# On Partial Sums of Chromatic Polynomials

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**Abstract.** In this paper we prove that the partial sums of the chromatic polynomial of a graph define an alternating sequence of upper and lower bounds.

We consider finite undirected graphs without loops or multiple edges. For such a graph $G$, let $n(G)$ resp. $m(G)$ denote its number of vertices resp. edges. The girth of $G$ is denoted by $g(G)$, where $g(G) := +\infty$ if $G$ is cycle-free. For any $\lambda \in \mathbb{N}$, let $P_G(\lambda)$ denote the number of mappings of the vertex-set of $G$ into $\{1, \ldots, \lambda\}$ such that any two vertices joined by an edge receive different values. By Birkhoff [1], $P_G(\lambda)$ is a monic polynomial in $\lambda$ of degree $n(G)$, i.e.,

$$P_G(\lambda) = \sum_{k=0}^{n(G)} a_k(G)\, \lambda^{n(G)-k}, \quad a_0(G) = 1. \tag{1}$$

This polynomial is called the *chromatic polynomial of G*. A good account of results and conjectures on it is given in the article of Read and Tutte [4].

Subsequently, we prove that the partial sums in Eq. (1) define an alternating sequence of upper and lower bounds.

**Theorem.** *For any graph $G$, $\lambda \in \mathbb{N}$ and $q = 0, \ldots, n(G)$ we have*

$$P_G(\lambda) \leq \sum_{k=0}^{q} a_k(G)\, \lambda^{n(G)-k} \qquad (q \text{ even}),$$

$$P_G(\lambda) \geq \sum_{k=0}^{q} a_k(G)\, \lambda^{n(G)-k} \qquad (q \text{ odd}).$$

**Proof:** We prove the theorem by induction on the number of edges. If $G$ has no edges, then $P_G(\lambda) = \lambda^{n(G)}$ and the statement holds. Now let $G$

have at least one edge $e$, and assume that the statement is true for graphs with fewer edges. Let $G-e$ resp. $G/e$ denote the graph obtained from $G$ by deleting resp. contracting $e$ and then, in the resulting multigraph, replacing each class of parallel edges by a single edge. Note that $n(G-e) = n(G)$ and $n(G/e) = n(G) - 1$. By the deletion-contraction formula (cf. [4]),

$$P_G(\lambda) \;=\; P_{G-e}(\lambda) - P_{G/e}(\lambda) \tag{2}$$

and consequently,

$$a_k(G) \;=\; a_k(G-e) - a_{k-1}(G/e) \qquad (k = 1, \ldots, n(G)). \tag{3}$$

The induction hypothesis applied to $G-e$ and $G/e$ gives

$$\begin{aligned} P_{G-e}(\lambda) \;&\leq\; \sum_{k=0}^{q} a_k(G-e)\,\lambda^{n(G-e)-k} \;=\; \sum_{k=0}^{q} a_k(G-e)\,\lambda^{n(G)-k}, \\ P_{G/e}(\lambda) \;&\geq\; \sum_{k=0}^{q-1} a_k(G/e)\,\lambda^{n(G/e)-k} \;=\; \sum_{k=1}^{q} a_{k-1}(G/e)\,\lambda^{n(G)-k}. \end{aligned}$$

in the case where $q$ is even. From this and Eq. (2) we conclude that

$$P_G(\lambda) \;\leq\; \lambda^{n(G)} + \sum_{k=1}^{q} (a_k(G-e) - a_{k-1}(G/e))\,\lambda^{n(G)-k}.$$

Now apply Eq. (3). The case where $q$ is odd is treated similarly. □

For $q < g(G) - 1$, a different proof of the above inequalities can be found in [2]. As a consequence of our result, we now deduce the main result of [2].

**Corollary.** *For any graph $G$, $\lambda \in \mathbb{N}$ and $q = 0, \ldots, \min\{n(G), g(G) - 1\}$,*

$$\begin{aligned} P_G(\lambda) \;&\leq\; \sum_{k=0}^{q} (-1)^k \binom{m(G)}{k} \lambda^{n(G)-k} \qquad (q \textit{ even}), \\ P_G(\lambda) \;&\geq\; \sum_{k=0}^{q} (-1)^k \binom{m(G)}{k} \lambda^{n(G)-k} \qquad (q \textit{ odd}). \end{aligned}$$

**Proof:** By a result of Meredith [3] we have

$$(-1)^k a_k(G) \;=\; \binom{m(G)}{k} \quad (k = 0, \ldots, q-1), \quad (-1)^q a_q(G) \;\leq\; \binom{m(G)}{q}.$$

By combining Meredith's result with our theorem the corollary follows. □

## References

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