Further Results on Minimal Rankings

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Abstract

Let G be a graph. A function $f : V(G) \to \{1, 2, \ldots, k\}$ is a k-ranking for G if $f(u) = f(v)$ implies that every $u - v$ path P contains a vertex w such that $f(w) > f(u)$. A function $f : V(G) \to \{1, 2, \ldots, k\}$ is a minimal k-ranking if f is a k-ranking and for any x such that $f(x) > 1$ the function $g(z) = f(z)$ for $z \neq x$ and $1 \leq g(x) < f(x)$ is not a k-ranking. This paper establishes further properties of minimal rankings, gives a procedure for constructing minimal rankings, and determines, for some classes of graphs, the minimum value and maximum value of k for which G has a minimal k-ranking. In addition we establish tighter bounds for the minimum value of k for which G has a k-ranking.

1. Introduction

For this paper we will assume we have a graph $G = (V, E)$ with $|V| = p$. The minimum and maximum degree of a vertex in G will be denoted by $\delta(G)$ and $\Delta(G)$, respectively. A function $f : V(G) \to \{1, 2, \ldots, k\}$ is a **proper vertex coloring (coloring)** if $f(u) = f(v)$ implies $(u, v) \notin E(G)$. The **chromatic number** of G, denoted $\chi(G)$, is the minimum value of k for which G has a proper vertex coloring. A **complete k-coloring** is a proper vertex coloring into classes $C_1, \ldots, C_k$ such that for every i, j where $1 \leq i < j \leq k$ there exists a vertices $v_i \in C_i$ and $v_j \in C_j$ such that $(v_i, v_j) \in E$. Note that every coloring of G using $\chi(G)$ colors is necessarily a $\chi(G)$-complete coloring. The **achromatic number** of G, denoted $\psi(G)$, is the maximum value of k for which G has a complete k-coloring [9].For other concepts not explicitly defined here, see [8].

Given a graph, $G = (V, E)$, a function $f : V(G) \to \{1, 2, \ldots, k\}$ is a **k-ranking** for G if $f(u) = f(v)$ implies that every $u - v$ path P contains a vertex w such that $f(w) > f(u)$. If we are not concerned about the actual value of k then f is referred to as a **ranking** of G. The concept of a ranking is studied, among other places, in [2] [5], [7].

From the definition of a ranking it is clear that every ranking is also a proper vertex coloring. It is also easy to see that, for any graph, assigning the vertices with different labels produces a ranking. This being the case it is natural to look at the minimum value of k for which a graph has a k-ranking. The **rank number** of G, denoted $\chi_r(G)$, is defined as:

$$\chi_r(G) = min\{k\text{: G has a } k\text{-ranking}\} \tag{1}$$

Given a graph $G = (V, E)$, a function $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ is a **minimal k-ranking** if the following two conditions hold:

1. f is a k-ranking
2. for all $x \in V(G)$ such that $f(x) > 1$ the function $g : V(G) \rightarrow \{1, 2, \ldots, k\}$ defined by $g(z) = f(z)$ for $z \neq x$ and $g(x) < f(x)$ is not a ranking.

Minimal rankings are introduced in [7]. As with rankings, we will often refer to a minimal k-ranking as simply a **minimal ranking** when the value of k is unimportant. Moreover, a (minimal) k-ranking of G will be called a **(minimal) $\chi_r(G)$-ranking** if $k = \chi_r(G)$. If there is no ambiguity over the graph with a ranking then this can be shortened to a **(minimal) χ_r-ranking.** For any vertex v, we refer to $f(v)$ as the **label of** v. If $f(v) = f(w)$ implies $v = w$ then $f(v)$ is a **distinct label.** Otherwise $f(v)$ is a **repeated label.**

In [7] it is observed that:

$$\chi_r(G) = min\{k\text{: G has a minimal } k\text{-ranking}\} \tag{2}$$

The authors then define the **arank number** of G, as follows:

$$\psi_r(G) = max\{k\text{: G has a minimal } k\text{-ranking}\}. \tag{3}$$

The examples in Figure 1 which illustrate the difference between rankings and minimal rankings are given in [7].

A ranking is given in (a) while (b) is an example of a χ_r-ranking which is not minimal. Example (c) depicts a minimal χ_r-ranking while (d) illustrates a minimal ranking which is not a χ_r-ranking.

The problem of finding the rank number of a graph has received a lot of attention recently because of the growing number of applications such as Cholesky factorizations of matrices in parallel [1], [6], [13], VLSI layout [12], [15], and scheduling problems of assembly steps in manufacturing systems [4], [11]. Although it is unknown if there are applications for the arank number, the concept of minimal rankings is, as we shall soon see, a valuable tool for studying the rank number of a graph.

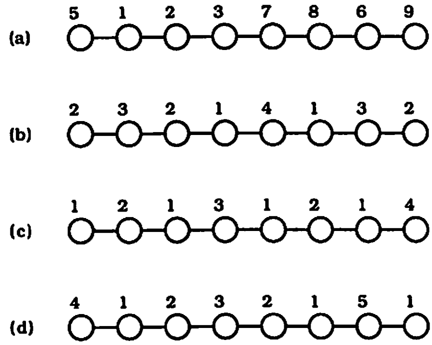


Figure 1: Different types of rankings

2. Background

Central to the idea of a ranking is the concept of a reduction. This is introduced in [7] as follows: Given a graph $G = (V, E)$ and a subset $S \subset V$ define a graph $G^* = (V-S, E^*)$ where $(u,v) \in E^*$ if and only if $u, v \in V-S$ and either $(u,v) \in E$ or there exists a path $u - w_1 - w_2 - \ldots - w_m - v$ in G where $w_i \in S$ for $1 \leq i \leq m$. We say that the graph $G^* = (V - S, E^*)$ is **the reduction of G by S** and denote this by the more compact notation G^*_S. An example of a reduction is given in Figure 2.

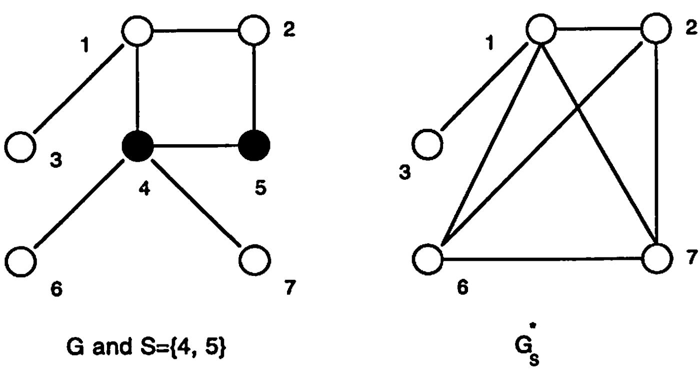


Figure 2: An example of a reduction.

Domination is an important part of our study of the rank number and arank number. If $G = (V, E)$ is a graph, a set $S \subseteq V$ is an **independent set** if $x, y \in S$ implies $(x,y) \notin E$. A set $S \subseteq V$ is a **dominating set** if for each vertex $y \in V - S$ there exists a vertex $x \in S$ such that $(x,y) \in E$. If $x \in V$ then $N(x) = \{y : (x,y) \in E\}$ and $N[x] = N(x) \cup \{x\}$. A set

$S \subseteq V$ is an **independent dominating set** if S is an independent set and a dominating set. The **independent domination number** of G, denoted $i(G)$, is the minimum cardinality of an independent dominating set for G. Finally, the **independence number** of G, denoted $\beta(G)$ is the maximum cardinality of an independent set for G. An independent dominating set containing $i(G)$ elements is called an $i(G)$**-set**. Similarly for the other parameters. Domination parameters are extensively studied, for example [10].

Before we begin our study of rankings we present some background that our paper will rely on. The following bounds for the rank number are immediate, where $\chi(G)$ is the **chromatic number** of G.

Lemma 1 *For any graph G, $\chi(G) \leq \chi_r(G) \leq p - \beta(G) + 1$.*

Our next Lemma is another immediate result which is cited in papers on rankings [5], [7].

Lemma 2 *If G is a connected graph then there exists a unique vertex with largest label.*

The following observation is made in [5].

Lemma 3 *If H is a subgraph of G then $\chi_r(H) \leq \chi_r(G)$.*

A **cograph** is defined recursively in [3] as follows:

1. A graph on a single vertex is a cograph.

2. If $G_1, G_2, \ldots, G_k$ are cographs then so is their union $G_1 \cup G_2 \ldots \cup G_k$.

3. If G is a cograph, then so is its complement $\overline{G}$.

In [3], the next result is established.

Theorem 1 *[3] Let $G = (V, E)$ be a graph. The following are equivalent.*

1. G is a cograph.

2. G does not contain P_4 as an induced subgraph.

3. every connected subgraph of G has diameter less than or equal to two.

3. Properties of Rankings

Our work on minimal rankings will rely on many results that have already been proven. Some properties of rankings and reductions established in [7] are given below.

1. If H is an induced subgraph of G then $\psi_r(H) \leq \psi_r(G)$.

2. $\psi_r(G) = p$ if and only if $\Delta(G) = p - 1$.

3. A minimal k-ranking is an onto function.

4. If f is a minimal k-ranking and $S_i = \{x : f(x) = i\}$ for $1 \leq i \leq k$ then $| S_1 | \geq | S_2 | \geq \ldots \geq | S_k |$.

5. In a ranking of G, the set R of vertices having a label that is repeated is a dominating set for G.

6. Let $G = (V, E)$ be a graph. For $S \subset V$, if A is an independent set of vertices in G_S^* then A is an independent set of vertices in G.

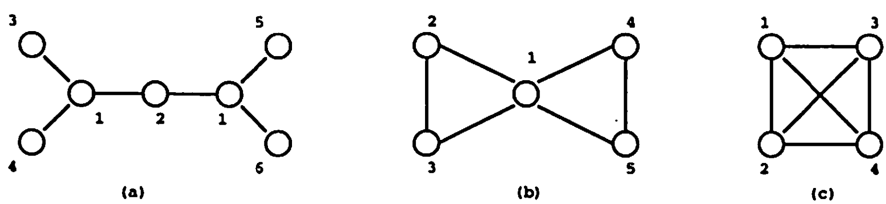
7. Let G be a graph. Let f be a minimal χ_r-ranking of G; if $S_1 = \{x : f(x) = 1\}$ then $\chi_r(G_{S_1}^*) = \chi_r(G) - 1$.

8. Let f be a minimal ψ_r-ranking of G. If $S_1 = \{x : f(x) = 1\}$ then $\psi_r(G_{S_1}^*) = \psi_r(G) - 1$.

9. $\chi_r(G) \geq 1 + \delta(G)$ and $\psi_r(G) \geq 1 + \Delta(G)$.

When refering to result j on the list above we will cite Property j.

One of the main results in [7] is that if G is a graph and f is a ranking of G then, if $S_1 = \{x : f(x) = 1\}$, the function $g(x) = f(x) - 1$ is a ranking for $G_{S_1}^*$. Indeed, we are able to say more. If the original ranking, f, is a (minimal) $\chi_r(G)$-ranking then g will be a minimal $\chi_r(G_{S_1}^*)$-ranking. If the original ranking is a minimal $\psi_r(G)$-ranking, as illustrated in Figure 3 (a), then we can produce a minimal $\psi_r(G_{S_1}^*)$-ranking by subtracting one from each label of the remaining vertices. This process can be repeated, as shown in (c), and is the essence of Property 7 and Property 8.



4. Further Properties of Minimal Rankings

We now establish several more properties of rankings; the following theorem presents a useful characterization of minimal rankings.

Theorem 2 *A k-ranking f is minimal if and only if for all u with $f(u) = i > 1$, for each j such that $1 \leq j < i$ either*

1. *there exist vertices x and y with $f(x) = f(y) \geq j$ and u is the only vertex on some $x - y$ path such that $f(u) > f(y)$,* *or*

2. *there exists a vertex w with $f(w) = j < i$ and there exists a $u - w$ path such that every vertex x on the path has $f(x) \leq f(w)$.*

Before proving this result we will first illustrate Theorem 2 with the example shown below.

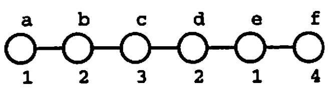


Figure 4: An illustration of Theorem 2.

To check whether the ranking above is minimal, one merely determines whether changing an existing label to a smaller value still results in a ranking. If the answer is no then the ranking is minimal. In this case, if any label is made smaller, then at least one of the two conditions of Theorem 2 is violated. Suppose, for example, vertex c in Figure 4 is assigned the label 1 (or 2) then vertices b and d have label 2, which is greater than or equal to 1 (or 2), and there is a path between them for which c is the only vertex with a label greater than 2. This is the first condition. The second condition implies that the vertex f cannot be labelled 1, 2, or 3 because in all cases there exists a path from the newly labelled vertex to another vertex with the same label through vertices with lower labels. Since we can, by inspection, see that no label can be made smaller and still have a ranking, the figure above is minimally ranked.

Proof: Suppose f is a minimal k-ranking of a graph G and let $x \in V(G)$ such that $f(x) > 1$. Define a function g on G by $g(z) = f(z)$ for $z \neq x$ and $g(x) = j < f(x) = i$. Since f is a minimal ranking, g is not a ranking and so there exist vertices u and v such that $g(u) = g(v)$ and there exists a $u - v$ path P_{uv} such that for all $w \in P_{uv}, g(w) \leq g(u)$. Now $g(z) = f(z)$ for $z \neq x$ implies $x \in P_{uv}$; otherwise f is not a ranking.

There are two cases to consider:

Case 1 Suppose x is an endpoint of P_{uv}; say $x = u$. Since $g(w) \leq g(x) = g(v)$ for all $w \in P_{uv}$ we have, by the way g is defined, that $f(w) \leq f(v) = g(x) < f(x)$. Hence Hence 2 holds.

Case 2 Suppose x is an interior point of P_{uv}. Then P_{uv} is a path where $f(u) = f(v)$ and since $g(w) \leq g(u) = g(v) = f(u) = f(v)$, condition 1 holds.

To prove the converse let f be any ranking of a graph G satisfying the conditions 1 and 2. Let $x \in V$ such that $f(x) > 1$ and define $g(z) = f(z)$ for $z \neq x$ and $1 \leq g(x) < f(x)$. Now $g(x) < f(x)$ so, by hypothesis, one of the following must hold.

Case 1 There exist distinct vertices u and v such that $f(u) = f(v) \geq g(x)$ and x is the only vertex on some $u-v$ path, P, such that $f(x) > f(u)$. Then from the way g is defined, $g(u) = g(v)$ and $g(w) \leq g(u)$ for all $w \in V(P)$; so g is not a ranking.

Case 2 There exists a vertex w with $f(w) = g(x)$ and there exists a $w-x$ path such that every vertex $z(\neq x)$ on the path has $f(z) \leq f(w)$. Since $g(w) = f(w)$ and $g(x) = f(w)$ it follows that for every vertex on the $w-x$ path, $g(z) = f(z) \leq f(w) = g(w)$. Hence g is not a ranking.

In all cases g is not a ranking so f is a minimal ranking. □

Lemma 4 *Let G be a graph and let f be a minimal k-ranking of G. If $u \in V(G)$ such that $f(u) > 1$ then there exists a $w \in N(x)$ such that $f(w) < f(x)$.*

Proof: If $k = 1$ then the result is vacuously true, so suppose $k \geq 2$. Then there exists an $x \in V(G)$ such that $f(x) > 1$ and since f is a minimal ranking, by Theorem 2 there are two cases to consider:

Case 1 There exist vertices x and y with $f(x) = f(y)$ and u is the only vertex on some $x-y$ path such that $f(u) > f(y)$. This implies that the vertices adjacent to u on this $x-y$ path have a label no bigger than $f(x)$ which is less than $f(u)$.

Case 2 There exists a vertex w with $f(w) = j$ and there exists a $u-w$ path such that for every vertex x on the path $f(x) \leq f(w)$. Since $f(w) < f(u)$, the neighbor of u on the $u-w$ path has label less than $f(u)$.

In both cases there exists a $y \in N(x)$ such that $f(y) < f(x)$. □

Corollary 1 *Let G be a graph and let f be a minimal k-ranking of G. If $x \in V(G)$ such that $f(x) = 2$ then there exists a $y \in N(x)$ such that $f(y) = 1$.*

Lemma 5 *Let G be a graph and let f be a ranking of G. Suppose x and y are two nonadjacent vertices such that $f(x) = f(y)$ and $P_1, P_2, \ldots, P_n$ are n internally disjoint $x-y$ paths. Let $m_i = max\{f(x) : x \in P_i\}$. If $m_i = m_j$ then $i = j$.*

Proof: Suppose $m_i = m_j$ and let $P_i = x - s_1 - s_2 - \ldots - s_k - y$ and $P_j = x - s_1' - s_2' - \ldots - s_r' - y$. Let s_a and s_b' be the largest labelled vertices on P_i and P_j respectively; i.e., $f(s_a) = m_i$ and $f(s_b) = m_j$. Then G contains the path $P = s_a - \ldots - s_1 - x - s_1' - s_2' - \ldots - s_b'$. Since $f(s_a) = f(s_b')$ and f is a ranking there exists a $v \in P$ such that $f(v) > f(s_a)$, contradicting the definition of s_a or s_b', unless $i = j$. □

This allows us to generalize Lemma 2.

Corollary 2 *If G is an n-connected graph and f is a ranking of G then the n largest labels are distinct.*

Lemma 6 *Let G be a graph and suppose f is a minimal k-ranking of G. If $S = \{x : f(x) > j\}$ where $1 \leq j < k$ and C is a connected component of $< V - S >$, the induced subgraph of $V - S$, then f_C, the restriction of f to C, is a minimal ranking of C.*

Proof: Suppose not; then there exists a maximal connected component C of $< V - S >$, a ranking g_C of C, and a $w \in V(C)$ such that $g_C(w) < f_C(w)$ and for all $x \in V(C) - \{w\}$, $g_C(x) = f_C(x)$. Look at the function g defined on G by $g(x) = f(x)$ if $x \neq w$ and $g(w) = g_C(w) < f_C(w) = f(x)$. Since f is a minimal ranking, g is not a ranking. Observe that g can be described by $g(x) = g_C(x)$ if $x \in V(C)$ and $g(x) = f(x)$ if $x \notin V(C)$. Since g is not a ranking there exists a u, v such that $g(u) = g(v)$ and a path P having the property that for all $y \in V(P)$, $g(y) \leq g(u)$. Now $g(x) = f(x)$ if $x \neq w$ and f a ranking implies $w \in V(P)$. From the alternate, but equal, way g is defined, $w \in V(C)$ and $g_C(u) = g_C(v)$ and $g_C(y) \leq g_C(u)$ for all $y \in V(P)$. Therefore g_C is not a ranking. This contradiction establishes the result. □

5. Constructing Minimal Rankings

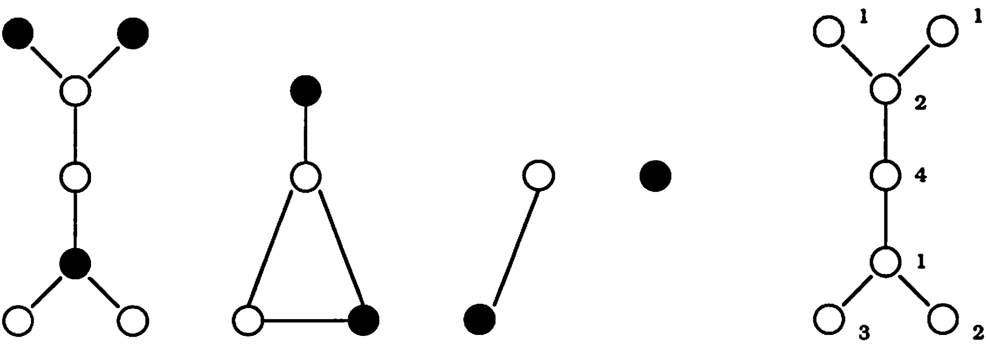
The next theorem provides a useful procedure for constructing minimal rankings for an arbitrary graph.

Theorem 3 *Given a graph, G, a minimal ranking can be constructed by the following procedure: Take an independent dominating set A_1 for G, then find an independent dominating set A_2 for $G^*_{A_1}$. Continue this process until a set A_k of isolates is obtained, which is an independent dominating set for $(((G^*_{A_1})^*_{A_2})\cdots)^*_{A_{k-1}} = G^*_{A_1 \cup A_2 \cup \ldots \cup A_{k-1}}$. Define the function f on G by $f(A_i) = i$ for $1 \leq i \leq k$. Then f is a minimal k-ranking of G.*

Proof: Let u and v be vertices such that $f(u) = f(v) = i > 1$. Since u and v are independent in $G^*_{A_1 \cup A_2 \cup \ldots \cup A_{i-1}}$, there does not exist a path between u and v with internal vertices entirely within $A_1 \cup A_2 \cup \ldots \cup A_{i-1}$. Moreover, u and v are independent in G by Property 6, hence f is a ranking.

To show that f is minimal let $x \in V(G)$ such that $f(x) > 1$ and suppose $g(z) = f(z)$ $[z \neq x]$ and $1 \leq g(x) = k < f(x)$. Look at $G^*_{A_1 \cup A_2 \cup \cdots \cup A_{k-1}}$, where $A_i = \{v : f(v) = i\}$; since A_k is an independent dominating set for $G^*_{A_1 \cup A_2 \cup \cdots \cup A_{k-1}}$ we know x is adjacent to some y of A_k. That is, $(x, y) \in E(G^*_{A_1 \cup A_2 \cup \cdots \cup A_{k-1}})$. This implies $(x, y) \in E(G)$ or there exists a path from x to y through $A_1 \cup A_2 \cup \cdots \cup A_{k-1}$. Since $g(x) = g(y)$ we have, in either case, that g is not a ranking. □

Figure 5 illustrates the process of constructing a minimal ranking using the theorem above.



(a) (b) (c) (d) (e)

Figure 5: A way to construct minimal rankings.

We start with the graph in (a), find an independent dominating set (which has been blackened). This is our set A_1. We then look at $G^*_{A_1}$, which is the graph in (b), and find an independent dominating set A_2. We again blacken our set A_2 and look at the graph $(G^*_{A_1})^*_{A_2}$ given in (c) and continue this process till we reach (d). At this point we now form a minimal ranking of G by labelling the vertices of A_i with the label i, this is shown in (e).

Not all minimal rankings are obtained by this process as Figure 6 illustrates, hence χ_r and ψ_r cannot necessarily be found utilizing the theorem.

6. Bounds on the Rank and Arank Numbers

For a graph $G = (V, E)$ a subset $F \subseteq E$ is a **matching** for G if no two edges in F have a vertex in common. A **strong matching** in a graph

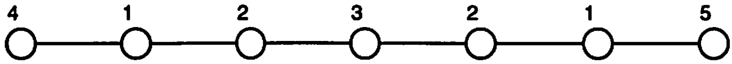


Figure 6: A minimal ranking of a graph where A_1 is not a dominating set.

$G = (V, E)$ is a matching in which no two edges are joined by an edge of G; that is, a strong matching, M, is an induced subgraph whose connected components are disjoint edges $e_1, e_2, \ldots, e_k$ where $k = \mid M \mid$.

$$\beta^*(G) = max\{k : G \text{ has a strong matching of } k \text{ edges}\} \tag{4}$$

$$b^*(G) = min\{k : G \text{ has a maximal strong matching of } k \text{ edges}\} \tag{5}$$

If $\mid E(G) \mid = 0$ then $\beta^*(G) = b^*(G) = 0$.

Let G be a graph and let f be any ranking of G. The set $S_1 = \{x : f(x) = 1\}$ is an independent set and so $\mid S_1 \mid \leq \beta(G)$. If we restrict our attention to minimal rankings we can obtain an upper bound on the number of vertices labelled 2.

Lemma 7 *Let G be a graph and let f be a minimal ranking of G. If $S_2 = \{x : f(x) = 2\}$ then $\mid S_2 \mid \leq \beta^*(G)$.*

Proof: Let $y_1, y_2, \ldots, y_k$ denote the vertices labelled 2. By Corollary 1, each y_i is adjacent to a vertex x_i labelled 1. Also, two distinct vertices y_i, y_j must be adjacent to two distinct nonadjacent vertices x_i and x_j, otherwise this would imply a path with labels $2 - 1 - 2$. Thus the set $\{(x_i, y_i)\}_{i=1}^{k}$ is a strong matching, hence $\mid S_2 \mid \leq \beta^*(G)$. □

We will use the idea of matchings to establish many bounds for χ_r and ψ_r. The following is an alternate upper bound for the rank number of a graph.

Theorem 4 *If G is a graph on p vertices then $\chi_r(G) \leq p - 2\beta^*(G) + 2$.*

Proof: Label the vertices of each edge in a $\beta^*(G)$-set with labels 1 and 2, and label the rest of the remaining $p - 2\beta^*(G)$ vertices with the labels 3 through $p - 2\beta^*(G) + 2$. If $u, v \in V$ such that $f(u) = f(v)$ then either $f(u) = 1$ or $f(u) = 2$. Since the set of vertices labelled 1 and the set of vertices labelled 2 are independent, the only way f can fail to be a ranking is if there is a path with labels $2 - 1 - 2$. This cannot happen because the vertices labelled 1 and 2 occur in a strong matching. □

For a graph G, depending on how $\beta^*(G)$ compares with $\beta(G)$, the upper bound given above may or may not be better than the one in Lemma 1. An example where $p - 2\beta^*(G) + 2$ provides a better bound than $p - \beta(G) + 1$

is given by the graph G in Figure 7. In this graph we have $p = 9, \beta^*(G) = 4, b^*(G) = 1$ and $\beta(G) = 4$. Thus $p - 2\beta^*(G) + 2 = 9 - 8 + 2 = 3$ while $p - \beta(G) + 1 = 9 - 4 + 1 = 5$. This bound is sharp for Figure 7 since $3 = \chi(G) \leq \chi_r(G) \leq 3$.

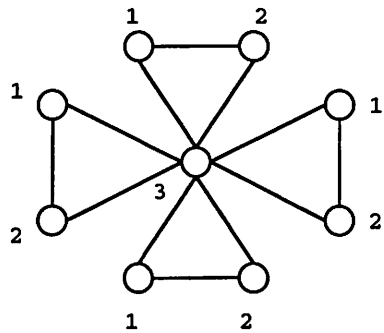


Figure 7: A graph where $\chi_r(G) \leq p - 2\beta^*(G) + 2$ provides a better estimate of the rank number than $\chi_r(G) \leq p - \beta(G) + 1$.

Using the idea of matchings we obtain the following result.

Theorem 5 *If G is a graph on p vertices such that $\beta^*(G) = 1$ then $\chi_r(G) = p - \beta(G) + 1$ and $\psi_r(G) = p - i(G) + 1$.*

Proof: Since $\beta^*(G) = 1$ and f is a minimal k-ranking we have by Lemma 7 and Property 4 that $| S_2 | = \ldots = | S_k | = 1$. Now use the fact that $p = | S_1 | + \ldots + | S_k |$ to conclude that $p = | S_1 | + k - 1$ or, equivalently, $k = p - | S_1 | + 1$. Now $\chi_r(G) = min\{k : G$ has a minimal k-ranking$\}$ implies $\chi_r(G) = min\{p - | S_1 | + 1 : f$ is a minimal ranking and $S_1 = \{x : f(x) = 1\}\}$. Since $| S_1 | \leq \beta(G)$ we have by Theorem 3 that $\chi_r(G) = p - \beta(G) + 1$. Likewise, $\psi_r(G) = max\{k : G$ has a minimal k-ranking$\}$ implies $\psi_r(G) = max\{p - | S_1 | + 1 : f$ is a minimal ranking and $S_1 = \{x : f(x) = 1\}\}$. Observe that $i(G) = 1$ implies $\Delta(G) = p - 1$ and so by Property 2, $\psi_r(G) = p = p - i(G) + 1$. Moreover $i(G) > 1$ implies $\psi_r(G) < p$ and so $| S_1 | > 1$. Then S_1 is the set of vertices whose labels are repeated, hence S_1 is a dominating set by Property 5. Since S_1 is independent as well, $i(G) \leq | S_1 |$. This implies $\psi_r(G) \leq p - i(G) + 1$ and then Theorem 3 allows us to conclude $\psi_r(G) = p - i(G) + 1$. □

Note that if $\beta^*(G) = 0$ then G is a collection of isolates and the result still holds. This will help establish the next corollary.

Corollary 3 *If G is the complement of a C_4-free graph then $\chi_r(G) = p - \beta(G) + 1$ and $\psi_r(G) = p - i(G) + 1$.*

Proof: It suffices to show that if G is the complement of a C_4-free then $\beta^*(G) \leq 1$. We establish the contrapositive. Suppose $\beta^*(G) > 1$ and let $(x_1, y_1), (x_2, y_2)$ be two edges in a strong matching. Therefore $(x_1, x_2), (x_2, y_1), (x_1, y_2)$, and (y_1, y_2) are not edges in G. This implies that the complement of G contains an induced C_4, establishing the result. □

Corollary 4 follows because a chordal graph is C_4-free while Corollary 5 and Corollary 6 follow because these graphs have $\beta^*(G) \leq 1$.

Corollary 4 *If G is the complement of a chordal graph then $\chi_r(G) = p - \beta(G) + 1$ and $\psi_r(G) = p - i(G) + 1$.*

Corollary 5 *[7] If G is a split graph on p vertices then $\chi_r(G) = p - \beta(G) + 1$ and $\psi_r(G) = p - i(G) + 1$.*

Corollary 6 *[7] If G is a complete multipartite graph on p vertices then $\chi_r(G) = p - \beta(G) + 1$ and $\psi_r(G) = p - i(G) + 1$.*

One should refer to the much longer proofs of Corollaries 5 and 6 in [7] to appreciate the connection between rankings and matchings.

We now establish many more bounds on χ_r and ψ_r. The following is a well known result/exercise in coloring theory. A proof of this result can be found in [8].

Theorem 6 *For any graph G, $\frac{p}{\beta(G)} \leq \chi(G) \leq p - \beta(G) + 1$.*

The idea for the next several theorems as well as their proofs comes from the proof of the lower bound for χ in Theorem 6.

Theorem 7 *For any graph G having at least one edge, $\chi_r(G) \geq \frac{p-\beta(G)}{\beta^*(G)} + 1$.*

Proof: Since G has an edge, $\beta^*(G) > 0$. Now use the fact that $\chi_r(G) = min\{k : G \text{ has a minimal } k\text{-ranking}\}$ and look at any minimal $\chi_r(G)$-ranking of G. By combining Lemma 7 and Property 4 we have that $\beta^*(G) \geq |S_2| \geq \ldots \geq |S_{\chi_r}|$. Now $p = |S_1| + \ldots + |S_{\chi_r}|$ and so it follows that $p - |S_1| \leq (\chi_r(G) - 1)\beta^*(G)$. Since $|S_1| \leq \beta(G)$ we can conclude that $\chi_r(G) \geq \frac{p-|S_1|}{\beta^*(G)} + 1 \geq \frac{p-\beta(G)}{\beta^*(G)} + 1$. □

Notice that Theorem 7 and Lemma 1 combine to prove part of Theorem 5; i.e., if $\beta^*(G) = 1$ then $\chi_r(G) = p - \beta(G) + 1$. It is easy to improve on the lower bound if the connectivity of the graph is known.

Theorem 8 *If G is an n-connected graph on p vertices then $\chi_r(G) \geq \frac{p-\beta(G)-n}{\beta^*(G)} + (n+1)$.*

Proof: Since $\chi_r(G) = min\{k : G \text{ has a minimal } k\text{-ranking}\}$ it suffices to look at any minimal $\chi_r(G)$-ranking of G. We know $| S_1 | + \ldots + | S_{\chi_r - n} | + | S_{\chi_r - n+1} | + \ldots + | S_{\chi_r} | = p$ and by Corollary 2, $| S_{\chi_r-n+1} | = \ldots = | S_{\chi_r} | = 1$. Combining these we get $| S_1 | + \ldots + | S_{\chi_r-n} | = p-n$. Therefore $p-n \leq (\chi_r(G)-n) | S_1 |$ which can be simplified down to $\chi_r(G) \geq n + \frac{p-n}{|S_1|}$. Since $\frac{p-n}{|S_1|} > 0$ and χ_r is an integer it must be that $\chi_r \geq n+1$ or, equivalently, $\chi_r - n + 1 \geq 2$. Thus in an n-connected graph the n largest labels are each greater than one. This allows us to write $p - | S_1 | - | S_{\chi_r-n+1} | - \ldots - | S_{\chi_r} | = | S_2 | + \ldots + | S_{\chi_r-n} |$ which, when combined with $\beta^*(G) \geq | S_2 | \geq \ldots \geq | S_{\chi_r} |$ gives $p - | S_1 | - n \leq \beta^*(G)(\chi_r(G) - n - 2 + 1)$. This can be simplified to $\chi_r(G) \geq \frac{p-|S_1|-n}{\beta^*(G)} + n + 1 \geq \frac{p-\beta(G)-n}{\beta^*(G)} + n + 1$, establishing the result. □

For $\psi_r(G)$, the following lower bounds are obtained.

Theorem 9 *For any graph* G, $\frac{p}{i(G)} \leq \psi_r(G)$.

Proof: Let f be a minimal k-ranking where S_1 is an $i(G)$-set; this is possible by Theorem 3. By Property 4 we have $i(G) = | S_1 | \geq \ldots \geq | S_k |$, hence $p = | S_1 | + \ldots + | S_k | \leq k(i(G))$. Therefore $\psi_r(G) \geq k \geq \frac{p}{i(G)}$.

Theorem 10 *For any graph* G *containing at least one edge,* $\psi_r(G) \geq \frac{p-i(G)}{\beta^*(G)} + 1$.

Proof: By Theorem 3 there exists a minimal k-ranking, f, of a graph G for which $| S_1 | = | \{x : f(x) = 1\} | = i(G)$. Now $p - | S_1 | = | S_2 | + \ldots + | S_k |$ and $\beta^*(G) \geq | S_2 | \geq \ldots \geq | S_k |$ implies that $p - i(G) \leq \beta^*(G)(k-1)$. Therefore $\psi_r(G) \geq k \geq \frac{p-i(G)}{\beta^*(G)} + 1$. □

Theorem 11 *For any* n*-connected graph* G *containing at least one edge,* $\psi_r(G) \geq \frac{p-i(G)-n}{\beta^*(G)} + (n+1)$.

Proof: By Theorem 3 there exists a minimal k-ranking, f, on $V(G)$ for which $| S_1 | = | \{x : f(x) = 1\} | = i(G)$. As in the proof of Theorem 8 we have $p - | S_1 | - | S_{k-n+1} | - \ldots - | S_k | = | S_2 | + \ldots + | S_{k-n} |$. Since G is n-connected and $\beta^*(G) \geq | S_2 | \ldots | S_k |$ it follows that $p - i(G) - n \leq \beta^*(G)(k - n - 2 + 1)$. This simplifies to $k \geq \frac{p-i(G)-n}{\beta^*(G)} + n + 1$ and since $\psi_r(G) = max\{k : G \text{ has a minimal } k\text{-ranking}\}$, $\psi_r(G) \geq \frac{p-i(G)-n}{\beta^*(G)} + (n+1)$. □

This idea is also useful in obtaining a similar lower bound.

Theorem 12 *For any graph,* G, *having at least one edge,* $\psi_r(G) \geq \frac{p-\beta(G)}{b^*(G)} + 1$.

Proof: Consider any $b^*(G)$-set, F, of edges and label the vertices of each edge 1 and 2. Now complete the labelling of G in any way such that the resulting ranking is a minimal k-ranking. By Property 4 we have $|S_2| = b^*(G) \geq |S_i|$ for $i > 2$. As in the previous theorems this means $p - |S_1| \leq b^*(G)(k-1)$ which simplifies to $k \geq \frac{p-|S_1|}{b^*(G)} + 1 \geq \frac{p-\beta(G)}{b^*(G)} + 1$. Hence $\psi_r(G) \geq \frac{p-\beta(G)}{b^*(G)} + 1$. □

We can use the idea of a strong matching to obtain an even stronger bound on the chromatic number.

Theorem 13 *For any graph, G, $\chi(G) \leq p - \beta(G) - \beta^*(G) + 2$.*

Proof: If G contains no edges then $\beta^*(G) = 0$ and the result is obviously true by Lemma 1. Therefore we suppose G contains at least one edge, hence $\beta^*(G) \geq 1$. Let S_1 be any $\beta(G)$-set and label the vertices of S_1 with 1. Consider any strong matching, F, containing $\beta^*(G)$ edges. Since S_1 is an independent set each edge of G has at most one vertex in S_1. From each edge of F, therefore, we can label one vertex not in S_1 with the label 2; call this set labelled 2 S_2. It is easy to see that S_2 is independent and $|S_2| = \beta^*(G)$. Now label the remaining $p - \beta(G) - \beta^*(G)$ vertices with 3 through $p - \beta(G) - \beta^*(G) + 2$. □

We note that Figure 7 is also an example where this upper bound improves upon the previous bound of $p - \beta(G) + 1$.

7. ψ_r-rankings of cographs

It is known how to determine the rank number of a cograph in linear time [14]. We will determine a formula for the arank number of a cograph with the help of the next theorem.

Theorem 14 *If S is an independent dominating set for a graph G and $\cap_{s\in S} N(s) \neq \emptyset$ then $\psi_r(G) \geq p - |S| + 1$.*

Proof: We will construct a minimal $(p - |S| + 1)$-ranking from which the result follows immediately. Let $i = |\cap_{s\in S} N(s)| > 0$. Label the vertices in S with 1 and the vertices in $\cap_{s\in S} N(s)$ with the labels 2 through $i+1$, where w is the vertex labelled 2. Finally, label the remaining $(p - |S| - i)$ vertices with $i+2$ through $(p - |S| - i) + (i+1) = p - |S| + 1$. Since 1 is the only repeated label and S is an independent set we have a ranking. Thus we need only show that the ranking is minimal. Let x be any vertex which is not labelled 1. Now S is an independent dominating set so x cannot be reassigned the label 1 and still be a ranking. If x is assigned the label 2 then x is adjacent to some vertex z in S. In this case, $x - z - w$ is a path labelled $2 - 1 - 2$, which cannot happen in a ranking. Finally, suppose x is assigned a smaller label which is greater than 2; then there exists another vertex y with the same label. Since S is a dominating set there exists an $s_1 \in S$ and an $s_2 \in S$ such that x is adjacent to s_1 and y is adjacent to s_2. Then under this new ranking g, the path $x - s_1 - w - s_2 - y$ has the property that $g(x) = g(y)$ and since $g(s_1) = g(s_2) = 1$ and $g(w) = 2$ we have that g is not a ranking. Since this x was arbitrary, the constructed ranking is minimal. Therefore $\chi_r(G) \leq p - \mid S \mid +1 \leq \psi_r(G)$. □

A graph, G, is **cobipartite** if the complement of G is bipartite.

Corollary 7 *If G is a connected cobipartite graph then $\psi_r(G) = p$ if $i(G) = 1$ and $\psi_r(G) = p - 1$ if $i(G) > 1$.*

Proof: For any graph G, if $i(G) = 1$ then $\Delta(G) = p - 1$ hence, by Property 2, $\psi_r(G) = p$. Therefore let G be a connected cobipartite graph with $i(G) > 1$. It is easy to see that G contains an independent dominating set $\{x, y\}$ with the property that $N(x) \cap N(y) \neq \emptyset$. By Theorem 14, $\psi_r(G) \geq p - 2 + 1$. Thus, by Property 2, $\psi_r(G) = p - 1$. □

Figure 6 shows a minimal ranking of P_7 in which S_1 is not a dominating set. However, the following theorem asserts that if a graph is P_5-free then S_1 is an independent dominating set for the minimal ranking.

Theorem 15 *If G is P_5-free and f is a minimal ranking of G then $S_1 = \{x : f(x) = 1\}$ is an independent dominating set for G.*

Proof: The proof is by induction on χ_r. If $\chi_r(G) = 1$ then the result is clearly true so suppose the claim is true for all graphs where $\chi_r \leq n$. Let G be any graph where $\chi_r(G) = n + 1$ and let f be a minimal $\chi_r(G)$-ranking. Let $S_{\chi_r} = \{x : f(x) = \chi_r\}$; since G is not necessarily connected the cardinality of S_{χ_r} is at least one. Now $H = < V(G) - S_{\chi_r} >$ is a P_5-free graph and f_H, the restriction of f to H, is a minimal $\chi_r(G) - 1$-ranking by Lemma 6. By the inductive hypothesis the set of vertices labelled 1 under f_H is a dominating set for H. Since the set of vertices labelled 1 under f_H is equal to $S_1 = \{x : f(x) = 1\}$ the set S_1 necessarily dominates all the vertices of G with the possible exception of vertices in S_{χ_r}. Take any vertex $u \in S_{\chi_r}$, since f is a minimal ranking of G, u cannot be reassigned the label 1 and still be a ranking. This implies, by the second condition of Theorem 2, that either u is adjacent to a vertex labelled 1, in which case there is nothing to prove, or the first condition of Theorem 2 holds. In this case there exist vertices $x, y \in V(G)$ with $f(x) = f(y)$ and u is the only vertex on some $x - y$ path such that $f(u) > f(y)$. Thus there is a path having labels $1 - f(x) - f(u) - f(y) - 1$. Since G is P_5-free and f is a ranking, the only possible edges must be from a vertex labelled 1 to the vertex u. □

Corollary 8 *If G is a cograph and f is a minimal ranking of G then $S_1 = \{x : f(x) = 1\}$ is an independent dominating set for G.*

Theorem 16 *If G is a connected cograph and S is an independent dominating set then $\cap_{s \in S} N(s) \neq \emptyset$.*

Proof: We proceed by induction on the cardinality of S. If $| S |= 1$ then the result is vacuously true. If $| S |= 2$ then, since connected cographs have diameter less than or equal to 2, the two vertices must share a common neighbor. Now suppose the result is true for all connected cographs with $| S |\leq n$. Look at any G for which $3 \leq| S |= n+1$. Label the elements of S by $\{s_1, s_2, \ldots, s_{n+1}\}$. Let $H_1 =< \cup_{i=1}^{n} N(s_i) >$, since $N(s_i) \cap N(s_j) \neq \emptyset$ if $i \neq j$, H_1 is a connected cograph and $\{s_1, s_2, \ldots, s_n\}$ is an independent dominating set for H_1 containing n elements. Likewise $H_2 =< \cup_{i=2}^{n+1} N(s_i) >$ is a connected cograph and $S - \{s_1\}$ is an independent dominating set for H_2 containing n elements. By the inductive hypothesis there exists a $z_2 \in \cap_{i=2}^{n+1} N(s_i)$. Since G is connected we have that $s_1 - z_1 - s_2 - z_2$ is a path. Now G is P_4 free and s_1 and s_2 are independent vertices so either $(s_1, z_2) \in E(G)$ or $(z_1, z_2) \in E(G)$. If $(s_1, z_2) \in E(G)$ then $z_2 \in \cap_{i=1}^{n+1} N(s_i)$ and the theorem is true; so suppose $(z_1, z_2) \in E(G)$. Then $s_1 - z_1 - z_2 - s_{n+1}$ is a path in G different from the previous one since $| S |\geq 3$. By an argument similar to the one above either $(z_1, s_{n+1}) \in E(G)$ or $(s_1, z_2) \in E(G)$. This implies either $z_1 \in \cap_{i=1}^{n+1} N(s_i)$ or $z_2 \in \cap_{i=1}^{n+1} N(s_i)$. In either case, $\cap_{i=1}^{n+1} N(s_i) \neq \emptyset$. □

Corollary 9 *If G is a connected cograph then $\psi_r(G) = p - i(G) + 1$.*

Proof: By Corollary 8, S_1 must be an independent dominating set for G. Therefore $| S_1 |\geq i(G)$ and so $\psi_r(G) \leq p - i(G) + 1$. By Theorems 16 and 14, $\psi_r(G) \geq p - i(G) + 1$, hence the result. □

8. Conclusion and Open Problems

We have seen that there is an analog between coloring theory and rankings. We have also established a connection between strong matchings and minimal rankings. The following are some open problems:

1. Is there an upper bound on $\psi_r(G)$ involving $\beta^*(G)$ so that when $\beta^*(G) = 1, \psi_r(G) = p - i(G) + 1$?

2. We know that for any graph G, $\chi_r(G) \geq \chi(G)$. If G is connected, is $\psi_r(G) \geq \psi(G)$?

3. Any algorithmic results for $\psi_r(G)$; especially trees and chordal graphs.

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References

[1] H.L. Bodlaender, J.R. Gilbert, H. Hafsteinsson, T. Kloks, Approximating Treewidth, Pathwidth and Minimum Elimination Tree Height, *Proc. 17th International Workshop on Graph Theoretic Concepts in Computer Science WG'91*. Springer -Verlag, Lecture Notes in Computer Science 570, (1992) 1-12.

[2] H.L. Bodlaender, J.S. Deogun, K. Jansen, T. Kloks, D. Kratsch, H. Muller, and Zs. Tuza, Rankings of Graphs, *Proceedings of WG' 94*, (eds. E.W. Mayr, G. Schmidt, and G. Tinhofer), Springer-Verlag, Berlin, LCNS 903. (1995) 292-304.

[3] D. G. Corneil, H. Lerchs, L. Stewart Burlingham, Complement Reducible Graphs, *Discrete Applied Mathematics*, **3**(1981), 163-174.

[4] P. de la Torre, R. Greenlaw and A.A. Schaeffer, Optimal ranking of trees in polynomial time, *Proc. 4th ACM Symp. on Discrete Algorithms*, Austin, Texas (1993) 138-144.

[5] J.S. Deogun, T. Kloks, D. Kratsch, and H. Muller, On Vertex Ranking for Permutation and Other Graphs, *Proceedings of the 11th Annual Symposium on Theoretical Aspects of Computer Science*, (eds. P. Enjalbert, E. W. Mayr, and K. W. Wagner) Lecture Notes in Computer Science 775, Springer Verlag, Berlin, (1994) 747-758.

[6] I.S. Duff, J.K. Reid. The multifrontal solution of indefinite sparse symmetric linear equations, *ACM Transactions on Mathematical Software*, **9** (1983) 302-325.

[7] J. Ghoshal, R. Laskar, D. Pillone, Minimal Rankings, *Networks*, **28**(1996), 45-53.

[8] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, New York, 1969.

[9] F. Harary, S.T. Hedetniemi. The Achromatic Number of a Graph, *Journal of Combinatorial Theory*, **8**(1970) 154-161.

[10] S.T. Hedetniemi and R. Laskar, Editors, *Annals of Discrete Mathematics: Topics on Domination 48*, New York, North Holland, 1991.

[11] A.V. Iyer, H.D. Ratliff, G. Vijayan, On edge ranking problems of trees and graphs, *Discrete Applied Mathematics*, **30** (1991) 43-52.

[12] C.E. Leiserson, Area efficient graph layouts for VLSI, *Proc. 21st Ann. IEEE Symp. FOCS*, (1980) 270-281.

[13] J.W.H. Liu, The role of elimination trees in sparse factorization, *SIAM Journal of Matrix Analysis and Applications*, **11** (1990) 134-172.

[14] P. Scheffler, Node Ranking and Search on Graphs (Abstract), in: U. Faighe and C. Hoede *Third Twente Workshop on Graphs and Combinatorial Optimization*, Memorandum No. 1132, (May 1993). Faculty of Applied Mathematics, University of Twente, The Netherlands.

[15] Arunabha Sen, Haiyong Deng, Sumanta Guha, On a graph partition problem with application to VLSI layout, *Information Processing Letters*, **43**(1992) 87-94.