ON 2-$(v,3)$ STEINER TRADES OF SMALL VOLUMES

G.B. KHOSROVSHAHI
Institute for Studies in
Theoretical Physics and Mathematics (IPM), and
The University of Tehran, Iran.
Email: rezagbk@zagros.ipm.ac.ir

H.R. MAIMANI
Institute for Studies in
Theoretical Physics and Mathematics (IPM), and
The University of Tehran, Iran.

Abstract

We consider all 2-$(v,3)$ trades in which every pair appears at most once in each part of the trade, and we call them Steiner Triple Trades $STT(v)$. We completely classify $STT(v)$ with $6 \leq \text{vol}(T) \leq 9$.

1. Introduction and Background Materials

For given v, k, and t, let $X = \{1, 2, \cdots, v\}$ and let $P_i(X)$ denote the set of all i-subsets of X, $2 \leq i \leq v$. The elements of $P_k(X)$ are called blocks.

A t-(v,k,λ) *design* is a collection of blocks in which every element of $P_t(X)$ is contained in exactly λ blocks. A t-(v,k) *trade* $T = (T_1, T_2)$, consists of two disjoint collections of blocks T_1 and T_2 such that for every $B \in P_t(X)$, the number of blocks containing B in T_1 is the same as the number of blocks containing B in T_2.

The number of blocks in $T_1(T_2)$ is called the *volume* of the trade T, and is denoted by $vol(T)$.

Clearly a t-(v,k) trade is also an i-(v,k) trade for $0 \leq i \leq t$. Therefore, in a trade T, both T_1 and T_2 must be based on the same subset of the elements of X. This subset is called the *foundation* of the trade T, and is denoted by $found\,(T)$.

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From here on, the set $\{x_1, x_2, \cdots, x_i\} \in P_i(X)$ will be denoted by $x_1x_2\cdots x_i$.

We call a t-(v,k) trade T a *minimal trade* if $\mathrm{vol}(T) = 2^t$, and $|\mathrm{found}(T)| = k+t+1$.

Let

$$r_x = \text{number of occurrences of } x \in \mathrm{found}(T) \text{ in } T_1(T_2).$$

$$\lambda_{T_1(xy)} = \lambda_{T_2(xy)} = \lambda_{xy} = \text{number of occurrences of } xy \text{ in } T_1(T_2).$$

$$E(i) = \{x | x \in \mathrm{found}(T), r_x = i\}.$$

Trades are very important and complicated combinatorial objects, and can be viewed as building blocks of t-(v,k,λ) designs. Their importance stems from the fact that the difference of any two t-(v,k,λ) designs is a trade [1,3]. If all the trades of all possible volumes could be catalogued, then the problem of existence of t-designs would become much more tractable [3,4].

It has been shown that for $t = 2$, trades of volume equal to 1,2,3, and 5 do not exist [2]. Therefore, in this paper for $k = 3$ and $\lambda_{xy} = 1$, we study and classify all of these types of trades with volume ranging from 6 to 9. The reason for this particular study is the fact that most of combinatorial structures of small orders behave usually rather peculiarly.

2. Some Useful Lemmas

Throughout, $T = (T_1, T_2)$ will be considered as a 2-$(v,3)$ trade, and for every pair $xy, \lambda_{xy} = 1$ or 0. In fact each part of this kind of a trade is a partial Steiner system. In this paper trades of this sort, T, with $\mathrm{vol}(T) = s$, and $|\mathrm{found}(T)| = f$ will be denoted by $TS(f,s)$.

In this section we mention some simple lemmas which are used in the remaining parts of the paper.

Lemma 1. For any $x \in \mathrm{found}(T), r_x \geq 2$.

Lemma 2. Let $T = (T_1, T_2)$ be a $TS(f,s)$, then $\frac{2}{3}f \leq s \leq \frac{f(f-1)}{6}$.

Proof. Since $2f \leq \sum_{x \in \mathrm{found}(T)} r_x = 3s$, the left bound is attained. The right bound follows by counting the pairs of the set $\{(x,B) | x \in B, B \in T_1\}$ in two ways. □

From the above lemma we conclude that

$$\begin{aligned} \mathrm{vol}(T) = 6, &\quad |\mathrm{found}(T)| \in \{7,8,9\}, \\ \mathrm{vol}(T) = 7, &\quad |\mathrm{found}(T)| \in \{7,8,9,10\}, \\ \mathrm{vol}(T) = 8, &\quad |\mathrm{found}(T)| \in \{8,9,10,11,12\}, \\ \mathrm{vol}(T) = 9, &\quad |\mathrm{found}(T)| \in \{8,9,10,11,12,13\}. \end{aligned}$$

Lemma 3. If $xyz \in T_1$ such that r_x, r_y, and r_z are greater than 3, then $\mathrm{vol}(T) \geq 7$.

Proof. This follows from the fact that $\lambda_{xy} = \lambda_{xy} = \lambda_{yz} = 1$. □

Lemma 4. Let abc, $ade \in T_1$ and $r_a = r_b = r_d = r_e = 2$. Then, a minimal trade is embedded in the structure of T.

Proof. Suppose that $r_a = r_b = r_d = r_e = 2$. Also let

$$abc, ade, bdx, cey \in T_1, \qquad abd, ace, bcx', dey' \in T_2,$$

where $x, y, x', y' \in \mathrm{found}(T)$. Since $r_b = r_d = 2$, then $x = x' = y'$, and since $r_e = 2$, then $y' = y$, and clearly there is a minimal trade in the structure of T. The other case can be established similarly.

Lemma 5. If T and T' are two trades, then $T - T'$ is also a trade.

Lemma 6. If $T = (T_1, T_2)$ and $T' = (T_1', T_2')$ are two trades with volumes s, and s', respectively, such that $T' \subseteq T$(i.e.,$T_1' \subseteq T_1, T_2' \subseteq T_2$), then $T - T'$ is a trade with volume $s - s'$.

Lemma 7. Let $x \in \mathrm{found}(T)$, and $r_x = m$, then $\mathrm{vol}(T) \geq 2m$, and $|\mathrm{found}(T)| \geq 2m + 1$.

Proof. Let $C = \{y | \lambda_{xy} = 1\}$. Clearly $C = \cup_{x \in B \in T_1} B - \{x\}$ and $C \cup \{x\} \subseteq \mathrm{found}(T)$. Consequently, we have $|C \cup \{x\}| \geq 2m + 1$. □

Let $x, y \in \text{found}(T)$, and $r_x = 3, r_y = 2$. We define
$F_x = \{(x,z) | z \in E(2), \lambda_{xz} = 1\}$,
and $D_y = \{(z,y) | z \in E(3), \lambda_{zy} = 1\}$. Then, clearly,

$$\sum_{x \in E(3)} |F_x| = \sum_{y \in E(2)} |D_y|. \qquad (1)$$

3. Classifying $TS(f, s)$ of Small Volumes

In this section, we completely classify the $TS(f, s)$ for $6 \le s \le 9$. Our argument is organized in 3 parts: nonexistence of some trades, uniqueness of some others, and finally existence of those trades which are not unique.

Lemma 8. $TS(8,7), TS(10,7), TS(12,9), TS(13,9), TS(9,6)$, and $TS(8,9)$ do not exist.

Proof. In the cases $TS(9,6)$, $TS(12,9)$, $TS(13,9)$, and $TS(10,7)$, there are two blocks B_1 and B_2 such that $|B_1 \cap B_2| = 1$, and four elements of $B_1 \cup B_2$ are in $E(2)$. So by Lemmas 4, and 5 we have a trade with volume 4 in the structure of these trades, and we obtain a contradiction by Lemma 6.

In the case $TS(8,9)$, we have $E(4) = \emptyset$, hence at most $8 \times 3 = 24$ elements can appear in this trade, but there are $9 \times 3 = 27$ spaces to be filled, therefore we have no trade in this case.

Suppose there exists a $TS(8,7)$. So we have $|E(3)| = 5$, and $|E(2)| = 3$. Clearly, there exists a block $xyz \in T_1$ such that $r_x = r_y = r_z = 3$. With no loss of generality, let that block be 123. So the trade T can possibly be of the following form:

$$\begin{aligned} T = (T_1, T_2) = (&\{123, 145, 167, 24x, 36y, 57z, uvw\}, \\ &\{124, 136, 157, 23x', 45y', 67z', u'v'w'\}) \end{aligned}$$

There is an element $s \in \{4,5,6,7\}$ such that $r_s = 2$ (since $|E(3)| = 5$). Suppose that $r_4 = 2$. Since $r_2 = r_3 = 3$, we have $z = 2$, or $z = 3$. Let $z = 2$, then $u = 3$. Clearly $x = y' \in \{6,8\}$. If $x = y' = 6$, then there exists a $TS(8,6)$ in the structure of this trade, and we obtain a contradiction by Lemma 6. Hence, $x = y' = 8$. Since $\lambda_{T_2(58)} = 1$, then $\lambda_{T_1(58)} = 1$. Since, the only choice for this pair is vw, hence $uvw = 358$.

Since $\lambda_{T_1(28)} = \lambda_{T_2(38)} = 1$, then $x' = 8$, and this is a contradiction. For $z = 3$, with a similar argument, again we obtain a contradiction. Hence this case does not offer any trade. □

Now we discuss about trades which are unique.

Proposition 9. Let T be a $TS(8,6)$ and $x, y \in \text{found}(T)$. If $r_x = r_y = 3$, then $\lambda_{xy} = 0$.

Proof. If $\lambda_{xy} = 1$, then there is a block $x'y'z' \in T_1$ such that $r_{x'} = r_{y'} = r_{z'} = 2$. Another block which contains x' (or y', or z') is $x'xw$ (or $yx'w$) where $r_w = 2$. From Lemma 4, there is a minimal trade in the structure of this trade, and by Lemmas 5 and 6, there is a trade of volume 2, and this is a contradiction. □

Lemma 10. The $TS(8,6)$, $TS(7,6)$, $TS(9,7)$, $TS(7,7)$, $TS(8,8)$, and $TS(12,8)$ are unique up to isomorphism.

Proof. $\boldsymbol{TS(7,6)}$. In this case $|E(2)| = 3$, and $|E(3)| = 4$. If $x \in \text{found}(T)$ and $r_x = 3$, then for each $y \in \text{found}(T)$, $\lambda_{xy} = 1$, and for each $x, y \in E(2)$, $\lambda_{xy} = 0$. Let $1 \in \text{found}(T)$, and $r_1 = 3$. This trade can be expressed as

$$T = (T_1, T_2) = (\{123, 145, 167, 24x, 36y, 57z\},\\ \{124, 136, 157, 23x', 45y', 67z'\}).$$

Each block contains only one element of $E(2)$. Suppose these elements are $2, 5$, and 6. Therefore, $x = x' = 7$, $y = z' = 4$, and $z = y' = 3$. Hence the trade is constructed.

$\boldsymbol{TS(8,6)}$. By Proposition 9, we assume that

$$T = (T_1, T_2) = (\{134, 156, 178, 35x, 47y, 68z\},\\ \{135, 147, 168, 34x', 56y', 78z'\}).$$

Since $\{3, 4, 5, 6, 7, 8\} \subseteq E(2)$, hence $x = x' = y = y' = z = z' = 2$.

$\boldsymbol{TS(7,7)}$. In this case, $|E(3)| = 7$, and for any $x, y \in \text{found}(T)$, $\lambda_{xy} = 1$. We can assume that T_1 is of the following unique form

$$T_1 = \{123, 145, 167, 246, 257, 356, 347\}.$$

But then T_2 can be of one of the followings:

$T_2 = \{124, 136, 157, 237, 256, 345, 467\}$, or $T_2 = \{124, 136, 157, 235, 267, 347, 456\}$.

So, these two trades are in fact isomorphic. ($\sigma = (24)(35)(67)$.) Hence we are left with only one trade.

***TS*(9, 7).** Here we have $|E(3)| = 3$, and $|E(2)| = 6$. Let $r_1 = 3$. The blocks of a possible trade T can be arranged as follows:

$$T = (T_1, T_2) = (\{123, 145, 167, 24x, 36y, 57z, uvw\},\\ \{124, 136, 157, 23x', 45y', 67z', u'v'w'\}).$$

One of the pairs 24, 36, and 57 is a subset of $E(2)$. Suppose that $24 \subseteq E(2)$. So $r_2 = r_4 = 2$. Therefore, $x = x' = y'$, and the only choices for these are 8, or 9. Let $x = x' = y' = 8$, then $u' = z'$. If $y = 8$, then $\lambda_{T_1(68)} = 1$ implies that $\lambda_{T_2(68)} = 1$. Therefore, $u'w' = 68$, and we conclude that the pair 69 appears in T_2 twice. This is a contradiction. Thus, $y \neq 8$. With a similar argument $z \neq 8$, hence we have $uv = 35$. So $w = 8$, $y = z = 9$, and $v'w' = 35$, and again we obtain a unique trade.

***TS*(8, 8).** Here, $|E(3)| = 8$. Suppose that the blocks of T are as follows:

$$T = (T_1, T_2) = (\{123, 145, 167, 24x, 36y, 57z, uvw, rst\},\\ \{124, 136, 157, 23x', 45y', 67z', u'v'w', r's't'\}).$$

Since $8 \in E(3)$, then one of the elements of $\{x, y, z\}$, and $\{x', y', z'\}$ is 8. Let $x = 8$, hence x' and $y' \neq 8$, and $z' = 8$. So $uv = 68$, $rs = 78$, $u'v' = 28$, $r's' = 48$. Clearly $z = 2$, $y' = 6$, $y = 4$, $x' = 7$, $w' = 5$, $t' = 3$, $w = 5$, and $t = 3$, hence only one trade can be obtained.

***TS*(12, 8).** For any $x \in \text{found}(T)$, $r_x = 2$. So this trade is a difference of two minimal trades with disjoint foundation sets. □

From here on we will denote the elements 10, 11, and 12 by A, B, and C, respectively.

Now we study $T = TS(9, 8)$.

Proposition 11. Suppose that there exists a $TS(9,8)$, and $r_x \leq 3$ for any $x \in \text{found}(T)$. Then there exists at least one $x \in E(3)$ such that $|F_x| = 1$.

Proof. Since $|E(3)| = 6$, so $|F_x| \geq 1$ for any $x \in E(3)$. Suppose that for any $x \in E(3)$, $|F_x| \geq 2$. Hence $\sum_{x \in E(3)} |F_x| \geq 12$. Since for any $y \in E(2)$, $|D_y| \leq 4$, therefore, $|D_y| = 4$. Let $E(2) = \{7,8,9\}$, and $E(3) = \{1,2,\cdots,6\}$. We can suppose that $D_7 = \{(1,7),(2,7),(3,7),(4,7)\}$, $D_8 = \{(1,8),(2,8),(5,8),(6,8)\}$, and $D_9 = \{(3,9),(4,9),(5,9),(6,9)\}$. Also we have

$$127, 347, 158, 268, 359, 469 \in T_1$$

Thus, the two other blocks of T_1 are 245 and 136. Hence, we can not construct T_2 of T.□

Lemma 12. There is a unique $TS(9,8)$ if for any $x \in \text{found}(T)$, $r_x \leq 3$.

Proof. By Proposition 11, suppose that $1 \in E(3)$, and $F_1 = \{(1,7)\}$. So the blocks of a $TS(9,8)$ can be expressed as follows:

$$T = (T_1, T_2) = (\{123, 145, 167, 24x, 36y, 57z, rst, uvw\},$$
$$\{124, 136, 157, 23x', 45y', 67z', r's't', u'v'w'\}).$$

$z = z'$ since $7 \in E(2)$. We have only three choices for z, namely 2, 8, and 9.

If $z = z' = 2$, then $x = 6, x' = 5$, and $y \in \{8,9\}$. We can assume that $y = 8$. So $r = u = 9$, and $s = 8$. Since $\lambda_{T_2(35)} = 1$, then $vw = 35$, and then $t = 4$. Therefore, $y' = 9$, $r's't' = 389$, and $u'v'w' = 468$. Hence the trade is constructed.

If $z = z' = 8$, then $\lambda_{T_2(68)} = 1$. So $y = 8$ or $rs = 68$. Since $5, 6 \in E(3)$, then $y, y' \neq 8$, so $rs = 68$. Therefore, $t = 2$, or 3 (since $2, 3 \in E(3)$). Let $t = 2$. Since $\lambda_{T_1(26)} = \lambda_{T_1(28)} = 1$, then $x' = 8$, $uv = 38$, and $u'v' = 26$. Clearly we conclude that $x = 9$, and hence $w' = 9$, so $y = 9$. But we have no choice for 9 in the blocks of T_2, and we obtain a contradiction. Hence, we are left with only one trade. □

Suppose that there exists a $TS(f,s)$, and for $x \in \text{found}(T)$, we have $r_x = 4$. Then the blocks of two parts of this trade which contain the element x have two nonisomorphic forms as follows:

$$(*) \qquad xbc, xde, xfg, xhl \in T_1, \quad xbd, xce, xfh, xgl \in T_2,$$
$$(**) \qquad xbc, xde, xfg, xhl \in T_1, \quad xbd, xcf, xeh, xgl \in T_2.$$

Lemma 13. The existence of an element $x \in \text{found}(T)$, with $r_x = 4$ implies the existence of a unique $TS(9,8)$.

Proof. Let $r_1 = 4$, and the blocks of T which contain 1 are of the form $(*)$. So we can show that the blocks of this trade are as follows:

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24x, 35y, 68z, 79w\},$$
$$\{124, 135, 168, 179, 23x', 45y', 67z', 89w'\}).$$

We assume that $r_2 = 2$. Hence $x = x' \in \{6,7,8,9\}$. Suppose $x = x' = 6$. If $y' = 6$, then $y = 6$, and we have no choice for z. So $y, y' \neq 6$. Only choices for z are 3 and 5. If $z = 3$, then $z' = w' = 3$, and therefore $w = 3$. Hence $y = y' = 6$. If $z = 5$, then $z' = w' = 3$, and $w = 5$. Therefore, $y = y' = 6$. For both choices we reach a contradiction, and hence, no trade exists.

Now, suppose that the blocks which contain 1 are of the form $(**)$. So we express the blocks of this trade as follows:

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24x, 36y, 58z, 79w\},$$
$$\{124, 136, 158, 179, 23x', 45y', 67z', 89w'\}).$$

Let $2 \in E(2)$. Hence $x = x' \in \{7,8,9\}$. Suppose that $x = x' = 7$. We have $z \in \{3,7\}$. If $z = 3$, then $y' = w' = 3$, and $z' = 4$, and we can express $T = TS(9,8)$ as follows:

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 247, 346, 358, 379\},$$
$$\{124, 136, 158, 179, 237, 345, 389, 467\}).$$

If $z = 7$, then $w = 3$, and therefore $w' = 3$, so we can express $T' = TS(9,8)$ as follows:

$$T' = (T_1', T_2') = (\{123, 145, 167, 189, 247, 368, 379, 578\},$$
$$\{124, 136, 158, 179, 237, 389, 457, 678\}).$$

But the permutation (37)(29)(48) transforms T to T'. Hence they are isomorphic. When $x = x' = 8$, or 9, we obtain a trade which is isomorphic with this trade. □

We summerize the above results in the following lemma.

Lemma 14. There are two nonisomorphic $TS(9,8)$'s.

Proposition 15. If for any $x \in$ found(T) in a $TS(10,8)$, $r_x \leq 3$, then, there is no block $xyz \in T_1$ with $r_x = r_y = r_z = 2$.

Proof. Suppose $123 \in T_1$ such that $r_1 = r_2 = r_3 = 2$. Also, suppose that $E(2) = \{1, 2, \cdots, 6\}$, and $E(3) = \{7, 8, 9, A\}$. So $12\alpha, 13\alpha'$, and $23\alpha''$ are three blocks of T_2. Therefore, $\alpha, \alpha', \alpha'' \in E(3)$ and are distinct (for $r_x \leq 3$, and Lemma 5). Then $\{\alpha, \alpha', \alpha''\} = \{7, 8, 9\}$. Hence $178, 279$, $389 \in T_1$. Notice that $789 \notin T_2$ (since $7, 8, 9 \in E(3)$.) Thus, $78m, 79n$, $89p \in T_2$. Only one of the elements of $\{m, n, p\}$ could be A, and the other two should be chosen from $\{4, 5, 6\}$ (say, it is 6). So $\lambda_{6A} = 2$, and this is a contradiction. □

Proposition 16. By the same assumption as in Proposition 15, there are exactly four blocks of $T_1(T_2)$ containing four pairs of the elements of $E(2)$, and if $E(2) = \{1, 2, \cdots, 6\}$, then the pairs are 12, 23, 45, 46.

Proof. There are 4 possible choices for these pairs:

(i) $12, 23, 45, 46$, (ii) $12, 13, 23, 45$, (iii) $12, 13, 34, 45$, (iv) $12, 13, 34, 56$.

If (ii) occurs, then $12\alpha, 13\alpha', 23\alpha'' \in T_1$, and therefore $\alpha = \alpha' = \alpha''$, and this is a contradiction. By similar arguments (iii), and (iv) can be ruled out. □

Lemma 17. By the same assumption as in Proposition 16, there exists a unique $TS(10,8)$.

Proof. Let $E(2) = \{1, 2, \cdots, 6\}$, and $E(3) = \{7, 8, 9, A\}$. By Propositions 15 and 16, we conclude that

$$12\alpha, 13\alpha', 45\alpha'', 46\alpha''' \in T_1, \quad 13\alpha, 12\alpha', 45\alpha''', 46'' \in T_2.$$

Hence $\{\alpha, \alpha', \alpha'', \alpha'''\} = \{7, 8, 9, A\}$. Therefore, we express the blocks of T as follows:

$$T = (T_1, T_2) = (\{127, 138, 459, 46A, 37x, 28y, 5Az, 69w\},$$
$$\{137, 128, 45A, 469, 27x', 38y', 59z', 6Aw'\}).$$

Hence $x' = y, x = y', z = z'$, and $w = w'$. So we can fill up the unoccupied spots in two ways by $w = 8$, or $w = A$. Therefore, two trades can be constructed. But these two trades are isomorphic ($\sigma = (56)(9A)$). □

Lemma 18. If for $x \in \text{found}(T)$, $x \in E(4)$, then there are 3 nonisomorphic $TS(10, 8)$'s.

Proof. Let $1 \in E(4)$, and the blocks containing 1 are of the form (∗). So we show that $TS(10, 8)$ is as follows:

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24x, 36y, 68z, 79w\},$$
$$\{124, 135, 168, 179, 23x', 45y', 67z', 89w'\}).$$

Clearly $A \in E(2)$ or $E(4)$. If $A \in E(2)$, we have $x = x' = y = y' = A$ (or $w = z = w' = z' = A$), and $z \in \{2, 4, 5, 6\}$. We can assume that $z = 2$. Therefore, a trade is constructed. If $A \in E(4)$, then all the unoccupied entries can be filled up with A. So we obtain two nonisomorphic trades.

Now suppose that the blocks containing 1 are of the form (∗∗), i.e., we have the following trade:

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24x, 36y, 58z, 79w\},$$
$$\{124, 136, 158, 179, 23x', 45y', 67z', 89w'\}).$$

Clearly $A \in E(4)$, so all the entries are A, and hence we are left with only one trade. □

Lemma 19. There are 4 nonisomorphic $TS(10, 8)$.

Lemma 20. Let $T = TS(11, 8)$. For any $x \in \text{found}(T)$, if $r_x \leq 3$, then there exists no $TS(11, 8)$, otherwise a unique $TS(11, 8)$ exists.

Proof. In this case $|E(3)| = 2$, and $|E(2)| = 9$. Suppose $1 \in E(3)$. Then we can assume that

$$123, 145, 167, 24x, 36y, 57z \in T_1, \quad 124, 136, 157, 23x', 45y', 67z' \in T_2.$$

Since at least five elements of $\{2, 3, \cdots, 7\}$ are in $E(2)$, so $x = y = z = x' = y' = z'$, and therefore, there must be a trade with volume 6 embedded in $TS(11, 8)$, which is impossible. Hence, there is no $TS(11, 8)$.

Now suppose that for an $\alpha \in \text{found}(T)$, we have $r_\alpha = 4$. Let $\alpha = 1$. First, assume that the blocks containing 1 are of the form (*). We show that the blocks of a possible $TS(11, 8)$ are as follows:

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24x, 35y, 68z, 79w\},\\ \{124, 135, 168, 179, 23x', 45y', 67z', 89w'\}).$$

Since, each of the elements A, or B appear at least twice, hence the $x = y = x' = y' = A$, and $z = w = z' = w' = B$, and so a trade is constructed.

Now suppose that the blocks containing 1 are of the form (**). Then T is as follows:

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24x, 36y, 58z, 79w\},\\ \{124, 136, 158, 179, 23x', 45y', 67z', 89w'\}).$$

If A occurs in one of the unoccupied spots, then it should occur in all of them, therefore B can not occur anywhere, which is impossible. □

Now we study $TS(f, 9)$ for $9 \le f \le 11$.

Definition. Let $x, y, z \in \text{found}(T)$, and $\lambda_{xy} = \lambda_{xz} = 0$. We say that x has Property (P) if $\lambda_{yz} = 1$.

Let T be a $TS(9, 9)$, and $r_x \le 3$ for any $x \in \text{found}(T)$. It follows that for any $x \in \text{found}(T)$, there are two elements $z, y \in \text{found}(T)$ such that $\lambda_{xy} = \lambda_{xz} = 0$.

First suppose that there exists $x \in \text{found}(T)$ with Property (P). Let $x = 1$, and $\lambda_{12} = \lambda_{13} = 0$, $\lambda_{23} = 1$. There exist $y, y' \in \text{found}(T)$ such that $\lambda_{2y} = \lambda_{3y'} = 0$. The following proposition is clear.

Proposition 21. $y \ne y'$, $1yy' \notin T_1$, and if $23y'' \in T_1$, then $yy'y'' \in T_1$.

Lemma 22. If there exists $\alpha \in \text{found}(T)$ with Property (P), and $r_\alpha \le 3$ for any $\alpha \in \text{found}(T)$, then there is a unique $TS(9, 9)$.

Proof. Let $\alpha = 1$, and $\lambda_{12} = \lambda_{13} = \lambda_{24} = \lambda_{37} = 0$. By the Proposition 21, we can express $TS(9,9)$ as follows:

$$T = (T_1, T_2) = (\{145, 167, 189, 23t, 27x, 2yz, 34w, 3uv, 47t\},$$
$$\{146, 158, 179, 23t', 27x', 2y'z', 34w', 3u'v', 47t'\}).$$

So $t \in \{8, 9\}$. If $t = 8$, then $\lambda_{T_2(58)} = 1$, and hence $\lambda_{T_1(58)} = 1$, but this pair can not appear in T_1, so $t = 9$. By a similar argument we have $t' = 5$. Now we can fill all the unoccupied entries uniquely, and obtain the following trade.

$$T = (T_1, T_2) = (\{145, 167, 189, 239, 257, 268, 346, 358, 479\},$$
$$\{146, 158, 179, 235, 267, 289, 349, 368, 457\}).\Box$$

Lemma 23. If $r_x \leq 3$ for any $x \in$ found(T), and if there exists no element in found(T) with Property (P), then there exists a unique $TS(9,9)$.
Proof. The pairs from the sets $\{1,2,3\}, \{4,5,6\}, \{7,8,9\}$ can not appear in T, and we can show that the possible T can be constructed as follows:

$$T = (T_1, T_2) = (\{146, 158, 179, 2-, 2-, 2-, 3-, 3-, 3-\},$$
$$\{148, 159, 167, 2-, 2-, 2-, 3-, 3-, 3-\}).$$

Now the blanks can be easily filled up as follows:

$$T = (T_1, T_2) = (\{147, 158, 169, 248, 259, 267, 349, 357, 368\},$$
$$\{148, 159, 167, 249, 257, 268, 347, 358, 369\}).\Box$$

Let T be a $TS(9,9)$, having an element $\alpha \in$ found(T) such that $r_\alpha = 4$, and the blocks containing α are of the form $(*)$. Let $\alpha = 1$, then T should be of the following form:

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24x, 35y, 68w, 79v, rst\},$$
$$\{124, 135, 168, 179, 23m, 45n, 67p, 89q, abc\}).$$

Proposition 24. By the above assumption, $|E(4)| > 1$.
Proof. Suppose that $|E(4)| = 1$. Let $2 \in E(2)$, then $x = m$. Clearly $x \in \{6, 7, 8, 9\}$. Let $x = m = 6$. Since $6 \in E(3)$, then $w = 3$, and $p = 4$. Therefore, $y \in \{7, 9\}$. If $y = 7$, then $378 \in T_2$. Since $\lambda_{T_2(78)} = 1$, hence,

$v = 8$. Therefore, $\lambda_{T_1(89)} = 2$, and this is a contradiction. If $y = 9$, then $q = 3$, and hence a trade with volume 8 is embedded in T which is clearly a contradiction. Therefore $|E(4)| \geq 2$. □

Lemma 25. By the assumption of Proposition 24, there is no $TS(9,9)$.
Proof. We know that $|E(4)| \geq 2$, and $1, 2 \in E(4)$. So for any $x \in$ found(T), $\lambda_{1x} = \lambda_{2x} = 1$. Since $\lambda_{25} = 1$, then $rs = ab = 25$. One of the elements v or w is 2. Let $v = 2$. Then $\{t, x\} = \{6, 8\}$. We can suppose that $t = 6$, and $x = 8$. Likewise one of the elements v' or w' is 2. In both cases we obtain a contradiction. By a similar argument the case, $w = 3$ is ruled out. So, we are left with no trade.□

Now suppose that for any $\alpha \in$ found(T), such that $r_\alpha = 4$, the blocks containing α are of the form (**). So we have

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24x, 36y, 38z, 79w, rst\},$$
$$\{124, 135, 158, 179, 23m, 45n, 67p, 89q, abc\}).$$

By a similar argument as in Proposition 24, we conclude that $|E(4)| \geq 2$. Clearly we have $|E(4)| \leq 3$.

Proposition 26. Let $|E(4)| = 3$ and $x, y, z \in E(4)$. Then

- $xyz \notin T_1(T_2)$,
- $\lambda_{xy} = \lambda_{xz} = \lambda_{yz} = 1$,
- if $xyv, xyv', yzv'' \in T_1$, then $v, v', v'' \in E(2)$.

Lemma 27. By the assumptions as in Proposition 26, there is a unique $TS(9,9)$.
Proof. We assume that $1, 2, 4 \in E(4)$, and by the Proposition 26, the blocks of a trade T are as follows:

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24x, 2yz, 2vw, 4xy, 4ab\},$$
$$\{125, 136, 148, 179, 24m, 2np, 2ql, 4ij, 4kh\}).$$

By Proposition 26, $3, 5, 8 \in E(2)$. Therefore, $x = 8$, and $m = 3$. We conclude that 289, $267 \in T_2$. From the other hand, since $3 \in E(2)$, we conclude that 346, 479 $\in T_1$, and hence $y = i = 5$. Also, since $5 \in E(2)$, we have $j = z \in \{6, 7, 9\}$. If $j = z = 6$, then 279 $\in T_1$, and $\lambda_{T_1(79)} = 2$, and this is a contradiction. By a similar argument in the case $j = z = 9$, we obtain a contradiction. So $j = z = 7$, and finally we conclude that $uv = kh = 69$, and hence, a trade is constructed. □

Now suppose $|E(4)| = 2$, and $1, 2 \in E(4)$. Then the blocks of T can have the following form:

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24x, 36y, 58z, 79w, rst\},$$
$$\{124, 136, 158, 179, 23m, 45n, 67p, 89q, abc\}).$$

To replace the letters, we need the following proposition.

Proposition 28. $3, 4 \notin E(2)$.

Proof. Suppose that $3 \in E(2)$. Hence $m = y$, and $m \in \{5, 8, 9\}$. If $m = 5$, since $5 \in E(3)$, we deduce that $z = 2$, and $n = 6$. Since $\lambda_{T_2(46)} = 1$, then $x = 6$ or $rs = 46$. In both cases we conclude that $w = 2$, and therefore $x = 6$. So a trade with volume 8 is embedded in this trade, and this is a contradiction, so $m \neq 5$. If $m = 8$, again by noting that $8 \in E(3)$, we deduce that $z = 2$, and $q = 6$. Since 2 appears twice, we conclude that $p = 2$, and $abc = 259$. Clearly $n = 7$. Since $\lambda_{T_1(57)} = \lambda_{T_1(59)} = 1$, then 579 $\in T_1$. Hence $w = x = 2$, and we have $\lambda_{T_1(47)} = 0$, but $\lambda_{T_2(47)} = 1$. By a similar argument $4 \notin E(2)$. □

Lemma 29. By the assumption $|E(4)| = 2$, there is a unique $TS(9, 9)$.

Proof. Clearly $5, 6 \notin E(2)$, otherwise we have $n = z$, and since $\lambda_{25} = 1$, then $n = z = 2$, and therefore $\lambda_{T_2(24)} = 2$.

Now we construct a trade. Two elements of each of the sets $\{z, w, r\}$ and $\{p, q, a\}$ are 2, and the third elements are 3, and 4, respectively. If $z = 3$, then $y = 4, w = 2, x = 8$, and $rs = 56$, and similarly $p = q = 2$, $n = 6$, $abc = 348$, and $m = 5$. Hence, we construct a trade. If $r = 3$, by a similar argument as above, we construct a trade which turns out to be isomorphic to the above trade.

If $w = 3$, then $y = 4$, $z = 2$, and $rst = 256$. This is a contradiction, to the fact that the pair 25 appears twice. □

Note. By Lemmas 23,24,28, and 30, we have 4 nonisomorphic $TS(9,9)$.

Now, we discuss the case of $TS(10,9)$.

Lemma 30. If for any $x \in \text{found}(T)$, $r_x \leq 3$, then there is a unique $TS(10,9)$.

Proof. In this case $|E(3)| = 7$, and $|E(2)| = 3$. By (1), there is an element $\alpha \in E(3)$, such that $0 \leq |F_\alpha| \leq 1$. Let $\alpha = 1$. We can express the blocks of a possible trade T as follows:

$$T = (T_1, T_2) = (\{123, 145, 167, 24x, 36y, 57z, uvw, rst, opq\}$$
$$\{124, 136, 157, 23x', 45y', 67z', u'v'w', r's't', o'p'q'\}).$$

First, suppose that $|F_1| = 0$. At least one of the elements of $\{x, y, z\}$ belongs to $\{2, 3, \cdots, 7\}$. Let x be such an element, and $x = 6$. Since every element of $\{2, 3, \cdots, 7\}$ belongs to $E(3)$, then x', $y' \neq 6$. We can assume that $z' = 2$, so $uv = 27$, and $u'v' = 46$. Hence $y = w' \in \{8, 9, A\}$. Let $y = 8$, hence, $rs = 48$, and $r's' = 38$. Therefore, $y' = t = t' \in \{9, A\}$. Suppose $y' = 9$. Since $5 \in E(3)$, so we have $opq = 359$. A must appear in T_1 twice, so $\lambda_{T_1(7A)} = 2$, and this is a contradiction, and so $|F_1| = 1$.

Let $2 \in E(2)$. Hence $x = x' \in \{7, 8, 9, A\}$. If $x = x' = 7$, then $z' = 4$, and $z = 3$. Therefore, $uv = 46$, and $u'v' = 35$. Let $8 \in E(3)$. Since $9, A \in E(2)$, then $\lambda_{T_1(8A)} = 2$, or $\lambda_{T_2(89)} = 2$, and this is a contradiction. So $x \neq 7$. Let $x = x' = 8$, and we have $uv = 38$, and $u'v' = 48$. Notice that $y \in \{4, 5, 9, A\}$. If $y = 4$, then $w' = 6$, $rs = 68$, $y' = 3$, and $w = 5$. Hence $r's' = 58$, and $rs = 68$. Since $9, A \in E(2)$, then with no loss of generality, we assume that $z = 9, z' = A, t = A$, and $t' = 9$. Clearly we can not fill the other blanks of this trade. So $y \neq 4$.

If $y = 5$, then $r's' = 35$, and $y' = 6$. Hence $rs = 46$, therefore, $t = w$, and $z = w = t'$. Assume that $z = 9$, then all the blanks can be filled uniquely, and we obtain the following trade,

$$T = (T_1, T_2) = (\{123, 145, 167, 248, 369, 378, 49A, 579, 68A\},$$
$$\{124, 136, 157, 238, 397, 48A, 459, 678, 69A\}).$$

If $y = 9$, then clearly we obtain a trade which is isomorphic with the above trade.□

Now suppose that there exists $\alpha \in$ found(T) such that $r_\alpha = 4$. Also suppose that blocks containing α are of the form (∗). Let $\alpha = 1$. We show that the trade T could be of the following form:

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24x, 35y, 68z, 79w, uvs\}$$
$$\{124, 135, 168, 179, 23x', 45y', 67z', 89w', u'v's'\}).$$

To replace the blanks we need the following proposition.

Proposition 31. $A \in E(2)$.

Proof. Suppose to the contrary, and let $A \in E(3)$ or $A \in E(4)$. If $A \in E(4)$, then $x = y = z = w = x' = y' = z' = w' = A$, and a trade with volume 8 is embedded in this trade. So $A \notin E(4)$. If $A \in E(3)$, then at least two elements of $\{x, y, z, w\}$ are A. We have two choices for these elements,

$$(i)\ \ x = y = A, \qquad (ii)\ \ x = z = A.$$

If (i) occurs, then clearly $x' = y' = A$, and there is a minimal trade embedded in this trade. So (i) is ruled out. If $x = z = A$, then x', or y' is equal A, and likewise z', or w' is equal A. With no loss of generality, suppose that $x' = z' = A$, then $u'v's' = 48A$, and $uvs = 37A$. Since $\lambda_{T_1(3A)} = 1$, then $\lambda_{T_2(3A)} = 1$, but we have no block to contain the pair 3A. □

Now we fill the blanks of the above $TS(10, 9)$.

Lemma 32. By the above assumption, there exists a unique $TS(10, 9)$.

Proof. By the Proposition 31, $A \in E(2)$. Hence, exactly one of the elements of $\{x, y, z, w\}$ is A. Let $x = A$. So x', or y' is A. Suppose $x' = A$. Therefore $r'v' = 4A, uv = 3A$, and $y \in \{6, 7, 8, 9\}$. Let $y = 6$. So $z' = 3$, $y' = 6$, and hence $z = 4$. We have two choices: (i) $w = 3$, and (ii) $s = s' = 7$. If $w = 3$, then $w' = 3$, and we have $s = s' = 8$. If $s = s' = 7$, then

$w = w' = 4$. So we obtain the following trades:

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24A, 356, 468, 479, 37A\}$$
$$\{124, 135, 168, 179, 23A, 456, 367, 489, 47A\}),$$

and

$$T' = (T_1', T_2') = (\{123, 145, 167, 189, 24A, 356, 468, 379, 38A\}$$
$$\{124, 135, 168, 179, 23A, 456, 367, 389, 48A\}).$$

But T and T' are isomorphic ($\sigma = (34)(78)$). □

Now Suppose that $r_1 = 4$, and the blocks containing 1 are of the form (**), then we can assume that:

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24x, 36y, 58z, 79w, uvs\},$$
$$\{124, 136, 158, 179, 23x', 45y', 67z', 89w', u'v's'\}).$$

Lemma 33. By the assumption as above, we have 3 nonisomorphic $TS(10,9)'s$.

Proof. Either $A \in E(2)$, or $A \in E(3)$. If $A \in E(2)$, then at least an element of $\{x, y, z, w\}$ is A. First, assume that only one element of $\{x, y, z, w\}$ is A. Let $x = A$. So, $x' = A$, or $y' = A$. With no loss of generality, suppose that $x' = A$, therefore $uv = 3A$, $u'v' = 4A$, and $s = s' \in \{7, 8, 9\}$. Clearly $s = 8$ or 7 can be ruled out. Hence $s = s' = 9$. So $w = 4$, and $w' = 3$. We have two choices, $y = 8$, or $z = 3$. With these choices, we can construct the following two trades:

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24A, 368, 39A, 578, 479\}$$
$$\{124, 136, 158, 179, 23A, 389, 457, 49A, 678\}),$$

and

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24A, 346, 39A, 358, 479\}$$
$$\{124, 136, 158, 179, 23A, 345, 389, 467, 49A\}).$$

Clearly, the above two trades are nonisomorphic, because the cardinality of $E(4)$ in these trades are not the same.

Now suppose that two elements of $\{x, y, z, w\}$ are A. Hence since at least one of the elements of $\{x', y', z', w'\}$ is A, with no loss of generality, let $x = y = x' = A$. Thus $u'v's' = 46A$, $uv = 46$, and $w \in \{8,9\}$. Clearly $s = 8$, then $\lambda_{T_2(48)} = \lambda_{T_2(68)} = 1$, and we have two choices, $w' = 6$, or $w' = 4$. If $w' = 6$, then $w = 6$, and $\lambda_{T_1(67)} = 2$, and this is a contradiction. Hence $w' = 4$, and $w = 4$, and we deduce that $x = 7$, $z = 7$, and $y' = 8$. Thus we have the following trade.

$$T = (T_1, T_2) = (\{123, 145, 167, 189, 24A, 36A, 468, 479, 578\},$$
$$\{124, 136, 158, 179, 23A, 457, 678, 489, 46A\}).$$

Now Suppose that $A \in E(3)$. If three elements of $\{x, y, z, w\}$ are A, we easily obtain a trade isomorphic to the trade of Lemma 32. If only two elements of $\{x, y, z, w\}$ are A, clearly we obtain a contradiction. □

Now we construct $TS(11, 9)$.

Lemma 34. Let T be a $TS(11, 9)$. If for any $x \in$ found $(T), r_x \leq 3$, then there exists a unique $TS(11, 9)$.

Proof. By (1) we conclude that there is an element $\alpha \in E(3)$, such that $3 \leq |F_\alpha| \leq 4$. Suppose that $\alpha = 1$. The blocks of $T = (T_1, T_2)$ containing the element 1 can be expressed as

$$123, 145, 167 \in T_1, \quad 124, 136, 157 \in T_2.$$

First assume that $|F_1| = 4$. Four elements of $E(2)$ which appear in the above blocks could be of the following forms:

$$(i) \quad \{2, 3, 4, 6\} \subseteq E(2),$$
$$(ii) \quad \{2, 3, 4, 7\} \subseteq E(2),$$
$$(iii) \quad \{2, 3, 5, 7\} \subseteq E(2).$$

(i) and (ii) can easily be ruled out. So we are left only with (iii). We can express T as follows:

$$T = (T_1, T_2) = (\{123, 145, 167, 24x, 36x, 57y, -, -, -\}$$
$$\{124, 136, 157, 23x, 45y, 67y, -, -, -\}).$$

Clearly $x = 8$, and $y = 9$. If $9 \in E(2)$, then 469 is a block of T_1, and the pair AB appears in the remaining blocks, and hence $\lambda_{T_1(AB)} = 2$, and this is a contraction. So, $9 \in E(3)$. Similarly $8 \in E(3)$. Thus we can easily construct a trade.

$$49A, 69B, 8AB \in T_1, \quad 9AB, 68B, 48A \in T_2.$$

Now suppose that for any $x \in \text{found}(T)$ such that $x \in E(3), |F_x| \leq 3$. Let $1 \in E(3)$, and $|F_1| = 3$. We consider the following cases,

$$\begin{aligned} (i) &\quad \{2,5,6\} \subseteq E(2), \\ (ii) &\quad \{2,4,6\} \subseteq E(2), \\ (iii) &\quad \{2,3,4\} \subseteq E(2). \end{aligned}$$

(i) leads us to the previous trade, and we do not obtain any new trade. (ii) and (iii) can be easily ruled out too. □

Lemma 35. Suppose that for at least one $\alpha \in \text{found}(T)$, $r_\alpha = 4$. Then there exists only one $TS(11, 9)$.

Proof. Let $r_1 = 4$. Suppose that the blocks containing 1 are of the form $(*)$, then we assume that

$$\begin{aligned} T = (T_1, T_2) = (&\{123, 145, 167, 189, 24x, 35y, 68z, 79w, rst\} \\ &\{124, 135, 168, 179, 23x', 45y', 67z', 89w', r's't'\}). \end{aligned}$$

Since $|E(2)| \geq 7$, at least 3 elements of $\{2,3,4,5\}$, or $\{7,8,9,A\}$ are in $E(2)$. Suppose that at least three elements of $\{2,3,4,5\}$ are in $E(2)$. So $x = y = x' = y'$, and then a minimal trade is embedded in this trade and this is a contradiction. So, we have no trade.

Now suppose that the blocks containing 1 are of the form $(**)$, then we can show the blocks of $TS(11, 9)$ are as follows:

$$\begin{aligned} T = (T_1, T_2) = (&\{123, 145, 167, 189, 24x, 36y, 58z, 79w, rst\} \\ &\{124, 136, 158, 179, 23x', 45y', 67z', 89w', r's't'\}). \end{aligned}$$

By the above argument, at least 3 elements of $\{2,3,4,5\}$ are in $E(2)$. Concerning the structure of this trade we consider two cases for the elements of $\{2,3,4,5\}$ which are in $E(2)$,

$$(i)\ \{2,3,4\} \subseteq E(2), \quad (ii)\ \{2,3,5\} \subseteq E(2).$$

If (i) occurs, then $x = x' = y' = y \in \{9, A, B\}$. The case $y = 9$ can be easily ruled out. So $x = x' = y = y' = A$. One of the elements z, or w is B. If $z = B$, then $w' = B$, $r's' = 5B$, and we conclude that $z' = A$, and $\lambda_{T_2(AB)} = 0$, so $w = A$. Since $\lambda_{T_1(9A)} = 1$, hence $\lambda_{T_2(9A)} = 1$, but we have no choice for this pair in the blocks of T_2. So $w = B$. By the same argument $w' = B$. Therefore, $rs = 8B$, $r's' = 7B$, and $t = t'$. Clearly we conclude that $z = A$. Since $\lambda_{T_1(8A)} = 1$, then $\lambda_{T_2(8A)} = 1$, but we have no choice for this pair in the blocks of T_2. So (i) can be ruled out.

If (ii) occurs, then $x = y = x'$, and $y' = z$. Also $y \neq y'$ (otherwise $\{2,3,4\} \subseteq E(2)$, and (i) will occur). Clearly $x = y = x' = A$, and $y' = z = B$. Therefore $w' = A$, $r's' = 4A$, and $rs = 4B$. We conclude that $w = B$ or $t = 9$. Choosing each of these elements, we obtain the following two trades:

$$\begin{aligned} T = (T_1, T_2) = (&\{123, 145, 167, 189, 24A, 36A, 49B, 58B, 79A\} \\ &\{124, 136, 158, 179, 23A, 45B, 49A, 67A, 89B\}), \end{aligned}$$

and

$$\begin{aligned} T' = (T_1', T_2') = (&\{123, 145, 167, 189, 24A, 36A, 46B, 58B, 79B\} \\ &\{124, 136, 158, 179, 23A, 45B, 4A6, 67B, 89B\}). \end{aligned}$$

But the above trades are isomorphic ($\sigma = (38)(25)(69)(A\ B)$). □

Note. By the Lemmas 34, and 35, there are 2 nonisomorphic $TS(11, 9)$.

4. The Summary of Results

We summarize the results of Section 3, in Tables 1 and 2.

Table 1. Numbers of nonisomorphic trades.

|found(T)| / vol(T)	7	8	9	A	11	12
6	1	1				
7	1		1			
8		1	2	4	1	1
9			4	5	2	

Table 2. A complete list of trades $T = (T_1, T_2)$ for $6 \leq \text{vol}(T) \leq 9$.

found(T)	7	8	7	9	8	9	9	10	10	10	10	11
T_1	123	123	123	123	123	123	123	127	123	123	123	123
	145	145	145	145	145	145	145	138	145	145	145	145
	167	167	167	167	167	167	167	28A	167	167	167	167
	247	248	246	248	248	246	189	379	189	189	189	189
	346	368	257	358	257	257	247	459	24A	24A	24A	24A
	357	578	356	369	346	359	346	46A	268	35A	36A	68B
			347	579	378	368	358	57A	279	68A	58A	79B
					568	489	379	689	35A	79A	79A	35A
T_2	124	124	124	124	124	124	124	128	124	124	124	124
	136	136	136	136	136	136	136	137	135	135	136	135
	157	157	157	157	157	157	158	27A	168	168	158	168
	237	238	237	238	237	235	179	389	179	179	179	179
	345	458	256	359	258	267	237	45A	23A	23A	23A	23A
	467	678	345	458	348	389	345	469	267	45A	45A	45A
			467	679	456	459	389	579	289	67A	67A	67B
					678	468	467	68A	45A	89A	89A	89B

found(T)	12	9	9	9	9	10	10	10	10	10	11	11
T_1	123	145	147	123	123	123	123	123	123	123	123	123
	145	167	158	145	145	145	145	145	145	145	145	145
	24A	189	169	167	167	167	167	167	167	167	167	167
	35A	239	248	189	189	248	189	189	189	189	248	189
	678	257	259	248	248	369	24A	24A	24A	24A	368	24A
	69B	268	267	257	256	378	356	368	346	36A	49A	36A
	79C	346	349	269	279	49A	37A	39A	358	468	579	49B
	8BC	358	357	346	346	579	468	479	39A	479	69B	58B
		479	368	479	358	68A	479	578	479	578	8AB	79A
T_2	124	146	148	125	124	124	124	124	124	124	124	124
	135	158	159	136	136	136	135	136	136	136	136	136
	23A	179	167	148	158	157	168	158	158	158	157	158
	45A	235	249	179	179	238	179	179	179	179	238	179
	67A	267	257	234	235	379	23A	23A	23A	23A	459	23A
	68B	289	268	267	267	459	367	389	345	457	679	45B
	78C	349	347	289	289	489	456	457	389	46A	48A	67A
	9BC	368	358	457	348	678	489	49A	467	489	68B	89B
		457	369	469	456	69A	47A	678	49A	678	9AB	49A

A=10, B=11, C=12.

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