Deleting Lines in Projective Planes

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ABSTRACT. Some properties of finite projective planes are used to obtain some new pairwise balanced designs with consecutive block sizes, by deleting configurations spanned by lines.

1 Introduction

A *finite projective plane of order n*, $n \geq 2$, is a collection of $n + 1$ subsets (called *lines*) of a $n^2 + n + 1$-set V of points, such that every two points occur together in exactly one of the lines.

It is known that a finite projective plane of order q exists for all $q = p^a$ where p is a prime and a is any positive integer.

If A and B are two points in a finite projective plane, we use the notation AB to represent the line defined by points A and B.

A *pairwise balanced design* (or PBD) is a pair $(V,\mathcal{B})$ where

1. V is a finite set of points,
2. $\mathcal{B}$ is a collection of subsets of V called blocks,
3. every pair of distinct points of V occurs in exactly one block.

We use the notation $PBD(v, K)$ when $|V| = v$ and $|B| \in K$ for all $B \in \mathcal{B}$. We denote $B(K)=\{v :$ there exists a PBD$(v,K)\}$.

The notation PBD$(v, K \cup k^*)$ is used for a PBD containing at least one block size k. If $k \notin K$, this indicates that there is exactly one block of size k in the PBD. On the other hand, if $k \in K$, then there is at least one block of size k in the PBD.

See [1] and [3] for background on PBDs and designs.

The objective of this paper is to show that certain line configurations can be removed from the projective plane to obtain some interesting PBDs. For example, we establish

$50, 51, 52, 53, 54 \in B(\{5, 6, 7\})$,

$72 \in B(\{6, 7, 8\})$,

$68, 69 \in B(\{5, 6, 7, 8\})$,

$82, 83, 84, 85, 86, 87, 88, 89 \in B(\{7, 8, 9\})$, and

$93, 94, 95, 110, 114 \in B(\{8, 9, 10\})$.

Numerous applications of PBDs with three but not four consecutive block sizes are given in [5, 6]. In determining existence of PBDs on v points with block sizes $\{k, k+1, k+2\}$, often the most difficult cases seem to arise when v is greater than $(k+2)^2$, but not much greater. For example, when $k = 7$, deletions of points in arcs of a projective plane of order 8, and of an affine plane of order 9, establish that if $63 \leq x \leq 81$, then $x \in B(\{7, 8, 9\})$. However, the range following this is not amenable to quite as simple a method (indeed, the next known member of $B(\{7, 8, 9\})$ was 91, from the (91,7,1) design). It is in this range that we find deletions of various configurations from finite projective planes to be most useful. While we have not been able to settle all open cases in $B(\{k, k+1, k+2\})$ for $k = 5, 6, 7, 8$ using the techniques described here, the extension of the initial sequence of values for which such PBDs are available both simplifies the determination of closure for these sets, and provide simple direct constructions for PBDs. For more complete results on closures of sets with three consecutive block sizes, see [6] and [5]. Naturally, the idea of employing configurations in finite planes to produce PBDs is far from new; see [2] and [3] for related results. The results here are general; while we illustrate them primarily with their consequences for $B(\{k, k+1, k+2\})$ when k is small, the goal is really to develop general observations about simple configurations in planes.

One particular importance of the line deletion techniques explored here is in the construction of incomplete transversal designs. Letting $N^*(k)$ be the number of idempotent mutually orthogonal latin squares of side k, it follows from $v \in B(\{k, k+1, k+2, a^*\})$ that a $TD(\ell, v) - TD(\ell, a)$ exists with $\ell = \min(N^*(k), N^*(k+1), N^*(k+2)) + 2$. Taking $k = 7$, we obtain $TD(7, v) - TD(7, a)$ whenever $v \in B(\{7, 8, 9, a^*\})$.

In providing motivation, we have concentrated on applications to the construction of various designs. It is perhaps important to remark that deleting any set of points at all in a projective plane yields a PBD of some kind. The only surprise, then, is that fairly simple considerations can be used to limit the block sizes to a small set. This goal of restricting the block sizes leads in some cases to interesting new geometric questions; we shall see that our goal of few block sizes leads to a notion of a scattering dual k-arc.

2 The Mia Configuration

A *Mia configuration* is a set of five lines l_1, l_2, l_3, l_4 and l_5 so that $l_2 \cap l_3$ and $l_4 \cap l_5$ are two distinct points on l_1. Figure 1 shows the Mia configuration.

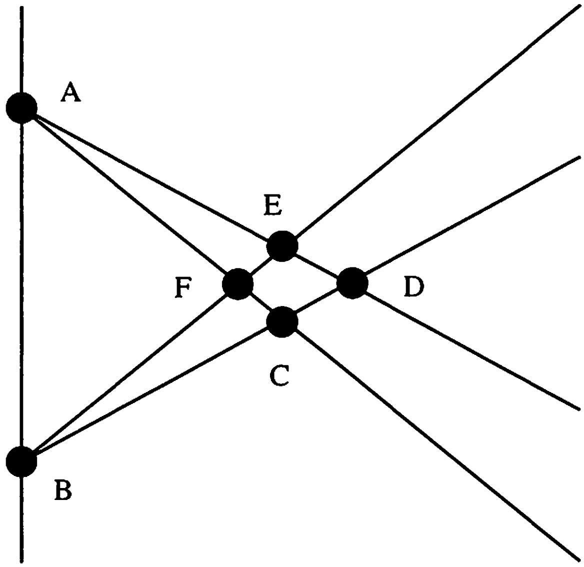

Figure 1. The Mia Configuration

Lemma 2.1. *The Mia configuration exists in any finite projective plane.*

Proof: Take a line l_1 in the plane, and identify two distinct points A, B on the line. For each of the two points, identity two distinct lines intersecting l_1 at that point. The intersections of the four lines define four more points in the finite projective plane. The five chosen lines form the Mia configuration. □

Now, we examine how each line intersects the Mia configuration.

Lemma 2.2. *Every line intersects the Mia configuration in either $q+1$,3,4 or 5 points where q is the order of the projective plane.*

Proof: The proof uses the labels in Figure 1. Trivially, any one of the five lines intersects the Mia configuration at $q+1$ points. If a line intersects the Mia configuration at point A, then it intersects line BE and line CD. Hence, the line intersects the Mia configuration at 3 points. The situation is similar if the line intersects point B. If the line intersects point C, then there are two possible cases. Either it intersects point E so the line intersects the Mia configuration at 3 points or it does not intersect E so the line intersects the configuration at 4 points. The situation is exactly the same by symmetry for points D, E and F. If a line does not hit any of the five lines except in the configuration, then it intersects the Mia configuration at five points; hence the result follows. □

Lemma 2.3. *If $q \geq 4$ is a prime power, then $q^2 - 4q + 4 \in B(\{q-4, q-3, q-2\})$.*

Proof: The Mia configuration has $5q - 3$ points. The result follows by removing the Mia configuration from a projective plane of order q and Lemma 2.2. □

We can also add back some points from the Mia configuration to obtain some interesting PBDs.

Lemma 2.4. *If $0 \leq a \leq q-3$ and $q \geq 4$ is a prime power, then $q^2 - 4q + 4 + a \in B(\{q-4, q-3, q-2, a^*\})$.*

Proof: From the proof of Lemma 2.2, we can add any a points on the line AB as long as we do not include the point of intersection of lines AB and CE or the point of intersection of lines AB and DF. □

As a consequence, we have the following corollary.

Corollary 2.5. $50, 54 \in B(\{5, 6, 7\})$.

Proof: Apply Lemma 2.4 with $q = 9$ and $a = 1, 5$. □

Corollary 2.6. $82, 88, 89 \in B(\{7, 8, 9\})$.

Proof: Apply Lemma 2.4 with $q = 11$ and $a = 1, 7, 8$. □

3 The Dual k-Arc

A *dual k-arc* is a set of k lines in a finite projective plane with the property that no three points of intersection of any two lines are concurrent. We begin with the existence of the dual k-arc in the finite projective plane. The *dual plane* of a projective plane π is the projective plane obtained by interchanging the role of lines and points in π.

Lemma 3.1. *For q a prime power, and any $1 \leq k \leq q+1$, there exists a projective plane of order q containing a dual k-arc.*

Proof: Every desarguesian projective plane contains k points such that no three of them are collinear. The result follows by taking the lines corresponding to the k points in the dual plane. □

Figure 2 shows a dual 6-arc.

We call P a *corner* point if P is on two of the k lines and Q, a *ray* point if Q is on exactly one of the k lines. Let $\mathcal{A}$ be any dual k-arc, and ℓ be any line of the plane not in $\mathcal{A}$. If a points on ℓ are ray points and b points on ℓ are corner points, we must have $a + 2b = k$. Using this observation, we have the following theorem.

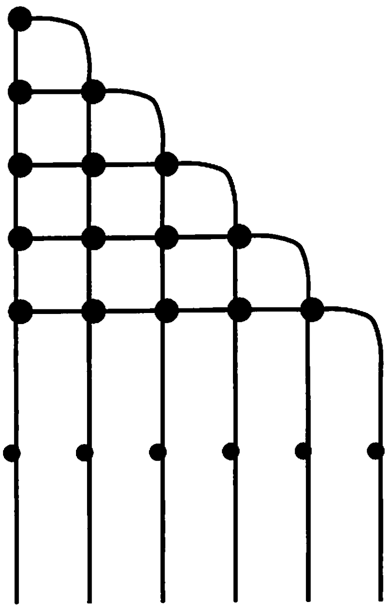

Figure 2. The Dual Arc Configuration

Theorem 3.2. *If q is a prime power and $1 \le k \le q+1$, then $q^2+q+1-k(q+1)+\frac{k(k-1)}{2} \in B(\{q+1-k, q+1-(k-1), \dots, q+1-\lceil\frac{k}{2}\rceil\})$.*

Proof: Take a desarguesian projective plane of order q. By Lemma 3.1, there exists a dual k-arc. There are $k(q+1)-\frac{k(k-1)}{2}$ points in the dual k-arc. If a points on l are ray points and b points on l are corner points, since $a+2b=k$, one has $a+b \in \{\lceil\frac{k}{2}\rceil, \lceil\frac{k}{2}\rceil+1, \dots, k\}$. The result follows by removing the points in the dual k-arc. □

As in Lemma 2.4, it is possible to identify some points in the dual k-arc whose retention does not increase the block sizes.

Theorem 3.3. *Let $k \ge 4$. If $q+1-k > \frac{(k-1)(k-2)(k-3)(k-4)}{8}$ and q is a prime power, then $q^2+q+1-k(q+1)+\frac{k(k-1)}{2}+1 \in B(\{q+1-k, q+1-(k-1), \dots, q+1-\lceil\frac{k}{2}\rceil\})$; in addition, if $q+1-k > \frac{(k-1)(k-2)(k-3)(k-4)}{8}+\frac{(k-2)(k-3)}{2}$ then $q^2+q+1-k(q+1)+\frac{k(k-1)}{2}+2 \in B(\{\{q+1-k, q+1-(k-1), \dots, q+1-\lceil\frac{k}{2}\rceil\})$.*

Proof: Choose a line ℓ_1 of the dual arc. There are $\frac{(k-1)(k-2)}{2}$ corner points not on ℓ_1, and $\frac{(k-1)(k-2)(k-3)(k-4)}{8}$ pairs of corner points defined by disjoint pairs of lines of the dual arc other than ℓ_1. Each such pair defines a line; the line so defined meets ℓ_1, and we call the intersection point *bad*. Under the stated requirement on q and k, one of the ray points, say p_1, is not bad. Adding p_1 therefore does not increase the size of any line whose size was already at least $q+1-k+2$.

Having chosen to add p_1, we next choose a line $\ell_2 \neq \ell_1$ from the dual arc. As before, pairs of corners make up to $\frac{(k-1)(k-2)(k-3)(k-4)}{8}$ of the ray points on ℓ_2 bad. In addition, in this case, a point is bad if it lies on a line defined by p_1 and one of the corners of the dual arc. Having fixed p_1 and ℓ_2, there are $\frac{(k-2)(k-3)}{2}$ ways to choose the corner, and hence at most this number of points are, in addition, classified as bad. Hence, under the stated condition, there remains a ray point p_2 that is not bad. Adding p_1 and p_2 does not increase the size of any line to more than $q+1-k+2$. □

Corollary 3.4. $51, 52 \in B(\{5,6,7\})$.

Proof: Apply Theorems 3.2 and 3.2 with $q = 9$ and $k = 5$. □

Corollary 3.5. $83, 84, 85 \in B(\{7,8,9\})$.

Proof: Apply Theorems 3.2 and 3.3 with $q = 11$ and $k = 5$. □

Corollary 3.6. $83 + a, 84 + a \in B(\{7, 8, 9, a^*\})$ *for* $0 \leq a \leq 6$.

Proof: Retain a points on one of the rays. □

Corollary 3.7. $93, 94, 95 \in B(\{8, 9, 10\})$.

Proof: Apply Theorems 3.2 and 3.3 with $q = 11$ and $k = 4$. □

If we only remove the ray points instead of all points in the dual k-arc, then we can also obtain some interesting PBDs.

Theorem 3.8. *If q is a prime power, then* $q^2 + q + 1 - k(q - k + 2) \in B(\{k-1, q+1-k, q+1-k+2, \ldots, q+1-\alpha\})$ *where* $\alpha \in \{0, 1\}$ *and* α *and* k *have the same parity.*

Proof: If a line intersects i corner points, then it intersects exactly $k - 2i$ ray points. So, by removing all the ray points, the result follows. □

Corollary 3.9. $48 \in B(\{4, 6, 8, 4^*\})$.

Proof: The corollary follows by taking $q = 8$ and $k = 5$. □

In Theorem 3.3 we have given a counting argument to ensure the presence of certain PBDs. However, it is possible that the bad points overlap to result in an overestimate of the number of bad points. We consider the cases when $q = 9$ and $q = 11$ to get a better result than Theorem 3.3.

Lemma 3.10. $53 \in B(\{5, 6, 7\})$.

Proof: A difference set for a projective plane of order 9 is

$$D = \{0, 1, 3, 9, 27, 49, 56, 61, 77, 81\}.$$

Let five lines be $D+0$, $D+1$, $D+3$, $D+5$ and $D+9$. One can check that the five lines form a dual 5-arc. Removing all points on the five lines except for 49 and 65, all lines have sizes 5,6 or 7. Hence, we obtain $53 \in B(\{5,6,7\})$. □

Lemma 3.11. $86, 87 \in B(\{7,8,9\})$.

Proof: A difference set for projective plane of order 11 is

$$D = \{1, 11, 16, 40, 41, 43, 52, 60, 74, 78, 121, 128\}.$$

Let five lines in the plane be $D+0$, $D+13$, $D+104$, $D+5$ and $D+39$. By removing all points in the five lines except 52, 53, 120 and 6, all lines have sizes 7, 8 or 9. This gives $87 \in B(\{7,8,9\})$. In addition, if we also remove the point 6, we also obtain $86 \in B(\{7,8,9\})$. □

So far, we have no restriction on the intersection pattern of the corners. However, if we restrict that no three corners in a dual k-arc are collinear, we can obtain some more PBDs with consecutive block sizes. We call a dual k-arc *scattering* if no three of the corner points obtained from six different lines are collinear. From Lemma 9.1.1 in [4], one obtains a necessary condition on scattering dual k-arcs.

Lemma 3.12. *A scattering dual k-arc in a projective plane of order q must satisfy* $k(k-1)(k-2)(k-3) + 8k \leq 8(q^2+q+1)$.

However, the necessary condition is not sufficient. A complete search was attempted for scattering dual 7-arcs in desarguesian projective planes of order 11 and 13. However, there is no scattering dual 7-arc in these projective planes. Also, there is no scattering dual 6-arc in the desarguesian projective plane of order 9. However, scattering dual 6-arcs exist in the desarguesian projective planes of order 11 and 13.

Lemma 3.13. *There exists a scattering dual 6-arc in a projective plane of order 11.*

Proof: A difference set for projective plane of order 11 is

$$D = \{1, 11, 16, 40, 41, 43, 52, 60, 74, 78, 121, 128\}.$$

Let the six lines be $D+0$, $D+13$, $D+104$, $D+39$, $D+1$ and $D+2$. It is a straightforward matter to check that these 6 lines form a scattering dual 6-arc. □

Lemma 3.14. *There exists a scattering dual 6-arc in a projective plane of order 13.*

Proof: A difference set for projective plane of order 13 is

$$D = \{0, 2, 3, 10, 26, 39, 43, 61, 109, 121, 130, 136, 155, 141\}.$$

Let the six lines be $D+0$, $D+1$, $D+4$, $D+5$, $D+6$ and $D+9$. One can check that these six lines from a scattering dual 6-arc. □

So far, we have only considered the existence of scattering dual k-arcs. Now, we show how to use them to obtain PBDs.

Theorem 3.15. *If there exists a scattering dual k-arc in a projective plane of order q then $q^2+q+1-k(q+1)+\frac{k(k-1)}{2} \in B(\{q+1-k, q+1-(k-1), q+1-(k-2)\})$.*

Proof: The proof of this theorem is parallel to Theorem 3.2 and thus omitted. □

Theorem 3.8 can also be generalized for the scattering dual k-arc.

Corollary 3.16. $68, 69 \in B(\{5,6,7,8\})$.

Proof: Apply Theorem 3.15 with the scattering dual 6-arc in Lemma 3.13 to obtain a PBD$(76, \{6,7,8\} \cup 8^*)$. The result follows by removing seven or eight points in a block of size eight. □

Corollary 3.17. $114 \in B(\{8,9,10\})$.

Proof: Apply Theorem 3.15 with the scattering dual 6-arc in Lemma 3.14 to obtain $114 \in B(\{8,9,10\})$. □

One general question is to decide when scattering dual k-arcs exist, as they appear to be very useful in constructing PBDs.

4 The Anti-Fano Configuration

Let π be a projective plane. Let A, B, C and D be 4 points such that no three are collinear. Let $G = AC \cap BD$, $E = AD \cap BC$ and $F = AB \cap CD$. The six lines AB, AC, AD, BC, BD and CD form an *anti-Fano configuration* if the three points E, F and G are non-collinear.

Lemma 4.1. *If q is an odd prime power, then there exists a projective plane of order q containing an anti-Fano configuration.*

Proof: It is known that the desarguesian projective plane of order q, q odd, does not contain a projective subplane of order 2 [1]. The result follows since if points E, F and G are collinear, then the seven points form a projective subplane of order 2 (a Fano configuration). □

Theorem 4.2. *If there exists a projective plane of order q containing an anti-Fano configuration, then $q^2-5q+6 \in B(\{q-5, q-4, q-3\})$.*

Proof: In the proof, we often refer to Figure 3. Let l be any line. If l does not intersect any of the seven vertices, then l intersects the configuration at precisely six points. If l intersects the configuration at any one of the A, B, C and D, then l does not hit any other vertices in the configuration.

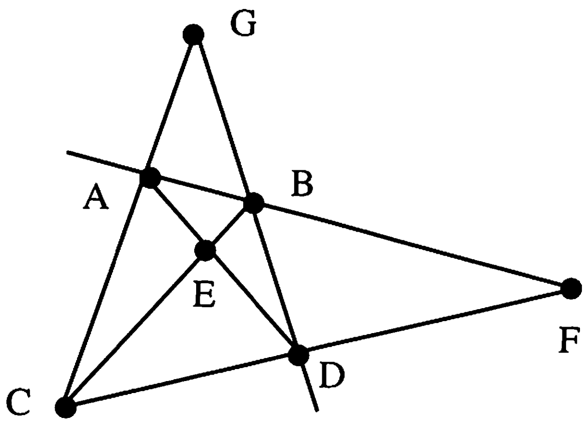

Figure 3. The Anti-Fano Configuration

Hence, l intersects the configuration at precisely four points. If l intersects one or two of E, F and G, then again by counting, it intersects precisely four or five times. Also, the number of points in the configuration is $6(q+1)-11$. We obtain the result by removing the anti-Fano configuration from the plane. □

Corollary 4.3. $72 \in B(\{6,7,8\})$.

Proof: Apply Theorem 4.2 with $q = 11$. □

Corollary 4.4. $110 \in B(\{8,9,10\})$.

Proof: Apply Theorem 4.2 with $q = 13$. □

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