Integral Matrices with Given Row and Column Sums

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Abstract

Let m and n be positive integers, and let $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ be nonnegative integral vectors with $r_1 + \cdots + r_m = s_1 + \cdots + s_n$. Let $Q = (q_{ij})$ be an $m \times n$ nonnegative integral matrix. Denote by $\mathfrak{A}^Q(R, S)$ the class of all $m \times n$ nonnegative integral matrices $A = (a_{ij})$ with row sum vector R and column sum vector S such that $a_{ij} \le q_{ij}$ for all i and j. We study a condition for the existence of a matrix in $\mathfrak{A}^Q(R, S)$. The well known existence theorem follows from the maxflow-mincut theorem. It contains an exponential number of inequalities. By generalizing the Gale-Ryser theorem, we obtain some conditions under which this exponential number of inequalities can be reduced to a polynomial number of inequalities. We build a kind of hierarchy of theorems: under weaker and weaker conditions, a (larger and larger) polynomial (in n) number of inequalities yield a necessary and sufficient condition for the existence of a matrix in $\mathfrak{A}^Q(R, S)$.

1 Introduction

Let m and n be positive integers, and let $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ be non-negative integral vectors with $r_1 + \cdots + r_m = s_1 + \cdots + s_n$. Let $Q = (q_{ij})$ be an $m \times n$ non-negative integral matrix. Denote by $\mathfrak{A}^Q(R, S)$ the class of all $m \times n$ non-negative integral matrices $A = (a_{ij})$ with row sum vector R and column sum vector S such that $a_{ij} \le q_{ij}$ for all i and all j, denoted by $A \le Q$. We use $\mathfrak{A}(R, S)$ to denote the class of all $(0, 1)$-matrices of size m by n with row sum vector R and column sum vector S, which corresponds to $\mathfrak{A}^Q(R, S)$ when Q is the matrix of 1's.

We can interpret $\mathfrak{A}^Q(R, S)$ in terms of network flows. The reader is referred to [3] for the basics of Network flow theory. Matrices in $\mathfrak{A}^Q(R, S)$ can be considered as integral flows of size $r_1 + \cdots + r_m$ in the following network. The vertices consist of a source s, a sink t, and vertices $R_1, \cdots, R_m, S_1, \cdots, S_n$. There is an edge from s to R_i with capacity r_i for $i = 1, \cdots, m$. There is an edge from S_j to t with capacity s_j for $j = 1, \cdots, n$. Finally there are edges from R_i to S_j with capacity q_{ij} for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. Suppose that there is an integral flow from s to t of size $r_1 + \cdots + r_m$. Then we can construct a matrix in $\mathfrak{A}^Q(R, S)$ from the flow. Let a_{ij} be the flow from R_i to S_j and let $A = (a_{ij})$. We easily deduce that $A \in \mathfrak{A}^Q(R, S)$. Conversely, from a matrix A in $\mathfrak{A}^Q(R, S)$ we can construct an integral flow of size $r_1 + \cdots + r_m$. Thus $\mathfrak{A}^Q(R, S)$ is nonempty if and only if there is an integral flow from s to t of size $r_1 + \cdots + r_m$. Now the maxflow-mincut theorem says that $\mathfrak{A}^Q(R, S)$ is nonempty if and only if no cut has capacity less than $r_1 + \cdots + r_m$. The number of cuts in the above network is 2^{m+n}. By Lemma 1 in Section 2, the number of cut inequalities can be reduced to 2^n inequalities, but it is still an exponential number.

D. Gale [5] and H. J. Ryser [6] independently obtained the result that in the special class $\mathfrak{A}(R, S)$ we can reduce the exponential number of inequalities to a linear number of them, that is only n inequalities. D. R. Fulkerson [4] observed that the result of Gale and Ryser can be generalized to the class of $(0, 1)$-matrices with given row and column sums and zero trace. Also R. P. Anstee [1] obtained the generalization to the class of $(0, 1)$-matrices with given row and column sums and with zeros in a prescribed set of positions consisting of at most one position per column. Recently W. Chen [2] generalized these results to the class $\mathfrak{A}^Q(R, S)$ of integral matrices satisfying a certain condition (see Section 2).

In Section 2, we state a generalization of the Gale-Ryser Theorem, that is if Q and S satisfy some 'regularity' condition, then we have a necessary and sufficient condition for the existence of a matrix in $\mathfrak{A}^Q(R, S)$ containing only n inequalities (Theorem 3). The results of Gale-Ryser, Fulkerson, Anstee and Chen can be regarded as special cases of our result, since they can easily be derived from it. We build a kind of hierarchy of theorems: under weaker and weaker conditions, a (larger and larger) polynomial (in n) number of inequalities yield a necessary and sufficient condition for the existence of a matrix in $\mathfrak{A}^Q(R, S)$ (Theorem 4). In Section 3, we give the definition of the auxiliary digraph for a matrix A, which is used to prove Theorem 4. In Section 4, we give the proof of Theorem 4 using an auxiliary digraph and standard Network flow theory. In Section 5, we remark on our next goal.

2 Theorems

Throughout this paper, we will use the following notation:

$$[a, b] = \begin{cases} \{a, a+1, a+2, \cdots, b\} & \text{if } a \le b \\ \emptyset & \text{otherwise} \end{cases}$$

$$a^+ = \max\ \{a, 0\}$$

By the maxflow-mincut theorem of networks, $\mathfrak{A}^Q(R,S)$ is nonempty if and only if for all $I \subseteq [1,m]$ and $J \subseteq [1,n]$,

$$\sum_{i\in I}\sum_{j\in J} q_{ij} \;\geq\; \sum_{i\in I} r_i - \sum_{j\notin J} s_j. \tag{1}$$

So we have 2^{m+n} inequalities. But the following lemma reduces these 2^{m+n} inequalities to 2^n inequalities.

Lemma 1 *$\mathfrak{A}^Q(R,S)$ is nonempty if and only if for all $J \subseteq [1,n]$,*

$$\sum_{i=1}^{m}(r_i - \sum_{j\in J} q_{ij})^+ \;\leq\; \sum_{j\notin J} s_j. \tag{2}$$

Proof. First, suppose that $\mathfrak{A}^Q(R,S)$ is nonempty. Let J be any subset of $[1,n]$. Let $I = \{i \mid r_i \geq \sum_{j\in J} q_{ij}\}$. Then

$$\sum_{i=1}^{m}(r_i - \sum_{j\in J} q_{ij})^+ = \sum_{i\in I} r_i - \sum_{i\in I}\sum_{j\in J} q_{ij}.$$

By (1),

$$\sum_{i\in I} r_i - \sum_{i\in I}\sum_{j\in J} q_{ij} \;\leq\; \sum_{j\notin J} s_j.$$

So we can obtain inequality (2).

Conversely, suppose that (2) holds for all $J \subseteq [1,n]$. For any $I \subseteq [1,m]$ and $J \subseteq [1,n]$,

$$\begin{aligned}\sum_{i\in I} r_i - \sum_{i\in I}\sum_{j\in J} q_{ij} &\leq \sum_{i=1}^{m}(r_i - \sum_{j\in J} q_{ij})^+ \\ &\leq \sum_{j\notin J} s_j.\end{aligned}$$

∎

In the general case of $\mathfrak{A}^Q(R,S)$, we have (2) for all subsets J of $[1,n]$, an exponential number of inequalities. Now n inequalities suffice for $\mathfrak{A}(R,S)$. We will state the Gale-Ryser theorem formally here in the notation of the above lemma.

Theorem 2 *[Gale-Ryser Theorem] [5] [6] Suppose that $s_1 \geq s_2 \geq \cdots \geq s_n$. Then there exists a matrix in $\mathfrak{A}(R,S)$ if and only if for all $J = [1,h]$ $(h \geq 1)$,*

$$\sum_{i=1}^{m}(r_i - h)^+ \;\geq\; \sum_{j\notin J} s_j. \tag{3}$$

The following theorem is seen to be a generalization of the Gale-Ryser theorem. We will not give its proof here, because this theorem is a special case of Theorem 4. But we state it separately because of its simplicity.

Theorem 3 *Suppose that*

$$\sum_{i=1}^{m}(q_{ij} - q_{ik})^+ \leq s_j - s_k + 1 \quad \text{for } 1 \leq j < k < n. \tag{4}$$

Then there exists a matrix in $\mathfrak{A}^Q(R, S)$ *if and only if for all* $J = [1, h]$ $(h \geq 1)$,

$$\sum_{i=1}^{m} (r_i - \sum_{j \in J} q_{ij})^+ \leq \sum_{j \notin J} s_j. \tag{5}$$

We can derive from the above theorem not only the Gale-Ryser Theorem, but also Fulkerson's result [4] on the existence of a matrix in $\mathfrak{A}(R, S)$ with zero trace and Anstee's result [1] on the existence of a matrix in $\mathfrak{A}(R, S)$ which has zeros in a prescribed set of positions consisting of at most one position per column. In [2], Chen gives the same theorem as the above theorem, except that instead of our condition (4) he gives the following condition:

$$m\Delta - \sum_{i=1}^{m} q_{ik} \leq s_j - s_k + 1 \qquad \text{for } 1 \leq j < k \leq n \tag{6}$$

where Δ is the maximum entry of Q. In his condition, each column sum of Q is required to be close to $m\Delta$. In our condition, each entry in column j is required to be close to the corresponding entry in column k. This makes our condition less restrictive than his condition. The following example shows the difference.

Example 1 Let $R = (4, 3, 2)$ and $S = (3, 3, 3)$. Let

$$Q = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

The matrix A is in $\mathfrak{A}^Q(R, S)$. The reader can easily see that Q and S satisfy our condition (7) but violate Chen's condition (6). ∎

The following theorem is a kind of hierarchy of theorems: under weaker and weaker conditions, a (larger and larger) polynomial number of cut inequalities yield a necessary and sufficient condition for the existence of a matrix in $\mathfrak{A}^Q(R, S)$. Its proof is given in Section 4.

Theorem 4 *Let l be a nonnegative integer. Suppose that*

$$\sum_{i=1}^{m}(q_{ij} - q_{ik})^{+} \leq s_j - s_k + 1 \quad \textit{for} \ 1 \leq j < k < \min\{j + \lfloor \frac{n}{l+1} \rfloor, n\}. \tag{7}$$

Then there exists a matrix in $\mathfrak{A}^{Q}(R, S)$ if and only if

$$\sum_{i=1}^{m} (r_i - \sum_{j \in J} q_{ij})^{+} \ \leq \ \sum_{j \notin J} s_j \tag{8}$$

where

$$\begin{aligned} J &= [1,h] \cup [h_1,h_2] \cup [h_3,h_4] \cup \cdots \cup [h_{2l-1},h_{2l}] \\ & \quad \textit{or} \\ & \quad [h_1,h_2] \cup [h_3,h_4] \cup \cdots \cup [h_{2l-1},h_{2l}] \\ & (1 \leq h \leq h_1 \leq h_2 \leq \cdots \leq h_{2l-1} \leq h_{2l} \leq n). \end{aligned}$$

In Theorem 4, a set J must be the union of at most l intervals, or the union of at most $l+1$ intervals if 1 is contained in J. If the inequality in (7) is required to hold for fewer and fewer pairs of columns, the inequality in (8) must hold for more and more subsets of $[1,n]$. If $l \geq \lfloor \frac{n}{2} \rfloor$, then $\lfloor \frac{n}{l+1} \rfloor \leq 1$ and so the inequality in (7) is not required to hold for any pair of columns. Also if $l \geq \lfloor \frac{n}{2} \rfloor$, then the inequality in (8) must hold for all subsets J of $[1,n]$. Thus the case of $l \geq \lfloor \frac{n}{2} \rfloor$ is equivalent to Lemma 1.

3 Auxiliary Digraph

In this section we give the definition of an auxiliary graph, which is used to prove Theorem 4. We can easily see that there exists a matrix A with row sum vector R and $A \leq Q$. If the column sum vector of A is S, $\mathfrak{A}^{Q}(R,S)$ is nonempty. Suppose that the column sum vector of A is not S. Then there exists an integer k such that the j^{th} column sum of A is s_j for $j = 1, 2, \cdots, k-1$, but the k^{th} column sum is not s_k. Without loss of generality, we may assume that a matrix A is chosen so that k is as large as possible and subject to that the difference between the k^{th} column sum and s_k is as small as possible. Using the auxiliary digraph and standard Network flow theory we show the following: if the k^{th} column sum is greater (resp. less) than s_k, then we can "shift one" from (resp. to) the k^{th} column to (resp. from) one of columns $k+1, \cdots, n$ keeping the row sum vector R and the j^{th} column sum $(1 \leq j \leq k-1)$ s_j and each entry of (i,j) less than or equal q_{ij}. After shifting the k^{th} column sum is closer to s_k and we arrive at a contradiction to the choice of A.

Definition 5 *Let* $A = (a_{ij})$ *be a matrix. We say* we shift one from column j to column k *if there is a sequence of entries* $(i_1, j_1), (i_1, j_2), (i_2, j_2), (i_2, j_3), \cdots, (i_t, j_t), (i_t, j_{t+1})$ *(possibly* $t = 1$*) where* $j_1 = j$, $j_{t+1} = k$ *and we alternately add and subtract one from the entries:* $a_{i_1 j_1} \leftarrow a_{i_1 j_1} - 1$, $a_{i_1 j_2} \leftarrow a_{i_1 j_2} + 1$, $\cdots$, $a_{i_t j_t} \leftarrow a_{i_t j_t} - 1$, $a_{i_t j_{t+1}} \leftarrow a_{i_t j_{t+1}} + 1$.

Let

$$\begin{aligned} I_1 &= \{i \mid a_{ik} \neq 0\} \\ I_2 &= \{i \mid a_{ij} < q_{ij} \text{ for some } j > k\} \end{aligned}$$

Construct a digraph $G = (V, D)$, where $V = \{v_1, \cdots, v_m\} \cup \{w_1, \cdots, w_{k-1}\} \cup \{s, t\}$ and D consists of the following arcs:

$$\begin{aligned} v_i &\longrightarrow w_j && \text{if} \quad a_{ij} < q_{ij} \\ w_j &\longrightarrow v_i && \text{if} \quad a_{ij} \neq 0 \\ s &\longrightarrow v_i && \text{if} \quad i \in I_1 \\ v_i &\longrightarrow t && \text{if} \quad i \in I_2 \end{aligned}$$

Call the digraph $G = (V, D)$ the *auxiliary digraph for* A.

In Section 4, it's shown that if the k^{th} column sum of A is greater than s_k then there exists a directed path P from s to t in the auxiliary digraph for A. By alternately adding and subtracting 1 from the entries along P, we can shift one from the k^{th} column to one of columns $k+1, \cdots, n$. If an arc $s \to v_i$ is in P, subtract 1 from a_{ik}. If an arc $v_i \to w_j$ is in P, add 1 to a_{ij}. If an arc $w_j \to v_i$ is in P, subtract 1 from a_{ij}. If an arc $v_i \to t$ is in P, find j ($> k$) such that $a_{ij} < q_{ij}$ and add 1 to a_{ij}. It is easy to see that after shifting the row sum vector is still R, the j^{th} column sum is s_j $(1 \leq j \leq k-1)$, each entry of (i, j) is less than or equal q_{ij} and the k^{th} column sum is closer to s_k.

Suppose that the k^{th} column sum of A is less than s_k. Shifting one from one of columns $k+1, \cdots, n$ to the k^{th} column in the matrix A is equivalent to shifting one from the k^{th} column to one of columns $k+1, \cdots, n$ in the matrix $Q - A$. In Section 4, it's shown that there exists a directed path from s to t in the auxiliary digraph for $Q - A$.

4 The proof of Theorem 3

Suppose that $\mathfrak{A}^Q(R, S)$ is nonempty. Then by Lemma 1, (8) holds. Now we will do the other direction of the proof by contradiction. Assume that condition (7) and (8) hold but $\mathfrak{A}^Q(R, S)$ is empty. Then there exist an integer $k < n$ and a matrix A with row sum vector R and $A \leq Q$ such that the j^{th} column sum of A is s_j for $j = 1, \cdots, k-1$ and the k^{th} column sum is not s_k (k may be 0). Choose a matrix A so that k is as large as possible and subject to that the difference between the k^{th} column sum and s_k is as small as possible. Let s'_j be the j^{th} column sum of A. If there exists a directed path P in the auxiliary digraph for A (resp. $Q - A$), we can shift one from (resp. to) the k^{th} column to (resp. from) one of columns $k+1, \cdots, n$ along P and we arrive at a contradiction to the choice of A. It's enough to show that there exists a desired directed path from s to t in the auxiliary digraph.

Case 1 $s'_k > s_k$:
Let $G = (V, D)$ be the auxiliary digraph for A. We will show that there is a directed path from s to t in G by contradiction. Assume that there is no such path. Let

$$\begin{aligned} I &= \{i \mid \exists \text{ a directed path from } s \text{ to } v_i\} \\ J_0 &= \{j < k \mid \exists \text{ a directed path from } s \text{ to } w_j\} \end{aligned}$$

Then

$$\begin{array}{lll} \forall i \in I \ \& \ \forall j \in [1, k-1] \setminus J_0 & & a_{ij} = q_{ij} \\ \forall i \notin I \ \& \ \forall j \in J_0 & & a_{ij} = 0 \end{array}$$

If a_{ik} is nonzero, then there is an arc $s \to v_i$ and i is in I. So $a_{ik} = 0$ for all $i \notin I$. If there is a pair (i, j) such that $i \in I$, $j > k$ and $a_{ij} < q_{ij}$, then there is a directed path from s to t. Thus $a_{ij} = q_{ij}$ for all $i \in I$ and all j $(> k)$. Thus

$$\begin{array}{lll} \forall i \in I \ \& \ \forall j \in [J_0 \cup \{k\}]^c & & a_{ij} = q_{ij} \\ \forall i \notin I \ \& \ \forall j \in J_0 \cup \{k\} & & a_{ij} = 0 \end{array}$$

where $[J_0 \cup \{k\}]^c$ is the set $[1, n] \setminus [J_0 \cup \{k\}]$. We use J^c to denote the set $[1, n] \setminus J$.

Suppose that $k \leq 2l$. The set $[1, k-1] \setminus J_0$ can be written as

$$[1, h] \cup [h_1, h_2] \cup \cdots \cup [h_{2p-1}, h_{2p}] \ , \quad \text{or}$$
$$[h_1, h_2] \cup \cdots \cup [h_{2p-1}, h_{2p}]$$

where $h_1 \neq 1$ and $p \leq l - 1$. Let $J = [J_0 \cup \{k\}]^c$. Then $J = ([1. k-1] \setminus J_0) \cup [k+1, n]$. So J is

$$[1, h] \cup [h_1, h_2] \cup \cdots \cup [h_{2p-1}, h_{2p}] \cup [h_{2p+1}, h_{2p+2}] \ , \quad \text{or}$$
$$[h_1, h_2] \cup \cdots \cup [h_{2p-1}, h_{2p}] \cup [h_{2p+1}, h_{2p+2}]$$

where $h_{2p+1} = k + 1$ and $h_{2p+2} = n$, and $p + 1 \leq l$. Note that $a_{ij} = q_{ij}$ for all $i \in I$ and all $j \in J$, and $s'_j = s_j$ for all $j \in J^c \setminus \{k\}$ and $s'_k > s_k$. We have the following inequality:

$$(*) \qquad \sum_{i=1}^{m} (r_i - \sum_{j \in J} q_{ij})^+ \ \geq \ \sum_{i \in I} (r_i - \sum_{j \in J} q_{ij})^+$$

$$\geq \sum_{i\in I}(r_i - \sum_{j\in J} q_{ij}) = \sum_{j\notin J} s'_j$$
$$> \sum_{j\notin J} s_j$$

This contradicts (8).

From now we suppose that $k > 2l$.

Claim 1 *There exists a pair (t, u) such that $t < u < k$, $a_{et} < q_{et}$ for some $e \in I$, and $a_{fu} \neq 0$ for some $f \notin I$.*

Proof. First, we will show that there exists u ($< k$) such that $a_{fu} \neq 0$ for some $f \notin I$. Assume that no such u exists. Then $a_{ij} = 0$ for all $i \notin I$ and all j ($< k$). Also $a_{ik} = 0$ for all $i \notin I$. Thus

$$\begin{array}{llll} \forall i \notin I & \& & \forall j < h_1 & a_{ij} = 0 \\ \forall i \in I & \& & \forall j \text{ with } h_1 \leq j \leq h_2 & a_{ij} = q_{ij} \end{array}$$

where $h_1 = k + 1$ and $h_2 = n$. Let $J = [h_1, h_2]$. By an argument similar to (*), we obtain a contradiction to (8).

Let u be the last column before column k that has a nonzero a_{fu} for some $f \notin I$. Now we will show that there exists t ($< u$) such that $a_{et} < q_{et}$ for some $e \in I$. Assume that there is no such t. Then $a_{ij} = q_{ij}$ for all $i \in I$ and all j ($< u$). Since $a_{fu} \neq 0$, $u \notin J_0$ and $a_{iu} = q_{iu}$ for all $i \in I$. Thus

$$\begin{array}{llll} \forall i \in I & \& & \forall j \in [1, h] \cup [h_1, h_2] & a_{ij} = q_{ij} \\ \forall i \notin I & \& & \forall j \text{ with } h < j < h_1 & a_{ij} = 0 \end{array}$$

where $h = u$ and h_i ($i = 1, 2$) is the same as above. Let $J = [1, h] \cup [h_1, h_2]$. By an argument similar to (*), we obtain a contradiction to (8). ∎

Claim 2 *There exist l pairs $(j_1, j_2), \cdots, (j_{2l-1}, j_{2l})$ of (t, u) satisfying conditions of Claim 1 where $j_1 < j_2 < \cdots < j_{2l-1} < j_{2l} < k$.*

Proof. We will prove it by an induction on the number of pairs of (t, u) we obtain. Let us suppose that we have p pairs (j_{2i-1}, j_{2i}) of (t, u) where $i = l, l-1, \cdots, l-p+1$. Now we want to obtain the $(p+1)^{st}$ pair. Without loss of generality, we can assume that

$$\begin{array}{llll} \forall i \in I & \& & \forall j \text{ with } j_{2(l-p)+1} < j \leq j_{2(l-p)+2} & a_{ij} = q_{ij} \\ \forall i \notin I & \& & \forall j \text{ with } j_{2(l-p)+2} < j \leq j_{2(l-p)+3} & a_{ij} = 0 \\ & \vdots & & \\ \forall i \in I & \& & \forall j \text{ with } j_{2l-1} < j \leq j_{2l} & a_{ij} = q_{ij} \\ \forall i \notin I & \& & \forall j \text{ with } j_{2l} < j \leq k & a_{ij} = 0 \end{array}$$

Now we will show that there is $j_{2(l-p)}$ such that $j_{2(l-p)} < j_{2(l-p)+1}$ and $a_{ej_{2(l-p)}} \neq 0$ for some $e \notin I$. Assume that there is no such $j_{2(l-p)}$. Then $a_{ij} = 0$ for all $i \notin I$ and all $j \leq j_{2(l-p)+1}$. Let

$$(**) \qquad h_1 = j_{2(l-p)+1} + 1, \quad h_2 = j_{2(l-p)+2}, \quad \cdots$$
$$h_{2p-1} = j_{2l-1} + 1, \quad h_{2p} = j_{2l}, \quad h_{2p+1} = k+1, \quad h_{2(p+1)} = n.$$

Let $J = [h_1, h_2] \cup \cdots \cup [h_{2p-1}, h_{2p}] \cup [h_{2p+1}, h_{2(p+1)}]$. By an argument similar to $(*)$, we obtain a contradiction to (8).

Now we will show that there is $j_{2(l-p)-1}$ such that $j_{2(l-p)-1} < j_{2(l-p)}$ and $a_{fj_{2(l-p)-1}} < q_{fj_{2(l-p)-1}}$ for some $f \in I$. Assume that there is no such $j_{2(l-p)-1}$. Then $a_{ij} = q_{ij}$ for all $i \in I$ and all $j \leq j_{2(l-p)}$. Let h be $j_{2(l-p)}$ and J be $[1, h] \cup [h_1, h_2] \cup \cdots \cup [h_{2p-1}, h_{2p}] \cup [h_{2p+1}, h_{2(p+1)}]$. By an argument similar to $(*)$, we obtain a contradiction to (8).

By repeating the above arguments, we can obtain l pairs of (t, u). This completes the proof of Claim 2. ∎

We have l pairs $(j_1, j_2), \cdots, (j_{2l-1}, j_{2l})$ satisfying conditions of Claim 2. For any j_{2p-1}, there exists $e \in I$ such that $a_{ej_{2p-1}} < q_{ej_{2p-1}}$. Also, $a_{ij_{2p-1}} = 0$ for all $i \notin I$. So,

$$\begin{aligned} \sum_{i \in I} q_{ij_{2p-1}} &\geq \sum_{i \in I} a_{ij_{2p-1}} + 1 \\ &= s_{j_{2p-1}} + 1. \end{aligned}$$

For any j_{2p}, there exists $e \notin I$ such that $a_{ej_{2p}} \neq 0$. Also, $a_{ij_{2p}} = q_{ij_{2p}}$ for all $i \in I$. So,

$$\begin{aligned} \sum_{i \in I} q_{ij_{2p}} &\leq \sum_{i \in I} a_{ij_{2p}} + (\sum_{i \notin I} a_{ij_{2p}} - 1) \\ &= s_{j_{2p}} - 1. \end{aligned}$$

Since $a_{ik} = 0$ for all $i \notin I$.

$$\sum_{i \in I} q_{ik} \geq s'_k \geq s_k + 1.$$

Since $s'_1 = s_1, \cdots, s'_{k-1} = s_{k-1}$ and $s'_k > s_k$, there exists $j > k$ such that $s'_j < s_j$. Since $a_{ij} = q_{ij}$ for all $i \in I$,

$$\sum_{i \in I} q_{ij} \leq s'_j \leq s_j - 1.$$

Let j_{2l+1} and j_{2l+2} be k and j, respectively. Then for any p with $1 \leq p \leq l+1$,

$$\sum_{i=1}^{m} (q_{ij_{2p-1}} - q_{ij_{2p}})^+ \geq \sum_{i \in I} (q_{ij_{2p-1}} - q_{ij_{2p}})^+$$

$$\begin{aligned} &\geq \sum_{i\in I} q_{ij_{2p-1}} - \sum_{i\in I} q_{ij_{2p}} \\ &\geq s_{j_{2p-1}} - s_{j_{2p}} + 2 \end{aligned}$$

Among $l+1$ pairs of (j_{2p-1}, j_{2p}), at least one pair has the difference less than or equal $\frac{n}{l+1} - 1$, that is $j_{2p} - j_{2p-1} < \lfloor \frac{n}{l+1} \rfloor$. Now we arrive at a contradiction to (7).

Case 2 $s'_k < s_k$:
Let $G = (V, D)$ be the auxiliary digraph for $Q - A$. By an argument similar to Case 1, we will show that there is a directed path from s to t in the digraph G. Assume that there is no directed path from s to t. Let

$$\begin{aligned} I_0 &= \{i \mid \exists \text{ a directed path from } s \text{ to } v_i\} \\ J_0 &= \{j < k \mid \exists \text{ a directed path from } s \text{ to } w_j\} \end{aligned}$$

Then

$$\begin{array}{llll} \forall i \in I_0 & \& & \forall j \in [J_0 \cup \{k\}]^c & q_{ij} - a_{ij} = q_{ij} \\ \forall i \notin I_0 & \& & \forall j \in J_0 \cup \{k\} & q_{ij} - a_{ij} = 0 \end{array}$$

Let $I = I_0^c$. Then

$$\begin{array}{llll} \forall i \notin I & \& & \forall j \in [J_0 \cup \{k\}]^c & a_{ij} = 0 \\ \forall i \in I & \& & \forall j \in J_0 \cup \{k\} & a_{ij} = q_{ij} \end{array}$$

Suppose that $k \leq 2l+1$. Then $J_0 \cup \{k\}$ can be written as either $[1, h] \cup [h_1, h_2] \cup \cdots \cup [h_{2p-1}, h_{2p}]$ or $[h_1, h_2] \cup \cdots \cup [h_{2p-1}, h_{2p}]$ where $p \leq l$. By an argument similar to (*), we arrive at a contradiction to (8) with $J = J_0 \cup \{k\}$.

From now, suppose that $k > 2l+1$.

Claim 3 *There exist v such that $v < k$ and $a_{iv} < q_{iv}$ for some $i \in I$.*

Proof. Assume that there is no such v. Then $a_{ij} = q_{ij}$ for all $i \in I$ and all j ($< k$). Also $a_{ik} = q_{ik}$ for all $i \in I$. By an argument similar to (*), we arrive at a contradiction to (8) with $J = [1, k]$. ∎

Now let v be the last column before column k satisfying the condition in Claim 3. By arguments similar to Claim 1 and 2, we can obtain l pairs $(j_1, j_2), \cdots, (j_{2l-1}, j_{2l})$ of (t, u) satisfying conditions of Claim 1 where $j_1 < j_2 < \cdots < j_{2l} < v$ (in (**) h_{2p+1} and $h_{2(p+1)}$ become $v+1$ and k, respectively). With these l pairs and the pair (v, k), we can obtain a contradiction to (7) in the same way as in Case 1.

The proof of Theorem 4 is completed.

5 Remark

In Chen's result, if there exists a column order satisfying condition (6) then a column order with s_j nonincreasing satisfies the condition as well. Thus we can check easily whether there exists such a column order. But at this moment we don't have a trivial way to check whether there exists a column order satisfying condition (7), and we believe that there is no trivial way to do. Our next goal is to modify condition (7) to be checked easily.

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