Domination Using Induced Cliques in Graphs

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ABSTRACT. Let $G = (V, E)$ be a graph and let $\mathcal{H}$ be a set of graphs. A set $S \subseteq V$ is $\mathcal{H}$-independent if for all $H \in \mathcal{H}$, $< S >$ contains no subgraph isomorphic to H. A set $S \subseteq V$ is an $\mathcal{H}$-dominating set of G if for every $v \in V - S$, $< S \cup \{v\} >$ contains a subgraph containing v which is isomorphic to some $H \in \mathcal{H}$.

The $\mathcal{H}$-domination number of a graph G, denoted by $\gamma_{\mathcal{H}}(G)$, is the minimum cardinality of an $\mathcal{H}$-dominating set of G and the $\mathcal{H}$-independent domination number of G, denoted by $i_{\mathcal{H}}(G)$, is the smallest cardinality of an $\mathcal{H}$-independent $\mathcal{H}$-dominating set of G.

A sequence of positive integers $a_2 \leq \cdots \leq a_m$ is said to be a domination sequence if there exists a graph G such that $\gamma_{\{K_k\}}(G) = a_k$ for $k = 2, \ldots, m$. In this paper, we find an upper bound for $\gamma_{\mathcal{H}}(G)$ and show that the problems of computing $\gamma_{\{K_n\}}$ and $i_{\{K_n\}}$ are NP-hard. Finally we characterise nondecreasing sequences of positive integers which are domination sequences, and provide a sufficient condition for equality of $\gamma_{\{K_n\}}(G)$ and $i_{\{K_n\}}(G)$.

1 Introduction

Let $G = (V, E)$ be a graph with p vertices and q edges. For a subgraph H of G, $p(H)$ and $q(H)$ denote the number of vertices and edges of H respectively. For any vertex $v \in V$, the *open neighbourhood of* v, denoted by $N(v)$, is defined by $\{u \in V : uv \in E\}$. The *closed neighbourhood of* v, denoted by $N[v]$, is the set $N(v) \cup \{v\}$. For $S \subseteq V$, the *open neighbourhood of* S, denoted by $N(S)$, is defined as $\bigcup_{v \in S} N(v)$, while the *closed neighbourhood of* S, denoted by $N[S]$, is defined by $\bigcup_{v \in S} N[v]$. The *private neighbours of a vertex* v *with respect to a set* S is denoted by $PN[v, S] = N[v] - N[S - \{v\}]$. the *clique number* of G, denoted $\omega(G)$, is the number of vertices in a maximum clique of G.

A set S is an *independent set* if no two vertices in S are adjacent; S is a *dominating set* if $N[S] = V$, or, equivalently, if for every vertex $v \in V - S$, there exists $u \in S$ such that $uv \in E(G)$.

The *domination number* of a graph G, denoted by $\gamma(G)$, is defined as $\min\{|S| : S \text{ is a dominating set of } G\}$. Similarly the upper domination number of a graph, $\Gamma(G)$, is the size of a largest minimal dominating set. The *independent domination number of* G, denoted by $i(G)$, is defined as $\min\{|S| : S \text{ is an independent dominating set of } G\}$. Equivalently, $i(G)$ can also be defined as the size of a smallest maximal independent set, while $\beta_0(G)$ is the size of a largest (maximal) independent set. The connection between these graph invariants can be seen by noting that a maximal independent set S satisfies two criteria:

1. S is an independent set.

2. For every $v \notin S$, there exists a vertex $u \in S$ such that u is adjacent to v.

If we focus on the second property, it is clear that any set which satisfies this property is a dominating set. Thus the maximality of independence is precisely the defining property for domination. It follows that every maximal independent set is a minimal dominating set and consequently the following inequality emerges for any graph G:

$$\gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G)$$

Before looking at a generalization of these ideas, it is helpful to restate the two items above by casting them in a different light. In any graph G, a set S is a maximal independent set if it satisfies two criteria:

1. There is no K_2 in S.

2. For every $v \notin S$, there exists a K_2 in $< S \cup \{v\} >$.

These concepts were generalised as follows in [3]. Let $G = (V, E)$ be a graph and let $\mathcal{H}$ be a set of graphs. For $u, v \in V$ and $S \subseteq V$, we say that u and v are *$\mathcal{H}$-adjacent in S* if there exists an $H \in \mathcal{H}$ such that $< S >$ contains a subgraph containing u and v which is isomorphic to H.

We say that a set $S \subseteq V$ is *$\mathcal{H}$-independent* if for all $H \in \mathcal{H}$, $< S >$ contains no subgraph isomorphic to H. A set $S \subseteq V$ is an *$\mathcal{H}$-dominating* set of G if for every $v \in V - S$, $< S \cup \{v\} >$ contains a subgraph containing v which is isomorphic to some $H \in \mathcal{H}$.

The *$\mathcal{H}$-domination number* of a graph G, denoted by $\gamma_{\mathcal{H}}(G)$, is defined as $\min\{|S| : S$ is an $\mathcal{H}$-dominating set of $G\}$ and the *$\mathcal{H}$-independent domination number of G*, denoted by $i_{\mathcal{H}}(G)$, is defined as $\min\{|S| : S$ is an $\mathcal{H}$-independent $\mathcal{H}$-dominating set of $G\}$.

Note that a maximal $\mathcal{H}$-independent set S again satisfies two criteria:

1. S is $\mathcal{H}$-independent.

2. For every $v \notin S$, there exists a subgraph isomorphic to some $H \in \mathcal{H}$ in $< S \cup \{v\} >$.

Thus any set which satisfies the maximality condition of $\mathcal{H}$-independence is an $\mathcal{H}$-dominating set. Consequently $\gamma_{\mathcal{H}}(G) \leq i_{\mathcal{H}}(G)$. Also, it may be noticed that in the special case when $\mathcal{H} = K_2$, these definitions reduce to the standard domination and independence parameters.

We establish next a few notational conveniences. Let $S \subseteq V$ be a set for which $|S| \geq \min\{p(H)-1 : H \in \mathcal{H}\}$ and $v \in S$. The *open $\mathcal{H}$-neighbourhood of v with respect to S*, denoted by $N_{\mathcal{H}}^{S}(v)$, is defined as $\{u \in V : u$ and v are $\mathcal{H}$-adjacent in $S \cup \{u\}\}$. The *closed $\mathcal{H}$-neighbourhood of v with respect to S*, denoted by $N_{\mathcal{H}}^{S}[v]$, is defined as $N_{\mathcal{H}}^{S}(v) \cup \{v\}$. Also, let $N_{\mathcal{H}}^{S}[S] = \bigcup_{v \in S} N_{\mathcal{H}}^{S}[v]$. For $S \subseteq V$ and $v \in S$, the set $N_{\mathcal{H}}^{S}[v] - N_{\mathcal{H}}^{S-\{v\}}[S - \{v\}]$ is called the set of *special $\mathcal{H}$-neighbours of v with respect to S* and is denoted by $PN_{\mathcal{H}}[v, S]$. We say that a vertex v is *$\mathcal{H}$-isolated* if v is in no subgraph H of G such that H is isomorphic to some member of $\mathcal{H}$. Generalizing the concept of an edge cover in G, we define an *$\mathcal{H}$-cover* as follows. A set $\mathcal{S} = \{H' \subseteq G :$ there is an $H \in \mathcal{H}$ such that $H' \cong H\}$ is an *$\mathcal{H}$-cover* for G if $\bigcup_{H \in \mathcal{S}} V(H') = V(G)$.

Subsequently, we denote $\{K_n\}$ by K_n, while $\gamma_{\{K_n\}}$ and $i_{\{K_n\}}$ will be denoted by γ_n and i_n respectively.

We organise the paper as follows. In Section 2, we characterise $\mathcal{H}$-dominating sets which are minimal and in Section 3, we determine an upper bound on $\gamma_{\mathcal{H}}(G)$. For the remainder of the paper we look specifically at $\mathcal{H} = K_n$. In Section 4, we show that the problems of computing γ_n and i_n are NP-hard. A sequence of positive integers $a_2 \leq \cdots \leq a_m$ is said to be a *domination sequence* if there exists a graph G such that $\gamma_k(G) = a_k$ for $k = 2, \ldots, m$. (Sequences related to other generalised parameters were characterised in [2] and [4].) In Section 5, we characterise nondecreasing sequences of positive integers which are domination sequences. We close by determining a sufficient condition for a graph G to satisfy $\gamma_n(G) = i_n(G)$.

2 Minimal $\mathcal{H}$-dominating sets

In this section we characterise $\mathcal{H}$-dominating sets which are minimal. Let $G = (V, E)$ be a graph and let P be a property enjoyed by some of the subsets of V. A subset of V with (without) property P is called a *P-set* (*$\overline{P}$-set*). A property P is *superhereditary* if each superset of a P-set is also a P-set. A subset S of V is called a *1-minimal P-set* if S has the property P, but $S - \{v\}$ is a $\overline{P}$-set for all $v \in S$. A subset S of V is called a *minimal P-set* if S has the property P, but all proper subsets $S' \subset S$ are $\overline{P}$-sets. For properties which are superhereditary 1-minimality and minimality are equivalent [3].

Notice that the property of being an $\mathcal{H}$-dominating set is a superhereditary property. We now characterise *minimal* $\mathcal{H}$-dominating sets or, equivalently, 1-*minimal* $\mathcal{H}$-dominating sets. This result generalises a result of Ore ([5]).

Proposition 1 *Let S be an $\mathcal{H}$-dominating set of a graph $G = (V, E)$.*

Then S is a minimal $\mathcal{H}$-dominating set of G if and only if every vertex $v \in S$ satisfies at least one of the following two properties:

P_1: *v is $\mathcal{H}$-adjacent to no other vertex of S.*

P_2: *There exists $w \in V - S$ such that $w \in PN_{\mathcal{H}}[v, S]$.*

Proof: Suppose first of all that S is a minimal $\mathcal{H}$-dominating set of G. Then, for each vertex $v \in S$, $S - \{v\}$ is not an $\mathcal{H}$-dominating set of G. Hence there is a vertex $w \in V - (S - \{v\})$ such that $w \notin N_{\mathcal{H}}^{S-\{v\}}[S - \{v\}]$. However, since S is an $\mathcal{H}$-dominating of G and since $w \notin N_{\mathcal{H}}^{S-\{v\}}[S - \{v\}]$, we must have that $w \in PN_{\mathcal{H}}[v, S]$. If $w = v$, then v has property P_1, while if $w \notin S$, v has property P_2. Conversely, if each vertex $v \in S$ has one of the properties P_1 or P_2, then for each such vertex v, $S - \{v\}$ is not an $\mathcal{H}$-dominating set of G, which implies that S is a *minimal* $\mathcal{H}$-dominating set of G. □

3 An upper bound on $\gamma_{\mathcal{H}}(G)$

Ore ([5]) showed that if G is a graph of order p containing no isolated vertices, then $\gamma(G) \leq \frac{p}{2}$. We now generalise this result, by establishing an upper bound on $\gamma_{\mathcal{H}}(G)$, where G is a graph containing no $\mathcal{H}$-isolated vertices.

Proposition 2 *Let G be a graph of order p and let $\mathcal{H}$ be a set of graphs such that for each $H \in \mathcal{H}$, there exists $H' \subseteq G$ such that $H \cong H'$. If G has no $\mathcal{H}$-isolated vertices, then*

$$\gamma_{\mathcal{H}}(G) \leq p - \frac{p}{\max\{p(H) : H \in \mathcal{H}\}}.$$

Proof: Let $\mathcal{S} = \{H_1', \ldots, H_m'\}$ be an $\mathcal{H}$-cover for G of minimum cardinality. Then, for each $i = 1, \ldots, m$, there exists $v_i \in V(H_i')$ such that $v_i \notin (\bigcup_{j=1}^{m} V(H_j')) - V(H_i')$, for otherwise $\mathcal{S} - \{H_i'\}$ is an $\mathcal{H}$-cover for G, contradicting our choice of $\mathcal{S}$. Also, since $\bigcup_{i=1}^{m} V(H_i') = V(G)$, it follows that $\sum_{i=1}^{m} p(H_i') \geq p$. Hence, $m(\max\{p(H) : H \in \mathcal{H}\}) \geq \sum_{i=1}^{m} p(H_i') \geq p$, so that $m \geq p/\max\{p(H) : H \in \mathcal{H}\}$. Since $V(G) - \{v_1, \ldots, v_m\}$ is an $\mathcal{H}$-dominating set of G, $\gamma_{\mathcal{H}}(G) \leq p - m \leq p - p/\max\{p(H) : H \in \mathcal{H}\}$. □

As an immediate consequence, we have

Corollary 3 *Let $n \geq 2$ be an integer. If G is a graph of order p containing no K_n-isolated vertices, then $\gamma_n(G) \leq \frac{n-1}{n} p$.* □

Note that if we take $n = 2$, then we obtain Ore's result. Let $n \geq 2$ and $k \geq 1$ be integers. Then $\gamma_n(K_n) = n-1 = \frac{n-1}{n} n = \frac{n-1}{n} p(K_n)$, which shows that the bound of Corollary 3 is best possible.

4 Complexity results

In this section we show that the following decision problems are NP-complete, by describing polynomial transformations from the problem 3SAT.

K_n-DOMINATING SET (KDS)
INSTANCE: A graph $G = (V, E)$, a positive integer n and a positive integer $k \leq |V|$.
QUESTION: Is there a K_n-dominating set of cardinality at most k?

K_n-INDEPENDENT DOMINATING SET (KIDS)
INSTANCE: A graph $G = (V, E)$, a positive integer n and a positive integer $k \leq |V|$.
QUESTION: Is there a K_n-independent dominating set of cardinality at most k?

Proposition 4 *The decision problem KDS is NP-complete.*

Proof: To see that KDS $\in$ NP, let $S \subseteq V$ be such that $|S| \leq k$ and consider $v \in V - S$. Let $S' = S \cap N(v)$. If $|S'| \leq n-2$, then S is not a K_n-dominating set of G. Hence, suppose that $|S'| \geq n-1$. Let $S'' \subseteq S'$ such that $|S''| = n-1$. It is easy to verify, in polynomial time, whether S'' is a clique. There are at most $\binom{\deg(v)}{n-1}$ such sets; hence, it takes a polynomial amount of time to verify whether S is a K_n-dominating set or not.

To show that KDS is an NP-complete problem, we will establish a polynomial transformation from the NP-complete problem 3SAT. Let I be an instance of 3SAT consisting of the finite set $C = \{C_1, \ldots, C_r\}$ of three literal clauses in the s variables $\{x_1, \ldots, x_s\}$. Transform I to the instance (G_I, k) of KDS in which $k = s(n-1)$ and G_I is the graph constructed as follows.

Let J be a graph isomorphic to K_n, select any two fixed vertices of J, say x and $\overline{x}$, and let $Y = V(J) - \{x, \overline{x}\}$. The graph H is constructed by taking a set Z of $n-1$ independent vertices and joining each vertex in Z to every vertex in J. Denote $\{x\} \cup Y$ by X and $\{\overline{x}\} \cup Y$ by $\overline{X}$. With each variable x_i, we associate the graph H_i, a copy of H. Let $X_i, \overline{X_i}, Z_i$ be the names of the vertices of H_i that are named $X, \overline{X}, Z$, respectively, in H. Corresponding to each clause C_j we associate the graph $c_j \cong K_1$. The construction of G_I is completed by joining the vertex c_j to the three special vertex sets that name the three literals in clause C_j (e.g., if $C_j = \{x_1, \overline{x}_3, x_4\}$, then c_j is joined to every vertex in the sets X_1, $\overline{X}_3$ and X_4).

It is easy to see that the construction of the graph G_I can be accomplished in polynomial time. All that remains, is to show that I has a satisfying truth assignment if and only if G_I has a K_n-dominating set of cardinality at most k.

First suppose that I has a satisfying truth assignment. We construct a K_n-dominating set S of cardinality at most k, as follows. For each $i =$

$1, \ldots, s$, do the following. If $x_i = T$, then include the vertex set X_i in the set S. On the other hand, if $x_i = F$, then include the vertex set $\overline{X}_i$ in the set S. Then it is straightforward to verify that S is a K_n-dominating set of cardinality k.

For the converse, suppose S is a K_n-dominating set of cardinality at most k. Before proceeding further, we prove the following claim.

Claim 1 $|V(H_i) \cap S| \geq n-1$ *for* $i = 1, \ldots, s$.

Proof: If $Z_i \subseteq S$, the assertion follows trivially. We suppose therefore that $z \in Z_i - S$. Then, since z must be K_n-dominated by S, $< S \cup \{z\} >$ contains a K_n containing z. Since $N(z) \subseteq V(H_i)$, it now follows that $|V(H_i) \cap S| \geq n-1$. □

Since $|V(H_i) \cap S| \geq n-1$ for $i = 1, \ldots, s$, we have that $|S| \geq k = s(n-1)$. But $|S| \leq s(n-1)$, so that $|S| = s(n-1)$. This implies that $|V(H_i) \cap S| = n-1$ for $i = 1, \ldots, s$. Note that, since $\{c_1, \ldots, c_r\} \cap S = \emptyset$ and c_j must be K_n-dominated by S, there exists $X_i \subseteq S$ or $\overline{X}_i \subseteq S$ such that $< X_i \cup \{c_j\} > \cong K_n$ or $< \overline{X}_i \cup \{c_j\} > \cong K_n$ (for some $i \in \{1, \ldots, s\}$). Define $f : \{x_1, \ldots, x_s\} \to \{T, F\}$ by $f(x_i) = T$ if and only if $X_i \subseteq S$ for $i = 1, \ldots, s$. Then f is a satisfying truth assignment for $C_1, \ldots, C_s$. If not, there is a clause, say C_i, that is not satisfied by f. If $C_i = \{x_{i_1}, x_{i_2}, x_{i_3}\}$, then $X_{i_t} \not\subseteq S$ for $t = 1, 2, 3$. But then $< S \cup \{c_i\} > \ncong K_n$, contradicting the fact that c_i is K_n-dominated by S. If $C_i = \{\overline{x}_{i_1}, x_{i_2}, x_{i_3}\}$, then $X_{i_1} \subseteq S$, while $X_{i_t} \not\subseteq S$ for $t = 2, 3$. As before, we obtain a contradiction. Similarly, contradictions are obtained when the cases $C_i = \{\overline{x}_{i_1}, \overline{x}_{i_2}, x_{i_3}\}$ and $C_i = \{\overline{x}_{i_1}, \overline{x}_{i_2}, \overline{x}_{i_3}\}$ are considered. □

A similar proof suffices to establish the following result.

Proposition 5 *The decision problem KIDS is NP-complete.* □

5 Domination sequences

In this section we characterise sequences $a_2 \leq \cdots \leq a_m$ of positive integers which are domination sequences.

We begin with two lemmas.

Lemma 6 *Let* $G = (V, E)$ *be a graph and let* s *and* t *be integers such that* $2 \leq s \leq t$. *Then* $\gamma_s(G) \leq \gamma_t(G)$.

Proof: Suppose $S \subseteq V$ is a K_t-dominating set of cardinality $\gamma_t(G)$. If $S \subset V$, then, for all $v \in V - S$, $< S \cup \{v\} >$ contains a K_t containing v, so that, for all $v \in V - S$, $< S \cup \{v\} >$ contains a K_s containing v. In this case S is also a K_s-dominating set of G, so that $\gamma_s(G) \leq \gamma_t(G)$. If $S = V$, then, clearly, $\gamma_s(G) \leq p(G) = \gamma_t(G)$. □

Lemma 7 *Let $G = (V, E)$ be a graph of order $p \geq 1$ such that $\gamma_t(G) = \gamma_{t+1}(G)$ for some $t \geq 2$. Then $\gamma_s(G) = \gamma_t(G) = p$ for all $s \geq t$.*

Proof: We will show that each vertex of G is K_t-isolated. For suppose vertex v is contained in some K_t. Clearly, $\gamma_t(G) \leq p - 1$. We now show that there is no clique with $t+1$ or more vertices. Suppose, to the contrary, that such a clique exists and let S be a K_{t+1}-dominating set of cardinality $\gamma_{t+1}(G)$. Since $\gamma_t(G) = \gamma_{t+1}(G)$, S is also a minimal K_t-dominating set of G. Let $U = \{u_1, \dots, u_{t+1}\} \subseteq V$ be such that $< U > \cong K_{t+1}$. If $|U \cap S| \geq t$, with, say, $u_1, \dots, u_t \in S$, then, since S is a minimal K_t-dominating set of G and u_1 is K_t-adjacent to u_t in S, Proposition 1 implies that there exists $w \in V - S$ such that $w \in PN_{K_t}[u_1, S]$. However, since S is also a K_{t+1}-dominating set of G, $< S \cup \{w\} >$ contains a K_{t+1} containing w. Hence, $< S \cup \{w\} - \{u_1\} >$ contains a K_t containing w, implying that $w \notin PN_{K_t}[u_1, S]$, which is a contradiction. We conclude that if U is the vertex set of a clique with $t+1$ vertices, then $|U \cap S| \leq t-1$. We now show that $V = S$. For suppose $v \in V - S$. Then $< S \cup \{v\} >$ contains a K_{t+1}, say K, containing v. But then $|V(K) \cap S| \geq t$, contradicting our earlier observation. Hence, $S = V$, so that $\gamma_{t+1}(G) = p > p - 1 \geq \gamma_t(G)$, which is a contradiction. This shows that every vertex of G is K_t-isolated and therefore K_s-isolated for all $s \geq t$. But then $\gamma_s(G) = \gamma_t(G)$ for all $s \geq t$, as required. □

We now show that the necessary conditions implied by the statements of Lemma 6 and Lemma 7 are also sufficient for a sequence of positive integers to be a domination sequence. In particular, we prove:

Proposition 8 *Let $a_2 \leq \cdots \leq a_m$ be a sequence of positive integers such that if $t \in \{2, \dots, m\}$ and $a_t = a_{t+1}$, then $a_s = a_t$ for all $s \in \{t, \dots, m\}$. Then there exists a graph G such that $\gamma_k(G) = a_k$ for all $k = 2, \dots, m$.*

Proof: If $a_t = a_{t+1}$ for some $t \in \{2, \dots, m\}$, let ℓ be the smallest t such that $a_t = a_{t+1}$; otherwise, let $\ell = m$. Let $U_2, \dots, U_\ell$ be disjoint sets of vertices such that $|U_2| = a_2 - 1$, while $|U_k| = a_k - a_{k-1} - 1$ for $k = 3, \dots, \ell$. Note that $|U_k| \geq 0$ for $k = 2, \dots, \ell$. Let $W = \{w_2, \dots, w_\ell\}$ be a set of vertices such that $W \cap U_k = \emptyset$ for $k = 2, \dots, \ell$. Furthermore, let $W_k = \{w_2, \dots, w_k\}$ for $k = 2, \dots, \ell - 1$ and let $U = \bigcup_{k=2}^{\ell} U_k$. Construct the graph G as follows: Let $V(G) = U \cup W$. Add edges so that $< W > \cong K_{\ell-1}$ and, for $k = 3, \dots, \ell$, join each vertex in U_k with every vertex in W_{k-1}. Note that U is an independent set and that $p(G) = a_\ell$. The graph corresponding to the sequence $a_2 = 2, a_3 = 4, a_4 = 6$ and $a_5 = 9$ is illustrated in Figure 1.

We now prove that $\gamma_k(G) = a_k$ for $k = 2, \dots, m$. Since $\omega(G) \leq \ell - 1$, every vertex in G is K_k-isolated for all $k \geq \ell$. This implies that $\gamma_k(G) = p(G) = a_\ell$ for all $k \geq \ell$, whence $\gamma_k(G) = a_k$ for all $k \in \{\ell, \dots, m\}$.

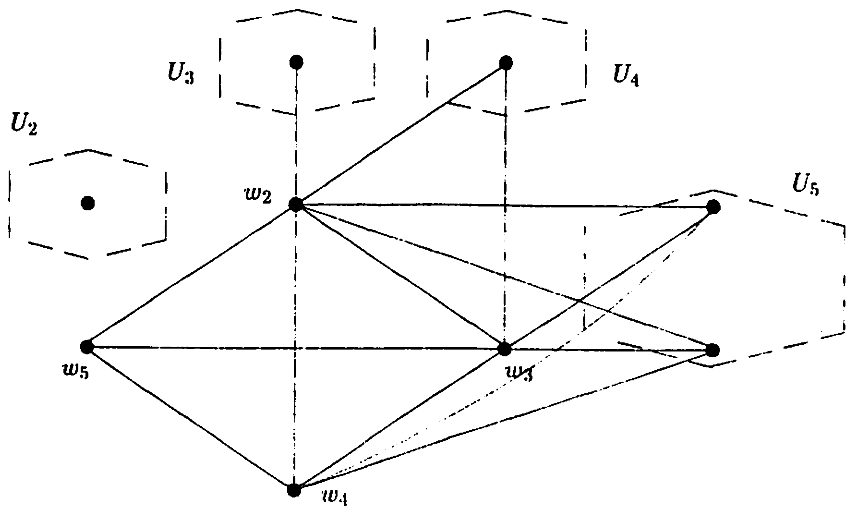

Figure 1. The graph corresponding to the sequence
$a_2 = 2, a_3 = 4, a_4 = 6$ and $a_5 = 9$

Let, therefore, $k \in \{2, \ldots, \ell\}$ and let $D_k = \bigcup_{i=2}^{k} U_i \cup W_i$. If $v \in V(G) - D_k$, then $< D_k \cup \{v\} >$ contains a K_k containing v, so that D_k is a K_k-dominating set of G. Hence $\gamma_k(G) \leq |D_k| = k - 1 + (a_2 - 1) + \cdots + (a_k - a_{k-1} - 1) = a_k$.

We now show that if S is a K_k-dominating set of G for a $k \in \{2, \ldots, \ell\}$, then $|S| \geq a_k$, implying that $\gamma_k(G) \geq a_k$.

We start by showing that $U' := \cup_{i=2}^{k} U_i \subseteq S$. Suppose, to the contrary, that $v \in U' - S$. Then $< S \cup \{v\} >$ contains a K_k containing v. This implies that v is adjacent to at least $k-1$ vertices in W, contradicting the fact that each vertex in U' is adjacent to at most $k-2$ vertices in W. It now follows that $|S \cap U'| \geq (a_2 - 1) + \cdots + (a_k - a_{k-1} - 1) = a_k - (k-1)$.

If $|S \cap W| \geq k - 1$, then $|S| \geq a_k$, and we are done. We therefore assume that $|S \cap W| \leq k - 2$. Since $k \leq \ell$, there exists a vertex $w \in W - S$. Then $< S \cup \{w\} >$ contains a K_k, say K, containing w. Since $|S \cap W| \leq k-2$, there exists a vertex $u \in S \cap U \cap V(K)$. It now follows that $\deg(u) \geq k - 1$. Hence $u \notin U'$. Also, since $\bigcup_{i=k+1}^{\ell} U_i$ is an independent set, $|V(K) \cap S \cap W| \geq k - 2$, which implies that $|S \cap W| \geq k - 2$. Hence, $|S \cap W| = k - 2$, $|S \cap U'| \geq a_k - (k-1)$ and $|S \cap \bigcup_{i=k+1}^{\ell} U_i| \geq 1$. This implies that $|S| \geq a_k$ and we are done. □

We have, therefore, the following result:

Theorem 9 *A sequence $a_2 \le \cdots \le a_m$ of positive integers is a domination sequence if and only if whenever $t \in \{2, \ldots, m\}$ and $a_t = a_{t+1}$, then $a_s = a_t$ for all $s \in \{t, \ldots, m\}$.* □

Let $a_2 \le \cdots \le a_m$ be a sequence of positive integers such that if $a_t = a_{t+1}$ for some $t \in \{2, \ldots, m\}$, then $a_s = a_t$ for all $s \in \{t, \ldots, m\}$. Let ℓ be the smallest integer t such that $a_t = a_{t+1}$; otherwise, let $\ell = m$. If G is a graph such that $\gamma_k(G) = a_k$ for $k = 2, \ldots, m$, then, clearly, $p(G) \ge a_\ell$. Since the graph constructed in the proof of Proposition 8 has order a_ℓ, a_ℓ is the minimum order of a graph with $a_2, \ldots, a_m$ as its domination sequence.

6 A sufficient condition on G for $\gamma_n(G)$ and $i_n(G)$ to be equal

A graph G is *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. Allan and Laskar [1] proved that if a graph G is claw-free, then $i(G) = \gamma(G)$. We now establish a sufficient condition on a graph G for $\gamma_n(G)$ and $i_n(G)$ to be equal.

Theorem 10 *If $n \ge 2$ is an integer and if the graph $G = (V, E)$ has no induced $K_{1,3}$ and no subgraph isomorphic to an induced $K_4 - e$, then $\gamma_n(G) = i_n(G)$.*

Proof: Let S be a K_n-dominating set of cardinality $\gamma_n(G)$ with a minimum number of induced K_n's in $< S >$. If $< S >$ contains no K_n's, then S is a K_n-free set and the result holds. So assume that $< S >$ contains a K_n, say K, and that $v \in V(K)$. Then, since S is a minimal K_n-dominating set, v has at least one special neighbour, say $v' \in V - S$. Then v and v' are in a K_n, say K', such that $K' \subseteq < S \cup \{v'\} >$ and v' is in no K_n in $< S \cup \{v'\} - \{v\} >$. Note further that all the remaining $n-2$ vertices of $V(K')$ are in S. At this point we claim that v' is not adjacent to any of the vertices in $V(K) - \{v\}$. For suppose to the contrary that v' is adjacent to $u \in V(K) - \{v\}$. Then, if there is a vertex $w \in V(K)$ with w not adjacent to v', then $< \{u, v', u, w\} > \cong K_4 - e$, contradicting our hypothesis. So v' must be adjacent to every vertex in $V(K)$. But this contradicts the fact that $v' \in PN_{K_n}[v, S]$. Hence, v' is not adjacent to any of the vertices in $V(K) - \{v\}$. Now suppose $v'' \in PN_{K_n}[v, S] - \{v'\}$. As before, v'' is not adjacent to any of the vertices in $V(K) - \{v\}$. We now prove that v' and v'' are adjacent. To see this, let u be a vertex, distinct from v, in K. Notice that v is adjacent to v', v'' and u and that neither v' nor v'' is adjacent to u. Thus, since G is $K_{1,3}$-free, we must have that v' is adjacent to v''. Finally, notice that every vertex of K' is adjacent to v'', for otherwise an induced $K_4 - e$ results.

Now consider the set $S' = S - \{v\} \cup \{v'\}$. Then $|S| = |S'|$. Since every vertex in $PN_{K_n}[v, S] - \{v'\}$ is K_n-adjacent to v', S' is a K_n-dominating set of G. Moreover, since $|V(K) \cap S'| = n-1$, $< S' >$ contains fewer induced K_n's than $< S >$, contradicting the choice of S. □

Of course, $\gamma_n(G)$ and $i_n(G)$ may differ on a graph G. To see this, let G be the graph constructed as follows. Let $H_i \cong K_{n-1}$ for $i = 1, \ldots, 6$ and add edges so that $< V(H_1) \cup V(H_2) > \cong$ $< V(H_2) \cup V(H_3) > \cong < V(H_2) \cup V(H_4) > \cong < V(H_4) \cup V(H_5) > \cong < V(H_4) \cup V(H_6) > \cong K_{2n-2}$. Then $V(H_2) \cup V(H_4)$ is a minimum K_n-dominating set of G and $V(H_1) \cup V(H_3) \cup V(H_4)$ is a minimum K_n-independent dominating set of G. Hence, $\gamma_n(G) = 2(n-1)$ and $i_n(G) = 3(n-1)$.

References

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