An Analysis of the Lines in the Three Dimensional Affine Space Over F_3

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ABSTRACT. Let $A(n,3)$ denote the n-dimensional affine space over the finite field of order three. In this paper, we use basic combinatorial principles to discuss some old and new results about the lines in $A(3,3)$. For $S \subset A(3,3)$, let $\|S\|_3$ and $\|S\|_{3,k}$ respectively denote the number of lines and the number of k-lines of $A(3,3)$ contained entirely in S. For each t, we compute $\alpha_3(t) = \min\{\|S\|_3 : |S| = t\}$ and $\omega_3(t) = \max\{\|S\|_3 : |S| = t\}$. We also give results about $\alpha_{3,k}(t) = \min\{\|S\|_{n,k} : |S| = t\}$, and $\omega_{3,k}(t) = \max\{\|S\|_{n,k} : |S| = t\}$ and results about 1-lines and n-lines in $A(n,3)$.

1 Affine lines and k-lines

Let $A(n,3)$ be the n-dimensional affine space over the finite field of order three. A line ℓ in $A(n,3)$ is a translate of a one dimensional subspace and thus consists of three distinct vectors, $\ell = \{(x_i),(y_i),(z_i)\}$, for which $(x_i)+(y_i)+(z_i) = (0_i)$ and $1 \le i \le n$. Let λ_n denote the number of affine lines in $A(n,3)$. Since every pair of vectors in $A(n,3)$ is on a unique line, it is easy to see that

$$\lambda_n = \frac{\binom{3^n}{2}}{3}.$$

For an integer k with $1 \le k \le n$, a *k-line* in $A(n,3)$ is a line $\ell = \{(x_i),(y_i),(z_i)\}$ in $A(n,3)$ for which $x_i \ne y_i \ne z_i$ for exactly k of the n indices. For example, $\{21011, 22010, 20012\}$ is a 2-line in $A(5,3)$. Let $\lambda_{n,k}$ denote the number of k-lines in $A(n,3)$. Henceforth, we will let x, y, and z denote vectors.

Let $x \in A(n,3)$ and let $V_n(x)$ and $V_{n,k}(x)$ respectively denote the set of lines and k-lines in $A(n,3)$ on the vector x. Because the cardinalities of these sets are respectively invariant, we let $|V_n(x)| = \nu_n$ and $|V_{n,k}(x)| = \nu_{n,k}$. It's not hard to see that $\nu_n = \frac{3^n-1}{2}$ and $\nu_{n,k} = \binom{n}{k}2^{k-1}$. Using $\nu_{n,k}$ one can easily compute that $\lambda_{n,k} = 3^{n-1}\binom{n}{k}2^{k-1}$. The (n,k)th entry in Table 1 is $\lambda_{n,k}$.

$n\backslash k$	1	2	3	4	5
1	1	-	-	-	-
2	6	6	-	-	
3	27	54	36	-	-
4	108	324	432	216	-
5	405	1610	3220	3220	1296

Table 1. $\lambda_{n,k}$

2 Basic results

For $S \in A(n,3)$, let $\|S\|_n$ and $\|S\|_{n,k}$ respectively denote the number of lines and the number of k-lines of $A(n,3)$ contained in S. Let

$$\begin{aligned}\alpha_n(t) &= \min\{\|S\|_n \colon |S| = t\},\\ \omega_n(t) &= \max\{\|S\|_n \colon |S| = t\},\\ \alpha_{n,k}(t) &= \min\{\|S\|_{n,k} \colon |S| = t\}, \text{ and}\\ \omega_{n,k}(t) &= \max\{\|S\|_{n,k} \colon |S| = t\}.\end{aligned}$$

A set $S \subset A(n,3)$ is respectively called *line free* or *k-line free* if $\|S\|_n = 0$ or $\|S\|_{n,k} = 0$. One question is: For a given n, what is the first value of t for which $\alpha_n(t) > 0$? This is a difficult question and the answer is known for only $n \leq 4$. (See [1] and [2].) Let τ_n (respectively $\tau_{n,k}$) denote the first value of t for which $\alpha_n(t) > 0$ (respectively $\alpha_{n,k}(t) > 0$). Then $\tau_2 = 5$, $\tau_3 = 10$, $\tau_4 = 21$.[1] We show below that $\tau_{n,1} = \tau_{n,n} = 2(3^{n-1}) + 1$ and that $\tau_{3,2} = 14$. For $n > 2$, it is interesting to note that $\tau_{n,1} = \tau_{n,n}$ even though $\lambda_{n,n} > \lambda_{n,1}$.

Proposition 1 demonstrates a connection between α_k and ω_k.

Proposition 1. *Let $S \subset A(n,3)$ with $|S| = t$, and let S' denote the complement of S in $A(n,3)$. Then:*

a) $\|S\|_n + \|S'\|_n = \lambda_n - t\frac{3^n-1}{2} + \binom{t}{2}$

b) $\alpha_n(t) + \omega_n(3^n - t) = \lambda_n - t\frac{3^n-1}{2} + \binom{t}{2}$

[1] It is not clear if a proof of this is in print. In [1] p.33, it is claimed without proof that $\tau_4 = 21$, but in [2] p. 205, it is claimed that $\tau_4 \geq 21$.

Proof: It is obvious that $\|S'\|_n = \lambda_n - \left|\bigcup_{x\in S} V_n(x)\right|$ and $\|S'\|_{n,k} = \lambda_{n,k} - \left|\bigcup_{x\in S} V_{n,k}(x)\right|$.

From inclusion-exclusion we have:

$$\begin{aligned}\left|\bigcup_{x\in S} V_n(x)\right| &= \sum_{x\in S} |V_n(x)| - \sum_{x,y\in S} |v_n(x)\cap v_n(y)| \\ &\quad + \sum_{x,y,z\in S} |v_n(x)\cap v_n(y)\cap v_n(z)| \\ &= t\nu_n - \binom{t}{2} + \|S\|_n \\ &= t\frac{3^n-1}{2} - \binom{t}{2} + \|S\|_n .\end{aligned}$$

By rearrangement part a) follows. From a) it is not hard to get b). □

An analogous analysis for k-lines is not as concise; but it does provide some information. For $(x_i), (y_i) \in A(n,3)$, we call the pair $\{(x_i),(y_i)\}$ a *k-pair* if $x_i \neq y_i$ for exactly k of the n indices. The unique line through a k-pair must be a k-line. For $S \subset A(n,3)$, let $D_{n,k}(S)$ be the collection of k-pairs contained entirely in S. By an argument analogous to that in the proof of Proposition 1, we have:

Proposition 2. *For any $S \subset A(n,3)$ with $|S| = t$,*

$$\begin{aligned}\|S\|_{n,k} + \|S'\|_{n,k} &= \lambda_{n,k} - t\binom{n}{k}2^{k-1} + |D_{n,k}(S)| \\ &= (3^{n-1} - t)\binom{n}{k}2^{k-1} + |D_{n,k}(S)| \\ &= (t - 2(3^{n-1}))\binom{n}{k}2^{k-1} + |D_{n,k}(S')|\end{aligned}$$

The latter equality in Proposition 2 comes from interchanging S and S' in the former. The main difference between Propositions 1 and 2 is that the left hand sum $\|S\|_{n,k}+\|S'\|_{n,k}$ in Proposition 2 is not invariant with respect to the cardinality of S. The reason for this is that different sets S_1 and S_2 with the same cardinality can have $|D_{n,k}(S_1)| \neq |D_{n,k}(S_2)|$. However, we have the following easy corollary.

Corollary 2.1. *For each $k \leq n$, $\tau_{n,k} \leq 2(3^{n-1}) + 1$.*

Proof: By Proposition 2, we have that $\|S\|_{n,k} = (t - 2(3^{n-1}))\binom{n}{k}2^{k-1} + |D_{n,k}(S')| - \|S'\|_{n,k}$. So if $|S| > 2(3^{n-1})$, then $\|S\|_{n,k} > 0$ for all k, with $1 \leq k \leq n$, because $|D_{n,k}(S')| - \|S'\|_{n,k} \geq 0$. (Indeed $|D_{n,k}(S')| - \|S'\|_{n,k} \geq 2\|S'\|_{n,k}$.) □

3 1-lines and n-lines in $A(n,3)$

Proposition 3. *For any n, $\tau_{n,1} = \tau_{n,n} = 2(3^{n-1}) + 1$.*

Proof: Since the set of vectors $\{(x_i) \in A(n,3) \colon x_1 = 0 \text{ or } 1\}$ is n-line free, we have by Corollary 2.1 that $\tau_{n,n} = 2(3^{n-1}) + 1$. To complete the proof, we need to demonstrate that for any n, there is a 1-line free $S \subset A(n,3)$ with $|S| = 2(3^{n-1})$.

It is easy to verify $\alpha_{2,1}(6) = 0$, so we have that the result for $n = 2$. We proceed by induction on n. Assume $\alpha_{n-1,1}(2(3^{n-2})) = 0$. Then there is a $S \subset A(n,3)$ with $|S| = 2(3^{n-2})$, and $\|S\|_{n-1,1} = 0$. Identify S with the subset $S_0 \subset A(n,3)$ defined by $S_0 = \{(x_i) \colon x_1 = 0, (x_{i-1}) \in S \text{ for } 2 \le i \le k\}$. Consider the translates of S_0 in $A(n,3)$ defined by $S_1 = S_0 + (1,0,\ldots,0,1)$, and $S_2 = S_0 + (2,0,\ldots,0,2)$. Let $S^* = S_0 \cup S_1 \cup S_2$. Then S^* has cardinality $2(3^{n-1})$, and we claim that S^* is 1-line free in $A(n,3)$.

Let $\{(x_i),(y_i),(z_i)\}$ be a line in S^*. It is easy to see that if $\{(x_i),(y_i),(z_i)\}$ is contained in S_0, S_1, or S_2, then $\{(x_i),(y_i),(z_i)\}$ is not a 1-line because S_0, S_1, and S_2 are all translates of a 1-line free set S. Therefore $\{(x_i),(y_i),(z_i)\}$ must intersect each of the sets S_0, S_1, and S_2. Without loss of generality, we assume $(x_i) \in S_0$, $(y_i) \in S_1$, and $(z_i) \in S_2$. Then there are (y_i'), and (z_i') in S_0 for which $(y_i) = (y_i') + (1,0,\ldots 0,1)$ and $(z_i) = (z_i') + (2,0,\ldots 0,2)$. Since $\{(x_i),(y_i),(z_i)\}$ is a line, then either $\{(x_i),(y_i'),(z_i')\}$ is a line in S_0, or $(x_i) = (y_i') = (z_i')$. If $(x_i) = (y_i') = (z_i')$ it is clear that $\{(x_i),(y_i),(z_i)\}$ is a 2-line in $A(n,3)$. If $\{(x_i),(y_i'),(z_i')\}$ is a line in S_0, then $\{(x_i),(y_i'),(z_i')\}$ is not a 1-line in S_0 because S_0 is 1-line free. It follows that (x_i) and (y_i') are different in (at least) two indices, i_1 and i_2, neither of which are equal to 1, because $x_1 = y_1' = 0$. So $x_{i_1} \ne y_{i_1}'$ and $x_{i_2} \ne y_{i_2}'$. Without loss of generality, we can assume $2 \le i_1 \le n-1$. It now follows that $\{(x_i),(y_i),(z_i)\}$ is not a 1-line since $x_1 \ne y_1$ and $x_{i_1} \ne y_{i_1}$. □

It is interesting to note that although there are sets $S \subset A(n,3)$ of cardinality $2(3^{n-1})$ that are 1-line or n-line free, the argument in the proof of Corollary 2.1 actually shows that $\alpha_{n,1}(2(3^{n-1}) + 1) \ge n$ and $\alpha_{n,n}(2(3^{n-1}) + 1) \ge 2^{n-1}$.

4 Lines and k-lines in $A(2,3)$ and $A(3,3)$

From Proposition 3, we have that $\tau_{2,1} = \tau_{2,2} = 7$. Using Proposition 1, we can generate Table 2. Observe that $\tau_2 = 5$.

$\alpha_2(0)=0$	$\omega_2(0)=0$	$\alpha_2(5)=1$	$\omega_2(5)=2$
$\alpha_2(1)=0$	$\omega_2(1)=0$	$\alpha_2(6)=2$	$\omega_2(6)=3$
$\alpha_2(2)=0$	$\omega_2(2)=0$	$\alpha_2(7)=5$	$\omega_2(7)=5$
$\alpha_2(3)=0$	$\omega_2(3)=1$	$\alpha_2(8)=8$	$\omega_2(8)=8$
$\alpha_2(4)=0$	$\omega_2(4)=1$	$\alpha_2(9)=12$	$\omega_2(9)=12$

Table 2

Consider the figures below. Label each box using the row letter and column number. Box $a2$ is marked with an $\star$.

	1	2	3
a	210 211 212	010 $\star$ 011 012	110 111 112
b	200 201 202	000 001 002	100 101 102
c	220 221 222	020 021 022	120 121 122

Figure 3

If we have three non-zero vectors x, y, and z such $y \neq 2x$ and $z \neq 2x+2y$, then we can form Figure 4 .

	1	2	3
a	z $2x+y+z$ $x+2y+z$	$x+z$ $y+z$ $2x+2y+z$	$2x+z$ $x+y+z$ $2y+z$
b	000 $2x+y$ $x+2y$	x y $2x+2y$	$2x$ $x+y$ $2y$
c	$2z$ $2x+y+2z$ $x+2y+2z$	$x+2z$ $y+2z$ $2x+2y+2z$	$2x+2z$ $x+y+2z$ $2y+2z$

Figure 4

For each figure, it is straightforward to verify that: 1) every vector in $A(3,3)$ appears exactly once, 2) every box contains a line in $A(3,3)$, and 3) every column, row and diagonal is a plane in $A(3,3)$. Moreover, if we take any three boxes such that no two come from the same row or column, then the nine vectors contained in those three boxes form a plane in $A(3,3)$.

A plane in $A(n,3)$ is a vector translate of 2-dimensional subspace. So given any set $S \subset A(3,3)$ of t vectors contained in a plane, then $\alpha_2(t) \leq \|S\|_3 \leq \omega_2(t)$ because a vector translate of a line is a line. We make use of this observation below.

Proposition 4. $\alpha_3(11) = 3$.

Proof: Suppose we have a set S of 11 vectors in $A(3,3)$. We want to show that $\|S\|_3 \geq 3$. First, we show that S is not line free. As is Figure 4, there must be vectors x and y in S. If $2x+2y$ is in S, then S is not line free. Suppose $2x+2y$ is not in S. Consider the following pairs of boxes from Figure 4: $\{a1, c3\}$, $\{a2, c2\}$, $\{b1, b3\}$, $\{a2, c2\}$. By the pigeon-hole principle, there must be a pair whose union contains three of the nine vectors in $S - \{x, y\}$. Without loss of generality, suppose three vectors are contained in the union of $a1$ and $c3$. Then there are five vectors in the union of $a1$, $b2$, and $c3$. Since this union forms a plane, and $\alpha_2(5) = 1$, we must have a line in S. Since the translated set $z+S$ contains three lines if and only if S does, then we can assume, without loss of generality, that S contains a line of the form $\{x, y, 2x+2y\}$ as in Figure 4.

Now, if S contains less than three lines, we show below that there is no way to distribute the remaining eight vectors in $S - \{x, y, 2x+2y\}$ among the boxes in Figure 4. Since the box $b2$ is full, at most one more box can be full; otherwise, S would contain three lines. We have two cases:

Case 1. One box other than $b2$ is full: Without loss of generality, we can assume that $a1$ is full. Since five vectors remain to be distributed, and no additional boxes can be full, there must be a box that contains only one vector. Without loss of generality, we can assume that box $a2$ contains exactly one vector from S. From here, it follows that boxes $a3$, $c2$, and $c3$ must be empty; because, if any one of them contains a vector, then S must contain three lines. Now there are four vectors left to be distributed and three boxes in which to distribute them; so one box must contain two vectors. Without loss of generality, we can assume that box $b1$ contains two vectors. Figure 5 depicts how the elements of S are distributed in the boxes of Figure 4 thus far.

	1	2	3
a	3	1	0
b	2	3	?
c	?	0	0

Figure 5

Now, no matter how the remaining two vectors are distributed, it follows that S must contain at least three lines.

Case 2. No box other than $b2$ is full: It is straightforward to verify that if each of the remaining boxes contains one vector, then S must contain three (indeed five) lines. So, without loss of generality, we can assume that $a1$ contains exactly two vectors. From here it follows that $c3$ must be empty; for if not, then S contains at least three lines. If each of the remaining boxes contains one vector, then S must contain at least three (indeed four) lines. This is because each of intersections $S \cap \{b1, b2, b3\}$, $S \cap \{a2, b2, c2\}$, and $S \cap \{a3, b2, c1\}$ must then contain exactly two lines. Figure 6 depicts this distribution.

2	1	1
1	3	1
1	1	0

Figure 6

Thus one of the remaining boxes must contain two vectors. Without loss of generality , suppose it is $a2$. Then box $c2$ must be empty; because if not, then S would contain three lines. The distribution of the elements thus far is depicted in Figure 7.

2	2	?
1	3	?
?	0	0

Figure 7

From here, it is not hard to see that, no matter how the remaining three vectors are distributed, S must contain three lines. Thus $\alpha_3(11) \geq 3$.

To finish the proof, we have to show that $\alpha_3(11) \leq 3$. Let $|S| = 11$. From Proposition 1a, we have that $\|S\|_3 + \|S'\|_3 = 29$. If S' is the set of vectors in parenthesis in Figure 8 (which is a copy of Figure 3), then S' contains at least (indeed exactly) 26 lines. Thus $\|S\|_3 \leq 3$. Hence $\alpha_3(11) = 3$. □

(210) **(211)** **(212)**	010 011 012	110 111 112
(200) **(201)** **(202)**	**(000)** 001 002	100 101 102
(220) **(221)** **(222)**	**(020)** **(021)** **(022)**	**(120)** **(121)** **(122)**

Figure 8

Corollary 4.1.

a) $\alpha_3(10) = 2$.

b) $\tau_3 = 10$.

c) $\alpha_3(12) = 4$.

d) $\alpha_3(13) = 7$.

Proof: a) Suppose $\alpha_3(10) \leq 1$. Then there is a set of vectors S in $A(3,3)$ with $|S| = 10$ and $\|S\|_3 \leq 1$. Let $x \in S'$ and consider $S \cup \{x\}$. By Proposition 4, it follows that $\|S \cup \{x\}\|_3 \geq 3$. We have two cases:

Case 1. $\|S\|_3 = 0$: It follows that x must be on at least three lines contained in $S \cup \{x\}$. Since this must be true for every $x \in S'$, it follows that there are $3|S'| = 51$ distinct lines in $A(3,3)$ that contain a pair of vectors from S. This is a contradiction because every pair of vectors in $A(3,3)$ defines a unique line, but S only has 45 distinct pairs.

Case 2. $\|S\|_3 = 1$: Let ℓ be the one line contained in S. It follows that x must be on at least two lines contained in $(S - \ell) \cup \{x\}$. Since this must be true for every $x \in S'$, it follows that there are $2|S'| = 34$ distinct lines that contain a pair of points from $(S - \ell)$. This is a contradiction because any pair of vectors defines a unique line and $(S - \ell)$ only has 21 pairs. □

b) It is easy to construct a line free collection of nine vectors in $A(3,3)$. □

c) Since $\alpha_3(11) = 3$, it follows that $\alpha_3(12) \geq 4$. If not, there is an S with $|S| = 12$ and $\|S\|_3 = 3$. Let ℓ be one of the lines contained in S and let $x \in \ell$. Then $|S - \{x\}| = 11$ and $\|S - \{x\}\|_3 \leq 2$. This contradicts $\alpha_3(11) = 3$.

Also $\alpha_3(12) \leq 4$. Let S' be the set of vectors in parenthesis in Figure 9. Then $\|S'\|_3 = 23$. From Proposition 1a, we have that $\|S\|_3 + \|S'\|_3 = 27$. Therefore $|S| = 12$ and $\|S\|_3 = 4$. Hence $\alpha_3(12) = 4$. □

(210)	010	110
(211)	011	111
(212)	012	112
(200)	000	100
(201)	001	101
(202)	002	102
(220)	(020)	(120)
(221)	(021)	(121)
(222)	(022)	(122)

Figure 9

d) By an argument similar to that in c), it is easy to see that $\alpha_3(13) \geq 5$. Let $|S| = 13$. If S' is the set of vectors in parenthesis in Figure 10, then $\|S'\|_3 = 19$. By Proposition 1a, it follows that $\alpha_3(13) \leq 7$. We now show that either $\alpha_3(13) = 5$ or $\alpha_3(13) = 6$ leads to a contradiction.

210	010	110
(211)	011	111
(212)	012	112
(200)	000	100
(201)	001	101
(202)	002	102
(220)	(020)	(120)
(221)	(021)	(121)
(222)	(022)	(122)

Figure 10

If $\alpha_3(13) = 5$, then there is an $S \subset A(3,3)$ with $|S| = 13$ and $\|S\|_3 = 5$. Then there must be an $x \in S$ that is contained in two of the five lines in S. Thus $\|S - \{x\}\|_3 \leq 3$. This contradicts $\alpha_3(12) = 4$.

If $\alpha_3(13) = 6$, then there is an S with $|S| = 13$ and $\|S\|_3 = 6$. Then there must be two vectors $x, y \in S$ that are contained in the union of four of the six lines in S. It follows that $\|S - \{x,y\}\|_3 \leq 2$. This contradicts $\alpha_3(11) = 3$. □

Table 5 gives the values of $\alpha\ell3(t)$ and $\omega_3(t)$ for all t with $0 \leq t \leq 27$. The values can be obtained using Propositions 1 and 4, and Corollary 4.1.

t	$\alpha_3(t)$	$\omega_3(t)$	t	$\alpha_3(t)$	$\omega_3(t)$	t	$\alpha_3(t)$	$\omega_3(t)$	t	$\alpha_3(t)$	$\omega_3(t)$
0	0	0	7	0	5	14	10	19	21	51	54
1	0	0	8	0	8	15	13	23	22	60	62
2	0	0	9	0	12	16	16	26	23	70	71
3	0	1	10	2	12	17	20	30	24	80	81
4	0	1	11	3	13	18	24	36	25	92	92
5	0	2	12	4	14	19	33	41	26	104	104
6	0	3	13	7	16	20	42	47	27	117	117

Table 5

Proposition 5. $\tau_{3,2} = 14$.

Proof (sketch): Consider the set of 13 vectors in parenthesis in Figure 12. Since it is 2-line free, then $\tau_{3,2} \geq 14$.

221 (202) (210)	(121) 102 (110)	(021) (002) 010
100 (111) (122)	(000) 011 (022)	(200) 211 (222)
012 020 (001)	212 220 022	112 120 101

Figure 12

To see that $\tau_{3,2} \leq 14$, we need to show that every set of 14 vectors in $A(3,3)$ contains a 2-line. By discussing a series of technical claims, we outline the main ideas. It is left to the reader to use the claims to construct the final argument.

Claim 1. If S is a set of at least 13 vectors in $A(3,3)$, then S contains either a 1-line or a 2-line.

Proof: By the pigeon-hole principle, we can assume, without loss of generality, that five of the vectors in S have their first coordinate equal to zero. Since $\tau_2 = 5$, it follows that there must be either a 1-line or a 2-line of $A(3,3)$ among those five vectors. □

Claim 2. If S is a set of at least 14 vectors in $A(3,3)$ and S contains a 1-line ℓ, then there is a plane P in $A(3,3)$ that contains a subset T of S with $|T| = 7$ and $\ell \subset T$.

Proof: Since the translate of a k-line is a k-line, we can assume, without loss of generality, that $\ell = \{x, y, 2x + 2y\}$ as in Figure 4. It is easy to see that no matter how the remaining 11 vectors in S are distributed, that there must be plane P that satisfies the claim. □

Claim 3. If seven vectors are contained in a plane P in $A(3,3)$ and for some i, the ith entry every of vector in P is the same, (e.g., the second entry of every vector in P is 1) , then those seven vectors contain a 2-line.

Proof: This follows from $\tau_{2,2} = 7$. □

Claim 4. If seven vectors are contained in a plane P in $A(3,3)$ and there is a 1-line contained in those seven vectors, then there is also a 2-line contained in those seven vectors.

Proof: If for some i, the ith entry of every vector in P is the same, then apply Claim 3. If not, then for each i, the number of vectors in P that have a 0 as their ith entry, and the number of vectors in P that have a 1 as their ith entry, and the number of vectors in P that have 2 as their ith entry, are all exactly three.

Let $\{x_1, x_2, x_3\}$ be a 1-line in P and let $\{y_1, y_2, y_3\}$ and $\{z_1, z_2, z_3\}$ be distinct lines in P which are parallel to $\{x_1, x_2, x_3\}$. Then $\{y_1, y_2, y_3\}$ and $\{z_1, z_2, z_3\}$ are also parallel. Moreover, since the (affine) planes in $A(3,3)$ have the Euclidean property, it follows that both $\{y_1, y_2, y_3\}$ and $\{z_1, z_2, z_3\}$ are translates of $\{x_1, x_2, x_3\}$. Thus $\{y_1, y_2, y_3\}$ and $\{z_1, z_2, z_3\}$ are also 1-lines. Consider the plane translate $P' = P + 2x_1$. Then $P' = \{0, x', 2x'\} \cup \{y'_1, y'_2, y'_3\} \cup \{z'_1, z'_2, z'_3\}$ (where 0 is the vector 000). It follows that $\{0, x', 2x'\}$, $\{y'_1, y'_2, y'_3\}$, and $\{z'_1, z'_2, z'_3\}$ are 1-lines and that P contains a 2-line if and only if P' does.

Now since $\{0, x', 2x'\}$ is a 1-line, it must be either $\{000, 001, 002\}$, $\{000, 010, 020\}$, or $\{000, 100, 200\}$. Suppose it is $\{000, 001, 002\}$. Since, as for P, the number of vectors in P' that have a 0 (or a 1, or a 2) as their ith entry is exactly three, it follows that $P' = \{000, 001, 002\} \cup \{120, 121, 122\} \cup \{210, 211, 212\}$ or $P' = \{000, 001, 002\} \cup \{110, 111, 112\} \cup \{220, 221, 222\}$. One can verify directly, that any seven points in one of these latter two planes must contain a 2-line. Like $\{000, 001, 002\}$, each of the other two possible values for $\{0, x', 2x'\}$ gives two possible outcomes for P'. One can argue these cases by symmetry. □

References

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