Colouring of distance graphs

Margit Voigt
Institut für Mathematik
TU Ilmenau
98684 Ilmenau
Germany
email: voigt@mathematik.tu-ilmenau.de

ABSTRACT. Let D be a set of natural numbers. The distance graph $G(D)$ has the integers as vertex set and two vertices u and v are adjacent if and only if $|u-v| \in D$.

In the eighties, there have been some results concerning the chromatic number $\chi(D)$ of these graphs especially by Eggleton, Erdős, Skilton and Walther. Most of these investigations are concentrated on distance graphs where the distance set D is a subset of primes.

This paper deals with the chromatic number of distance graphs of 3-element distance sets without further restrictions for the elements of D.

1 Introduction

This paper considers chromatic properties of distance graphs on the integers. Let Z be the set of integers and N be the set of natural numbers. Any finite or infinite subset $D \subseteq N$ determines a *distance graph* $G(D)$ on the integers: $G(D)$ is the graph with the integers as vertex set, $V(G) = Z$, and with edge set $E(G)$ comprising all pairs of integers with difference in the *distance set* D, so $E(G) = \{\{u, v\} : u, v \in Z,\ |u-v| \in D\}$.

In earlier studies, such as [1] and [5], distance graphs have been defined with more general subsets of euclidian space as their vertex set, and with arbitrary subsets of the positive reals as their distance sets.

A number of results for distance graphs on the real line were established in [1]. That paper also introduced prime distance graphs (distance graphs on the integers with distance sets of prime numbers), which have been the focus of attention in most subsequent published work on distance graphs (e.g. [2] - [6], [9] - [13]).

But it is natural to consider distance sets without such restrictions. For example, it was shown in [14] that every finite graph can be represented as a distance graph on the integers, in the following sense: for every finite graph H there is a distance set D such that $G(D)$ contains an induced subgraph isomorphic to H.

In the following we are concerned with distance sets $D = \{r, s, t\}$ of three elements where $r < s < t$. The chromatic numbers of distance graphs with only one or two elements in their distance sets were determined in [1]. It was noted for any $r \in N$ that $G(r)$ is isomorphic to $G(1)$, so $\chi(r) = \chi(1) = 2$. For any $r, s \in N$ it was shown that $\chi(r, s) = 2$ or 3 according as r and s are both odd or have opposite parity and it was noted that all cases follow from these since $G(r, s) \cong G(r/d, s/d)$, where $d = gcd\{r, s\}$.

These results can readily deduced using the inequality $\chi(D) \leq |D|+1$ for any $D \subseteq N$, given in [13], and its generalization, given in [11], asserting that $\chi(D) \leq n(|D_n|+1)$ for every $n \in N$, where D_n is the set of multiples of n in D. Noting the presence of odd cycles in $G(r, s)$ when r, s have opposite parity completes the demonstration that $\chi(r, s) = 3$ in this case.

These same inequalities and the case where r, s have opposite parity imply the following results for distance graphs with distance sets $\{r, s, t\}$:

1. $\chi(r, s, t) \leq 4$.

2. $\chi(r, s, t) = 2$ if r, s, t are all odd.

3. $\chi(r, s, t) \leq 3$ if non of r, s, t is a multiple of 3.

4. $\chi(r, s, t) \geq 3$ if r, s, t include numbers of opposite parity.

Note also that it suffices to restrict attention to cases in which $\{r, s, t\}$ is a relative prime set, that is, $gcd\{r, s, t\} = 1$, since $G(r, s, t) \cong G(r/d, s/d, t/d)$, where $d = gcd\{r, s, t\}$.

Now, we will give some definitions which are very useful for the following investigations.

For any set C with $|C| = k$ a *k-colouring* of a distance graph on the integers, with colour set C, is a mapping $f : Z \rightarrow C$. Such a colouring is *periodic*, with *period p*, if $f(i+p) = f(i)$ for every $i \in Z$.

In the following, colours are denoted by $a, b, c, \ldots$. As introduced in [13] a periodic colouring is described by a "P" indexed by the length of the period and the order of colours, for example: $P_5 := aabcc$. A block of i colours of the same type is sometimes indicated by the exponent i, that means $aabcc = a^2bc^2$.

It is well-known that any periodic sequence has a least period, and that all periods are multiples of that least period. Thus, with periodic colourings we will normally assume that p denotes the least period.

Following [5] and [13], for any subset $M \in N$ we say that a colouring $f : Z \to C$ is *M-consistent* if $f(i+m) \neq f(i)$ for every $i \in Z$ and every $m \in M$. If $M = \{m\}$, we shall simply say that f is *m-consistent*. Thus, if f has to be a proper vertex-colouring of the distance graph $G(D)$, then f must be D-consistent. As noted in [12], if f is an m-consistent periodic colouring with period $p > m$, then f is $(np \pm m)$-consistent for every $n \in N$.

2 Distance graphs with chromatic number 4

In view of the results for $\chi(r,s,t)$ mentioned in the Introduction, we already know the chromatic numbers of many distance graphs $G(r,s,t)$, and the only unresolved cases are among those where the following restrictions hold:

(a) $\gcd\{r,s,t\} = 1$;

(b) at least one of r,s,t is even;

(c) at least one of r,s,t is a multiple of 3.

For all such cases we know $3 \leq \chi(r,s,t) \leq 4$ because of the results given in the introduction. In the present section we shall establish some instances in which $\chi(r,s,t) = 4$.

Theorem 1 *If t is any multiple of 3, then* $\chi(\{1,2,t\}) = 4$.

Proof: Suppose $G(1,2,t)$ has a proper 3-colouring $f : Z \to \{a,b,c\}$. Then f is a proper 3-colouring of the subgraph $G(1,2)$, so $f = P_3 = abc$ without loss of generality. But P_3 is not m-consistent, if m is any multiple of 3. Therefore P_3 is not t-consistent, so it is not a proper colouring of $G(1,2,t)$. By contradiction, $\chi(1,2,t) = 4$. □

Theorem 2 *If $gcd\{r,s\} = 1$ and $r \not\equiv s$ (mod 3), then* $\chi(r,s,r+s) = 4$.

Proof: Please note that at least one of the elements r, s and $r+s$ is even and $gcd\{r,s\} = 1$ if and only if $gcd\{r,s,r+s\} = 1$. The additional assumption $r \not\equiv s$ (mod 3) is fulfilled if and only if at least one of the elements $\{r,s,t\}$ is a multiple of 3.

Suppose $G(r,s,r+s)$ has a proper 3-colouring $f : Z \to \{a,b,c\}$. Then $f(0) = a$ and $f(r) = b$ without loss of generality. Put $t := r+s$, so $f(t) = f(r+s) = c$ and $f(s) = b$, by considering the 3-cycles $0rt0$ and $0st0$. Further consideration of 3-cycles yields $f(r+t) = a$, $f(s+t) = a$, $f(2t) =$

$b, f(r+2t) = c, f(s+2t) = c, f(3t) = a$. Generalizing this reasoning, it follows that $f(i+r) = f(i+s)$ and $f(i+3t) = f(i)$ for all $i \in \boldsymbol{Z}$, so f is an periodic colouring, and $s-r$ and $3t$ are periods of f.

Let p be the smallest period of f. Then $p|(s-r)$ and $p|3t$. But f must be t-consistent, so t cannot be a multiple of p, hence $p|3t$ implies $3|p$. Then $3|(r-s)$, so $r \equiv s \pmod 3$, a contradiction to the assumption. Consequently, we obtain $\chi(r, s, r+s) = 4$. □

The 4-chromatic distance graphs discussed in Theorems 1 and 2 contain many triangles. In the next section we shall investigate the other cases in which $G(r, s, t)$ contain triangles: $s = 2r, t = 2r$ and $t = 2s$. It turns out that none of these graphs is 4-chromatic.

3 Remaining cases with triangles

Theorem 3 *If $gcd\{r, s\} = 1$ and $r > 1$, then $\chi(r, 2r, s) = 3$.*

Proof: We construct periodic 3-colourings $f : \boldsymbol{Z} \to \{a, b, c\}$ with period $3r$. For each positive integer $i \le r$, let j be specified by $j \equiv r \pmod{2i}$ and $0 \le j < 2i$.

First, define a block $A_r^{xy}(i)$ of r terms, comprising alternate blocks of i terms equal to x and i terms equal to y, and terminating either with a block of j terms equal to x (when $0 \le j \le i$) or with a block of $j-i$ terms equal to y (when $i < j < 2i$), thus:

$$A_r^{xy}(i) := \begin{cases} \underbrace{\underbrace{x\ldots x}_{i}\underbrace{y\ldots y}_{i}\ldots\underbrace{x\ldots x}_{i}\underbrace{y\ldots y}_{i}\underbrace{x\ldots x}_{j}}_{r} & \text{if } 0 \le j \le i \\ \underbrace{\underbrace{x\ldots x}_{i}\underbrace{y\ldots y}_{i}\ldots\underbrace{x\ldots x}_{i}\underbrace{y\ldots y}_{i}\underbrace{x\ldots x}_{i}\underbrace{y\ldots y}_{j-i}}_{r} & \text{otherwise} \end{cases}$$

Let $P_{3r}(i)$ be the periodic colouring

$$P_{3r}(i) := A_r^{ac}(i)A_r^{ba}(i)A_r^{cb}(i)$$

with period $3r$. A routine check shows that $P_{3r}(i)$ is $\{r, 2r, i\}$ consistent. Furthermore, it is also easy to ascertain that $P_{3r}(r) = a^r b^r c^r$ is k-consistent for any positive integer k satisfying $r < k < 2r$.

Now, let k be the unique positive integer satisfying $k \equiv s \pmod{3r}$ and $0 < k < 3r$. Please note that $k \notin \{0, r, 2r\}$ because s is not a multiple of r. It follows from the remarks about periodic m-consistent colourings in the Introduction that

1. $P_{3r}(k)$ is $\{r, 2r, s\}$-consistent if $0 < k < r$,
2. $P_{3r}(r)$ is $\{r, 2r, s\}$-consistent if $r < k < 2r$,
3. $P_{3r}(3r - k)$ is k-consistent and consequently it is $\{r, 2r, s\}$-consistent if $2r < k < 3r$.

In all cases we have produced a proper 3-colouring, so $\chi(r, 2r, s) = 3$. □

Usually, throughout the paper it is assumed that $r < s < t$. However, Theorem 3 and the construction contained in its proof does not specify any relation between the magnitudes of r and s. Hence, it follows from Theorem 3 that $\chi(r, 2r, t) = 3$ with $2r < t$, $\chi(r, s, 2r) = 3$ with $r < s < 2r$ and $\chi(r, s, 2s) = 3$ with $1 \leq r < s$. Thus, Theorem 3 covers all cases of distance graphs with 3-element distance sets which contain isosceles triangles.

All remaining distance graphs of 3-element distance sets for which the chromatic number is unknown so far does not contain triangles. This fact leads up to the following conjecture:

Conjecture. *If $\{r, s, t\} \subseteq N$ with $r < s < t$ satisfies $gcd\{r, s, t\} = 1$, then the chromatic number of the distance graph $G(r, s, t)$ on the integers is $\chi(r, s, t) = 4$ if $r = 1, s = 2, t \equiv 0$ (mod 3) or $t = r + s$ and $r \not\equiv s$ (mod 3), and in all other cases $\chi(r, s, t) \leq 3$.*

However, the conjecture is not inevitably. In 1986 Eggleton, Erdős and Skilton conjectured (see [2]) that the chromatic number of prime distance graphs (the elements of the distance set are primes) is 4 if and only if D contains $2, 3$ and at least a pair of twin primes. Obviously, these distance graphs are exactly the prime distance graphs with triangles. This conjecture was disproved by presention of 4-element prime distance sets D without pairs of twin primes and $\chi(D) = 4$ (e.g. $D = \{2, 3, 11, 19\}$). This result was published first in [4] by the same authors.

However, it is known that there is exactly one 3-element prime distance set D ($D = \{2, 3, 5\}$ such that $\chi(D) = 4$) (see [1]). Furthermore it exists only a finite number of 4-element prime distance sets D (exactly 8) which does not produce distance graphs with triangles and $\chi(D) = 4$ (see [12]).

The results of the remaining sections of this paper add further evidence in support of our conjecture, and show many classes of 3-chromatic distance graphs with 3-element distance sets.

Furthermore, we will notice that it is sufficient to investigate a finite set of vertices to obtain results about the chromatic number of a distance graph on the integers.

4 Restriction to a finite vertex set

Computer experiments have been made to investigate chromatic numbers of distance graphs on the integers (see [6]). Such experiments have also been conducted in the author's institute. A key question in such investigations is: In any given case, how many vertices have to be investigated to determine the chromatic number.

In [6] and [13] it is proved: If a distance graph has a proper k-colouring, then it has a periodic k-colouring. Consequently, it is sufficient to investigate a finite number of vertices.

However, the proofs are concentrated on existence arguments. The authors construct periods, but they use the Pigeonhole Principle in a way which does not lead up to tight bounds. The following result gives a practical bound for 3-element distance sets which happen to be 3-chromatic.

Theorem 4 *Suppose $\{r, s, t\} \subset N$ admits a 3-colouring $f : Z \rightarrow \{a, b, c\}$ which is a proper 3-colouring of $r+s+t$ consecutive vertices of the distance graph $G(r, s, t)$. Then $\chi(r, s, t) \leq 3$.*

Proof: Without loss of generality let the vertices $1, 2, 3, \ldots, r+s+t$ be properly coloured by f. Now, we colour successively the integers outwards from this interval (i.e. $r+s+t+1, r+s+t+2, \ldots$ and $0, -1, -2, \ldots$) with a colour which is not used for the already coloured neighbours of the coresponding vertex.

Assume that such an extension of f is not possible. Without loss of generality let v be the smallest integer such that all colours a, b and c are used already for the neighbors of v and assume $f(v-r) = a, f(v-s) = b, f(v-t) = c$. It follows: $f(v-(r+s)) = c, f(v-(r+t)) = b, f(v-(s+t)) = a$. Consequently, there is an already coloured vertex $v-(r+s+t) > 1$ which cannot be coloured with a, b or c contradicting the choice of v as the first such vertex. □

It follows a similar result for distance sets with more than three elements:

Corollary. *Suppose $D := \{d_1, d_2, \ldots, d_k\} \subseteq N$ with $d_1 < d_2 < \cdots < d_k$ admits a 3-colouring $f : Z \rightarrow \{a, b, c\}$ which is a proper 3-colouring of $d_{k-2} + d_{k-1} + d_k$ consecutive vertices of the distance graph $G(D)$. Then $\chi(D) \leq 3$.* □

Please note that this result does not give any information about a smallest period of a proper colouring of $G(D)$.

5 Further classes with chromatic number 3

In [12] it is proved: There are exactly 8 prime distance sets (excluded the sets with a twin of primes) with 4 elements and chromatic number 4. For all other such distance sets there were constructed 3-colourings for the distance graphs. In the following, we attempt to construct colourings for distance sets with 3 elements in a similar way.
First, we give a result for a special type of distance graphs.

Theorem 5 *Suppose $r, s, t \subset N$ has the properties r, s odd, $r < s$, t even, $2s \leq t$ and $gcd\{r, s, t\} = 1$. Then $\chi(r, s, t) = 3$*

Proof: First note that $G(r, s, t)$ contains odd cycles, since r and t have opposite parity, so $\chi(r, s, t) \geq 3$. For suitable chosen k, we construct periodic 3-colourings with period $6k + 3$ which are proper 3-colourings of $G(r, s, t)$. For any $k \in N$ let P_{6k+3} be the following periodic colouring:

$$P_{6k+3} := (ab)^k a(bc)^k b(ca)^k c = \underbrace{abab\ldots aba}_{2k+1}\underbrace{bcbc\ldots bcb}_{2k+1}\underbrace{caca\ldots cac}_{2k+1}$$

It is routine to check that P_{6k+3} is $(2i+1)$-consistent for all i with $0 \leq i \leq k$ and also $(4k + 2)$- and $(4k + 4)$-consistent. Now we consider two cases for t.

1. $t \equiv 0 \pmod 4$
   Put $k := \frac{t-4}{4}$. Then $t = 4k + 4$, and the odd numbers r, s satisfy $r < s < \frac{t}{2} = 2k + 2$. Hence, P_{6k+3} is $\{r, s, t\}$-consistent.
2. $t \equiv 2 \pmod 4$
   Put $k := \frac{t-2}{4}$. Then $t = 4k + 2$, and the odd numbers r, s satisfy $r < s \leq \frac{t}{2} = 2k + 1$. Hence, P_{6k+3} is $\{r, s, t\}$-consistent.

In both cases P_{6k+3} is a proper 3-colouring of $G(r, s, t)$, so $\chi(r, s, t) = 3$. □

Next, for relative prime distance sets $\{r, s, t\}$ with at least one even element and $r < s < t$, let $s \equiv k \pmod{3r}$ where $0 \leq k < 3r$. We shall show that $\chi(r, s, t) = 3$ if $0 < k < 2r$ and t is big enough in relation to r and s. It is convenient to split the discussion into two cases, the first with $0 < k \leq r$, and the second with $r < k < 2r$. The next two theorems accord with these cases.

Theorem 6 *Suppose $\{r, s, t\} \subset N$ with $r < s < t$ has the properties $t \geq 9s^2 - 10s + 2$, $gcd\{r, s, t\} = 1$, at least one of r, s, t is even and $s \equiv k \pmod{3r}$ with $0 < k \leq r$. Then $\chi(r, s, t) = 3$.*

Proof: First note that $G(r, s, t)$ contains odd cycles since $\{r, s, t\}$ contains elements of opposite parity, so $\chi(r, s, t) \geq 3$. To show that $\chi(r, s, t) = 3$, we construct a periodic 3-colouring P_{s+t} which is a proper 3-colouring of $G(r, s, t)$.

Choose $k, q \in N$ with $0 < k \leq r$, so that $s = 3rq + k$. For $i \in \{0, 1\}$ define

$$A^{xyz}_{s-i} := (x^r y^r z^r)^q x^{k-i} = \underbrace{x\ldots x}_{r}\underbrace{y\ldots y}_{r}\underbrace{z\ldots z}_{r}\ldots\underbrace{z\ldots z}_{r}\underbrace{x\ldots x}_{k-i}$$

and

$$B_{3s-i} := A_s^{abc} A_s^{bca} A_{s-i}^{cab}.$$

Now let $m, n \in N$ satisfy $3sm+(3s-1)n = s+t$. Since $gcd\{3s, 3s-1\} = 1$, there always exist such m and n provided $s+t \geq 9s^2 - 9s + 2$, by a classical result of Sylvester (see [7], [8]). Define the periodic 3-colouring

$$P_{s+t} := B_{3s}^m B_{3s-1}^n.$$

It is routine to check that P_{s+t} is $\{r, s\}$-consistent. Since it has period $s+t$ and $t = (s+t) - s$ it follows that it is also t-consistent. Hence P_{s+t} is a proper 3-colouring of $G(r, s, t)$ and $\chi(r, s, t) = 3$. □

Theorem 7 *Suppose $\{r, s, t\} \subset N$ with $r < s < t$ has the properties $t \geq (r+s)^2 - (r+2s)$, $gcd\{r, s, t\} = 1$, at least one of r, s, t is even and $s \equiv k \pmod{3r}$ with $r < k < 2r$. Then $\chi(r, s, t) = 3$.*

Proof: First note that $G(r, s, t)$ contains odd cycles since $\{r, s, t\}$ contains elements of opposite parity, so $\chi(r, s, t) \geq 3$. To show that $\chi(r, s, t) = 3$, we construct a periodic 3-colouring P_{s+t} which is a proper 3-colouring of $G(r, s, t)$.

Choose $k, q \in N$ with $0 < k \leq r$, so that $s = 3rq + k$. For $i \in \{0, 1\}$ define

$$A_{r+s+i} := (a^r b^r c^r)^q a^r b^r c^{k-r+i} = \underbrace{a\ldots a}_{r}\underbrace{b\ldots b}_{r}\underbrace{c\ldots c}_{r}\ldots\underbrace{a\ldots a}_{r}\underbrace{b\ldots b}_{r}\underbrace{c\ldots c}_{k-r+i}$$

Now let $m, n \in N$ satisfy $(r+s)m+(r+s+1)n = s+t$. Since $gcd\{r+s, r+s+1\} = 1$, there always exist such m and n provided $s+t \geq (r+s)^2 - (r+s)$ (see [7], [8]). Define the periodic 3-colouring

$$P_{s+t} := A_{r+s}^m A_{r+s+1}^n.$$

It is routine to check that P_{s+t} is r-consistent.

Next we shall show that P_{s+t} is s-consistent. Let $f : Z \to \{a, b, c\}$ be a 3-colouring corresponding to P_{s+t} such that the integers $v = 1, 2, \ldots, r+s+i$ are coloured by the colours of a block A_{r+s+i} of P_{s+t}. Obviously, it is sufficient to show that $f(v) \neq f(v+s)$ for all $v \in \{1, 2, \ldots, r+s+i\}$ where $i \in \{0, 1\}$ according to the block A_{r+s+i} which is assigned to these vertices. First assume that $1 \leq v \leq r+i$. It follows $f(v) = f(v+3rq) = f(v+(s-k))$ because of the structure of A_{r+s+i}. Furthermore it is easy to check that $f(v+s-k) \neq f(v+s)$ and hence $f(v) \neq f(v+s)$.
Now let v be one of the integers $\{r+i+1, r+i+2\ldots r+s+i\}$. It is routine to check that $f(v) \neq f(v-(r+i))$ because of the structure of A_{r+s+i}.

Furthermore it is easy to see that $f(v-(r+i)) = f(v-(r+i)+(r+s+i)) = f(v+s)$ and it follows $f(v) \neq f(v+s)$.

Since P_{s+t} has period $s+t$ and $t = (s+t) - s$ it follows that it is also t-consistent. Hence P_{s+t} is a proper 3-colouring of $G(r,s,t)$ and $\chi(r,s,t) = 3$. □

The case with $s \equiv k \pmod{3r}$ and $2r \leq s \leq 3r$ remains unresolved, though the presented Conjecture suggests that these distance sets produce again 3-chromatic distance graphs. In addition to this tantalizing open problem, other problems which remain include the following:

1. Determine the chromatic numbers of distance graphs on the integers with distance sets of more than 3 elements.
2. Is there a simple generalization of Theorem 4 ?
3. Investigate relationships of distance graphs to other kinds of graphs, such as indicated in [12] and [14].

Acknowledgement. The author is very grateful to the referee for his help in preparing the revised version of the paper.

References

[1] R.B. Eggleton, P. Erdős and D.K. Skilton: Colouring the real line, *Journal of Combinatorial Theory* **39B** (1985), 86–100, to erratum **41** (1986), 139.

[2] R.B. Eggleton, P. Erdős and D.K. Skilton: Research Problem 77, *Discrete Math.* **58** (1986), 323.

[3] R.B. Eggleton: Three unsolved problems in graph theory, *Ars Combinatoria* **23A** (1987), 112–132.

[4] R.B. Eggleton, P. Erdős and D.K. Skilton: Update on Research Problem 77, *Discrete Math.* **69** (1988), 105–106.

[5] R.B. Eggleton: New results on 3-chromatic prime distance graphs, *Ars Combinatoria* **26B** (1988), 153–180.

[6] R.B. Eggleton, P. Erdős and D.K. Skilton: Colouring prime distance graphs, *Graphs and Combinatorics* **6** (1990), S. 17–32.

[7] R.K. Guy: *Unsolved Problems in Number Theory*, Springer, 1981

[8] Sylvester: *Math. Quest. Educ. Times* **41** (1884), 21.

[9] M. Voigt, H. Walther: On the chromatic number of special distance graphs, *Discrete Mathematics* **97** (1991), 395–397.

[10] M. Voigt, H. Walther: Chromatic number of prime distance graphs, *Discrete Applied Mathematics* **51** (1994), 197–209.

[11] M. Voigt: On the chromatic number of distance graphs, *J. Inform. Process. Cybernet.* **EIK 28** (1992), 1, 21–28.

[12] M. Voigt: Über die chromatische Zahl einer speziellen Klasse unendlicher Graphen, Dissertationsschrift, TU Ilmenau, 1992, p.98.

[13] H. Walther: Über eine spezielle Klasse unendlicher Graphen, in: K.Wagner and R.Bodendiek, *Graphentheorie II*, Bibl. Inst. Mannheim 1990, 268–295.

[14] H. Walther: Graph representation in a special family of infinite graphs, Memorandum no.831, University of Twente 1989, p.10.