' = x y\, \overleftarrow{C_2}\, u_2 v_2\, \overleftarrow{C_2}\, y'' x \text{ and } P_0' = u_1\, \overrightarrow{P_0}\, y.$$

Then in each case, $V(P_1') \cup V(C_2') = V(P_1) \cup V(C_2) = T_1 \cup T_2$, $V(P_1') \cap V(C_2') = \{x\}$, $y = x^{+(C_2')}$ and $l(P_0') < l(P_0)$. This contradicts the choice of (P_1, C_2, P_0). □

Therefore, we have completed the proof of Lemma 3. ■

5. Proof of the Main Theorems

In order to prove Theorem 1 and Theorem 2, we prove a stronger statement. Let G' be a graph obtained from a claw-free graph by local completion at a vertex. Using Lemma 3, we prove that for each set of k disjoint paths in G' there exists a set of k disjoint paths in G which contains it. We can also impose some restrictions on the sum of the lengths of these k paths of G. Since the "only if" part of Theorem 1 is trivial, Theorem 1 is a consequence of the following result.

Theorem 4. *Let G be a claw-free graph and let x be a locally connected vertex of G. Let G' be the graph obtained from G by local completion at x. Then for each set of k disjoint paths $\{P'_1, \dots, P'_k\}$ in G' there exists a set of k disjoint paths $\{Q_1, \dots, Q_k\}$ in G with $\bigcup_{i=1}^k V(P'_i) \subset \bigcup_{i=1}^k V(Q_i)$ and $\sum_{i=1}^k l(P'_i) \le \sum_{i=1}^k l(Q_i) \le (\sum_{i=1}^k l(P'_i)) + 3$.*

Proof. Let $B = E(G') - E(G)$. If $(\bigcup_{i=1}^k E(P'_i)) \cap B = \emptyset$, then $\{P'_1, \dots, P'_k\}$ is a required set of k disjoint paths. Hence we may assume $(\bigcup_{i=1}^k E(P'_i)) \cap B \neq \emptyset$.

Let $S'_0 = (\bigcup_{i=1}^k V(P'_i)) \cup \{x\}$. If $x \in \bigcup_{i=1}^k V(P'_i)$, then $\{P'_1, \dots, P'_k\}$ is a path-factor of $G'[S'_0]$ with k components.

Suppose $x \notin \bigcup_{i=1}^k V(P'_i)$. Let $e = uv \in B \cap E(P'_j)$ $(1 \le j \le k)$. We may assume $j = 1$ and $v = u^{+(P'_1)}$. Let s and t be the starting and terminal vertices of P'_1, respectively. Note that $\{u, v\} \subset N_G(x)$. Let $P''_1 = s \overrightarrow{P'_1} u x v \overrightarrow{P'_1} t$. Then $\{P''_1, P'_2, \dots, P'_k\}$ is a path-factor of $G'[S'_0]$ with k components.

Now choose a path-factor $\{Q'_1, \dots, Q'_k\}$ of $G'[S'_0]$ with k components so that

(a) $|(\bigcup_{i=1}^k E(Q'_i)) \cap B|$ is as small as possible.

In either case, we have a set of k disjoint paths $\{Q'_1, \dots, Q'_k\}$ in G' with

$$(\bigcup_{i=1}^{k} V(P'_i)) \cup \{x\} \subset \bigcup_{i=1}^{k} V(Q'_i) \text{ and}$$

$$\sum_{i=1}^{k} l(P_i') \leq \sum_{i=1}^{k} l(Q_i') \leq (\sum_{i=1}^{k} l(P_i')) + 1.$$

Let $F = \bigcup_{i=1}^{k} E(Q_i')$. If $F \cap B = o$, then $\{Q_1', \ldots, Q_k'\}$ is a required set of k disjoint paths. Hence we may assume $F \cap B \neq o$. We denote the starting and terminal vertices of Q_i' by s_i' and t_i' $(1 \leq i \leq k)$, respectively.

Suppose $x \in V(Q_1')$. Then we have the following fact.

Claim 1. $(\bigcup_{i=2}^{k} E(Q_i')) \cap B = o$

Proof. Assume, to the contrary, $E(Q_i') \cap B \neq o$, say $f = u'v' \in E(Q_i') \cap B$, $(2 \leq i \leq k)$. We may assume $i = 2$ and $v' = u'^{+(Q_2')}$.

If $x = s_1'$, let

$$Q_1'' = t_2' \overline{Q_2'} v' x \overline{Q_1'} t_1', \quad Q_2'' = s_2' \overline{Q_2'} u'$$

and $F' = F - (E(Q_1') \cup E(Q_2')) \cup (E(Q_1'') \cup E(Q_2''))$. Then F' is the edge set of a path-factor of $G'[S_0']$ with k components and $|F' \cap B| = |F \cap B| - 1$. This contradicts the minimality of $|F \cap B|$. Therefore, we have $x \neq s_1'$. By symmetry, we have $x \neq t_1'$.

Let $x^+ = x^{+(Q_1')}$. Since x and $\{x^+, u', v'\}$ do not form a claw in G, $\{x^+u', x^+v'\} \cap E(G) \neq o$. If $x^+u' \in E(G)$, let

$$Q_1'' = s_1' \overline{Q_1'} x v' \overline{Q_2'} t_2', \quad Q_2'' = s_2' \overline{Q_2'} u' x^+ \overline{Q_1'} t_1'$$

and $F' = F - (E(Q_1') \cup E(Q_2')) \cup (E(Q_1'') \cup E(Q_2''))$. Then F' is the edge set of a path-factor of $G'[S_0']$ with k components and $|F' \cap B| = |F \cap B| - 1$. This contradicts the minimality of $|F \cap B|$. By symmetry, we have a contradiction if $x^+v' \in E(G)$. □

Since $F \cap B \neq o$, $E(Q_1') \cap B \neq o$. Furthermore, we have the following fact.

Claim 2. $|E(Q_1') \cap B| = 1$

Proof. Assume, to the contrary, $|E(Q_1') \cap B| \geq 2$, say $e_1, e_2 \in E(Q_1') \cap B$, $e_1 \neq e_2$. Let $e_i = x_i y_i$ $(i = 1, 2)$. Then, by symmetry, we have only to consider the following two cases.

Case 1. s_1', x, x_1, y_1, x_2, y_2 and t_1' appear in this order along Q_1'.

Then x^+, y_1 and y_2 are distinct vertices in $N_G(x)$. Since x and $\{x^+, y_1, y_2\}$ do not form a claw in G, $\{y_1y_2, x^+y_1, x^+y_2\} \cap E(G) \neq \emptyset$. If $y_1y_2 \in E(G)$, let

$$Q_1'' = s_1' \overline{Q_1'} x_1 x_2 \overline{Q_1'} y_1 y_2 \overline{Q_1'} t_1' \text{ (note } x_1x_2 \in E(G')\text{)}.$$

Then $V(Q_1'') = V(Q_1')$ and $E(Q_1'') = E(Q_1') - \{x_1y_1, x_2y_2\} \cup \{x_1x_2, y_1y_2\}$. This implies $|E(Q_1'') \cap B| < |E(Q_1') \cap B|$, which contradicts the minimality of $|F \cap B|$. If $x^+y_1 \in E(G)$, let

$$Q_1'' = s_1' \overline{Q_1'} x x_1 \overline{Q_1'} x^+ y_1 \overline{Q_1'} t_1'.$$

Then $V(Q_1'') = V(Q_1')$ and $E(Q_1'') = E(Q_1') - \{xx^+, x_1y_1\} \cup \{xx_1, x^+y_1\}$. Since $xx_1 \in E(G)$, we have $|E(Q_1'') \cap B| < |E(Q_1') \cap B|$, again a contradiction. We have a similar contradiction if $x^+y_2 \in E(G)$.

Case 2. $s_1', x_1, y_1, x, x_2, y_2$ and t_1' appear in this order along Q_1'.

Similarly as in Case 1, x^+, y_1 and y_2 are distinct vertices in $N_G(x)$, and $\{y_1y_2, x^+y_1, x^+y_2\} \cap E(G) \neq \emptyset$. In each case, by a similar argument, we have a contradiction. □

Let $E(Q_1') \cap B = \{x_1y_1\}$. We may assume s_1', x, x_1, y_1 and t_1' appear in this order along Q_1'. Let

$$T_1 = s_1' \overline{Q_1'} x \cup y_1 \overline{Q_1'} t_1' \text{ and } T_2 = x \overrightarrow{Q_1'} x_1.$$

Then $T_1 \cup T_2 = V(Q_1')$ and $T_1 \cap T_2 = \{x\}$. Since $x_1y_1 \in B$, $xx_1, xy_1 \in E(G)$. Hence $s_1' \overline{Q_1'} xy_1 \overline{Q_1'} t_1'$ is a hamiltonian path in $G[T_1]$. If $x^+ \neq x_1$, then $x \overline{Q_1'} x_1 x$ is a hamiltonian cycle in $G[T_2]$. If $x^+ = x_1$, then $G[T_2]$ is isomorphic to K_2. Note that, by the minimality of $|F \cap B|$, $G[T_1 \cup T_2]$ is not traceable.

We consider a path P_1 and a cycle or an edge C_2 in $G[T_1 \cup T_2]$ with $V(P_1) \cup V(C_2) = T_1 \cup T_2 = V(Q_1')$ and $V(P_1) \cap V(C_2) = \{x\}$, and a path P_0 in $G[N_G(x)]$ with starting vertex in $\{x^{+(P_1)}, x^{-(P_1)}\}$ and terminal vertex in $\{x^{+(C_2)}, x^{-(C_2)}\}$ (if C_2 is a cycle) or $V(C_2) - \{x\}$ (if C_2 is an edge). Note that, since x is a locally connected vertex of G, $G[N_G(x)]$ has such a path P_0.

Now choose $\{Q_1', \dots, Q_k'\}$, P_1, C_2, and P_0 so that
(b) P_0 is as short as possible, subject to (a).
Then, by Lemma 3, $2 \le l(P_0) \le 3$ and $\mathrm{int}(P_0) \cap (V(P_1) \cup V(C_2)) = \emptyset$.

Let s_1 and t_1 be the starting and terminal vertices of P_1, respectively. Since $G[T_1 \cup T_2]$ is not traceable, we have $x \notin \{s_1, t_1\}$, which implies that both $x^{-(P_1)}$ and $x^{+(P_1)}$ exist on P_1. Let $u_1 = x^{-(P_1)}$ and $v_1 = x^{+(P_1)}$. If C_2 is a cycle, let $u_2 = x^{+(C_2)}$ and $v_2 = x^{-(C_2)}$. If C_2 is an edge, let $C_2 = u_2 x$. (An edge $u_2 x$ is also denoted by $u_2 \overline{C_2} x$ in the subsequent arguments.) In either case, we may assume that the starting and terminal vertices of P_0 are u_1 and u_2, respectively.

Since $G[T_1 \cup T_2]$ is not traceable, we have $u_1 u_2, u_1 v_2, v_1 u_2, v_1 v_2 \notin E(G)$ if C_2 is a cycle, $u_1 u_2, v_1 u_2 \notin E(G)$ if C_2 is an edge. Since x and $\{u_1, v_1, u_2\}$ do not form a claw in G, $u_1 v_1 \in E(G)$. Similarly, $u_2 v_2 \in E(G)$ if C_2 is a cycle. (See Figure 2.)

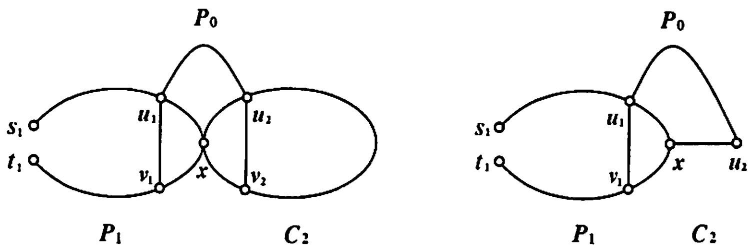

Figure 2:

Let $w = u_1^{+(P_0)}$ and $y = u_2^{-(P_0)}$. Then we have the following two facts.

Claim 3. $w \notin \bigcup_{i=2}^{k} V(Q_i')$

Proof. Assume, to the contrary, $w \in \bigcup_{i=2}^{k} V(Q_i')$. Suppose $w \in V(Q_2')$.

If $w = s_2'$, let

$$Q_1'' = u_2 \overline{C_2} x \overline{P_1} t_1, \quad Q_2'' = s_1 \overline{P_1} u_1 w \overline{Q_2'} t_2'$$

and $F' = F - (E(Q_1') \cup E(Q_2')) \cup (E(Q_1'') \cup E(Q_2''))$. Then F' is the edge set of a path-factor of $G'[S_0']$ with k components and $|F' \cap B| = |F \cap B| - 1 = 0$.

This contradicts the minimality of $|F \cap B|$. Therefore, we have $w \neq s'_2$. By symmetry, we have $w \neq t'_2$.

Let $w^+ = w^{+(Q'_2)}$ and $w^- = w^{-(Q'_2)}$. Assume $w^+x \in E(G)$. Since $u_1u_2 \notin E(G)$ and x and $\{w^+, u_1, u_2\}$ do not form a claw in G, $\{w^+u_1, w^+u_2\} \cap E(G) \neq \emptyset$. If $w^+u_1 \in E(G)$, let

$$Q''_1 = s'_2 \overrightarrow{Q'_2}\, w u_2 \overleftarrow{C_2}\, x \overrightarrow{P_1}\, t_1,\quad Q''_2 = s_1 \overrightarrow{P_1}\, u_1 w^+ \overrightarrow{Q'_2}\, t'_2$$

and $F' = F - (E(Q'_1) \cup E(Q'_2)) \cup (E(Q''_1) \cup E(Q''_2))$. Then F' is the edge set of a path-factor of $G'[S'_0]$ with k components and $F' \cap B \subset \{wu_2\}$. By the minimality of $|F \cap B|$, $F' \cap B = \{wu_2\}$. Furthermore, $P'_1 = s'_2 \overrightarrow{Q'_2}\, wx \overrightarrow{P_1}\, t_1$ is a path and $C'_2 = C_2$ is a cycle or an edge in G with $V(P'_1) \cup V(C'_2) = V(Q''_1)$ and $V(P'_1) \cap V(C'_2) = \{x\}$. Since $w \overrightarrow{P_0}\, u_2$ is shorter than P_0, this contradicts the choice of $\{Q'_1, \ldots, Q'_k\}$, P_1, C_2 and P_0 given in (b). If $w^+u_2 \in E(G)$, let

$$Q''_1 = t'_2 \overleftarrow{Q'_2}\, w^+ u_2 \overleftarrow{C_2}\, x \overrightarrow{P_1}\, t_1,\quad Q''_2 = s_1 \overrightarrow{P_1}\, u_1 w \overleftarrow{Q'_2}\, s'_2$$

and $F' = F - (E(Q'_1) \cup E(Q'_2)) \cup (E(Q''_1) \cup E(Q''_2))$. Then F' is the edge set of a path-factor of $G'[S'_0]$ with k components and $|F' \cap B| = |F \cap B| - 1 = 0$. This contradicts the minimality of $|F \cap B|$. Therefore, we have $w^+x \notin E(G)$. By symmetry, we have $w^-x \notin E(G)$.

Since w and $\{w^+, w^-, x\}$ do not form a claw in G, $w^+w^- \in E(G)$. Let

$$Q''_1 = s_1 \overrightarrow{P_1}\, u_1 w u_2 \overleftarrow{C_2}\, x \overrightarrow{P_1}\, t_1,\quad Q''_2 = s'_2 \overrightarrow{Q'_2}\, w^- w^+ \overrightarrow{Q'_2}\, t'_2$$

and $F' = F - (E(Q'_1) \cup E(Q'_2)) \cup (E(Q''_1) \cup E(Q''_2))$. Then F' is the edge set of a path-factor of $G'[S'_0]$ with k components and $F' \cap B \subset \{wu_2\}$. By the minimality of $|F \cap B|$, $F' \cap B = \{wu_2\}$. Furthermore, $P'_1 = s_1 \overrightarrow{P_1}\, u_1 wx \overrightarrow{P_1}\, t_1$ is a path and $C'_2 = C_2$ is a cycle or an edge in G with $V(P'_1) \cup V(C'_2) = V(Q''_1)$ and $V(P'_1) \cap V(C'_2) = \{x\}$. Since $w \overrightarrow{P_0}\, u_2$ is shorter than P_0, this contradicts the choice of $\{Q'_1, \ldots, Q'_k\}$, P_1, C_2 and P_0 given in (b). □

Claim 4. $y \notin \bigcup_{i=2}^{k} V(Q'_i)$

Proof. Assume, to the contrary, $y \in \bigcup_{i=2}^{k} V(Q'_i)$. Suppose $y \in V(Q'_2)$. By a similar argument as in the proof of Claim 3, we have $y \notin \{s'_2, t'_2\}$.

Let $y^+ = y^{+(Q'_2)}$ and $y^- = y^{-(Q'_2)}$. Assume $y^+x \in E(G)$. Let

$$Q''_1 = s'_2 \overrightarrow{Q'_2}\, y u_2 \overleftarrow{C_2}\, x y^+ \overrightarrow{Q_2}\, t'_2,\quad Q''_2 = s_1 \overrightarrow{P_1}\, u_1 v_1 \overrightarrow{P_1}\, t_1$$

and $F' = F - (E(Q'_1) \cup E(Q'_2)) \cup (E(Q''_1) \cup E(Q''_2))$. Then F' is the edge set of a path-factor of $G'[S'_0]$ with k components and $|F' \cap B| = |F \cap B| - 1 = 0$. This contradicts the minimality of $|F \cap B|$. Therefore, we have $y^+x \notin E(G)$. By symmetry, we have $y^-x \notin E(G)$.

Since y and $\{y^+, y^-, x\}$ do not form a claw in G, $y^+y^- \in E(G)$. Let

$$Q''_1 = s_1 \overline{P_1} u_1 y u_2 \overline{C_2} x \overline{P_1} t_1, \quad Q''_2 = s'_2 \overline{Q'_2} y^- y^+ \overline{Q'_2} t'_2$$

and $F' = F - (E(Q'_1) \cup E(Q'_2)) \cup (E(Q''_1) \cup E(Q''_2))$. Then F' is the edge set of a path-factor of $G'[S'_0]$ with k components and $F' \cap B \subset \{u_1 y\}$. By the minimality of $|F \cap B|$, $F' \cap B = \{u_1 y\}$. Furthermore, $P'_1 = P_1$ is a path and $C'_2 = x y u_2 \overline{C_2} x$ is a cycle in G with $V(P'_1) \cup V(C'_2) = V(Q''_1)$ and $V(P'_1) \cap V(C'_2) = \{x\}$. Since $u_1 \overline{P_0} y$ is shorter than P_0, this contradicts the choice of $\{Q'_1, \ldots, Q'_k\}$, P_1, C_2 and P_0 given in (b). □

Now we complete the proof of Theorem 4. By Lemma 3, Claim 3 and Claim 4, we have $\mathrm{int}(P_0) \cap \bigcup_{i=1}^{k} V(Q'_i) = \emptyset$. Let

$$Q_1 = s_1 \overline{P_1} u_1 \overline{P_0} u_2 \overline{C_2} x \overline{P_1} t_1 \quad \text{and} \quad Q_i = Q'_i \ (2 \le i \le k).$$

Then $\{Q_1, \ldots, Q_k\}$ is a set of k disjoint paths in G with

$$\bigcup_{i=1}^{k} V(Q'_i) \subset \bigcup_{i=1}^{k} V(Q_i) \quad \text{and}$$

$$\sum_{i=1}^{k} l(Q'_i) \le \sum_{i=1}^{k} l(Q_i) \le \left(\sum_{i=1}^{k} l(Q'_i)\right) + 2.$$

Therefore,

$$\bigcup_{i=1}^{k} V(P'_i) \subset \bigcup_{i=1}^{k} V(Q'_i) \subset \bigcup_{i=1}^{k} V(Q_i) \quad \text{and}$$

$$\sum_{i=1}^{k} l(P'_i) \le \sum_{i=1}^{k} l(Q'_i) \le \sum_{i=1}^{k} l(Q_i) \le \left(\sum_{i=1}^{k} l(Q'_i)\right) + 2 \le \left(\sum_{i=1}^{k} l(P'_i)\right) + 3. \blacksquare$$

Theorem 2 is a consequence of the following corollary of Theorem 4.

Corollary 5. *Let G be a claw-free graph and let x be a locally connected vertex of G. Let G' be the graph obtained from G by local completion at x. Then G is covered by k paths if and only if G' is covered by k paths.*

Proof. Since the "only if" part is trivial, we have only to prove the "if" part of the corollary.

Suppose G' is covered by k paths, say $V(G') = V(P_1') \cup \cdots \cup V(P_k')$ for paths $P_1', \ldots, P_k'$ in G'. By Theorem 4, for each P_i' there exists a path Q_i in G with $V(P_i') \subset V(Q_i)$ $(1 \leq i \leq k)$. Then $V(G) = V(Q_1) \cup \cdots \cup V(Q_k)$. ∎

6. Concluding Remarks

Let T be a set of vertices in a claw-free graph G. Then, by Theorem 4, the minimum number of disjoint paths covering T in G is the same as the minimum number of disjoint paths covering T in $\mathrm{cl}(G)$. Furthermore, by Corollary 5, the minimum number of paths covering T in G is the same as the minimum number of paths covering T in $\mathrm{cl}(G)$.

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