Maximum Size of a Connected Graph with Given Domination Parameters

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Abstract. Let G be a connected (p,q)-graph. Let γ_c denote the connected domination number of G. In this paper, we prove that $q \leq \lfloor p(p-\gamma_c)/2 \rfloor$ and equality holds if and only if $G = C_p$ or K_p or $K_p - Q$ where Q is a minimum edge cover of K_p. We obtain similar bounds on q for graphs with given total domination number γ_t, clique domination number γ_k, edge domination number γ' and connected edge domination number γ'_c and for each of these parameters, characterize the class of graphs attaining the corresponding bound.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. Terms not defined here are used in the sense of Harary [3].

A subset S of V is called a *dominating set* in G if every vertex not in S is adjacent to some vertex in S. The *domination number* $\gamma = \gamma(G)$ is the minimum cardinality taken over all dominating sets in G.

The concept of connected domination was introduced by Sampathkumar and Walikar [6]. A dominating set S of a connected graph G is called a *connected dominating set* if the induced subgraph $\langle S \rangle$ is connected. The *connected domination number* $\gamma_c = \gamma_c(G)$ is the minimum cardinality taken over all connected dominating sets in G.

The concept of total domination was introduced by Cockayne, Dawes and Hedetniemi [1]. Let G be a graph without isolated vertices. A subset

S of V is called a *total dominating set* if every vertex in V is adjacent to some vertex in S. The *total domination number* $\gamma_t = \gamma_t(G)$ is the minimum cardinality taken over all total dominating sets in G.

The concept of clique domination was introduced by Cozzens and Kelleher [2]. A dominating set S of a connected graph G is called a *clique dominating set* if the induced subgraph $\langle S \rangle$ is complete. The *clique domination number* $\gamma_k = \gamma_k(G)$ is the minimum cardinality taken over all clique dominating sets in G.

The concept of edge domination was introduced by Mitchell and Hedetniemi [5]. A subset X of E is called an *edge dominating set* if every edge not in X is adjacent to some edge in X. The *edge domination number* $\gamma' = \gamma'(G)$ is the minimum cardinality taken over all edge dominating sets in G. In a similar way we can define the connected edge domination number γ'_c of a connected graph G.

A subset Q of E is said to be an *edge cover* of G if every vertex of G is incident with an edge of Q; if there is no edge cover S with $|S| < |Q|$, then Q is said to be *minimum edge cover*.

For any real number x, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. If n and r are two positive integers with $r \leq n$, then $\binom{n}{r}$ is the binomial coefficient that stands for the number of ways of choosing r objects out of n objects. For a graph G, we denote by G^+ the graph obtained from G by adjunction of a vertex v' for every vertex v in G and joining v and v'. If G is a connected graph and m is any positive integer, then mG stands for the graph with m components each isomorphic to G.

In [8] Vizing gives an upper bound for the number of edges in a graph with a given number of vertices and a given domination number.

Theorem 1.1 [8]. *If G is a graph with p vertices and domination number γ where $2 \leq \gamma \leq p$, then the number of edges of G is at most $\lfloor (p-\gamma)(p-\gamma+2)/2 \rfloor$. Equality occurs if and only if G is the disjoint union of $\gamma - 2$ isolated vertices and $K_{p-\gamma+2} - Q$ where Q is a minimum edge cover of $K_{p-\gamma+2}$.*

The graph which attains the bound in Theorem 1.1 has isolated vertices. L. A. Sanchis [7] has obtained an upper bound for the number of edges in a graph G with a given number of vertices and given domination number and having no isolated vertex. In this paper we obtain an upper bound for the number of edges in a connected graph with a given number of vertices and given connected domination number and characterize the class of graphs for which the bound is attained. We also obtain similar results for total domination number, clique domination number, edge domination number and connected edge domination number. We need the following theorems.

Theorem 1.2 [4]. *Let G be a connected (p,q)-graph with connected domination number γ_c. Then $\gamma_c \leq p - \Delta$ where Δ denotes the maximum degree of a vertex in G.*

Theorem 1.3 [1]. (a) *If G is a graph without isolated vertices then $\gamma_t \leq p - \Delta + 1$.* (b) *If G is connected and $\Delta < p - 1$, then $\gamma_t \leq p - \Delta$.*

2. Main Results

Theorem 2.1. *Let G be a connected (p,q)-graph with connected domination number γ_c. Then $q \leq \lfloor p(p-\gamma_c)/2 \rfloor$ and equality holds if and only if $G = C_p$ or K_p or $K_p - Q$ where Q is a minimum edge cover of K_p.*

Proof: Since $2q \leq p\Delta$ and $\gamma_c \leq p - \Delta$, we have $q \leq \lfloor p(p-\gamma_c)/2 \rfloor$.
For $G = C_p$, $\gamma_c = p - 2$ and $q = p$.
For $G = K_p$, $\gamma_c = 1$ and $q = p(p-1)/2$.
For $G = K_p - Q$ where Q is a minimum edge cover of K_p, $\gamma_c = 2$ and $q = \lfloor p(p-2)/2 \rfloor$.
Thus in all these cases we have $q = \lfloor p(p-\gamma_c)/2 \rfloor$.

Now suppose G is any connected graph for which $q = \lfloor p(p-\gamma_c)/2 \rfloor$. We can assume that $p \geq 2$. Then by using $2q \leq p\Delta \leq p(p-\gamma_c)$ and $q = \lfloor p(p-\gamma_c)/2 \rfloor$ we get that $\gamma_c = p - \Delta$ and $q = \lfloor p\Delta/2 \rfloor$. Hence any spanning tree of G has at most Δ pendant vertices. Further G is regular of degree Δ or G has $p-1$ vertices of degree Δ and one vertex of degree

$\Delta - 1$. We claim that $\gamma_c = 1$ or 2 or $p-2$. Suppose $3 \leq \gamma_c \leq p-3$. Then $3 \leq \Delta \leq p-3$. Let u be a vertex of maximum degree Δ. Let T be a spanning tree in G with $\deg_T(u) = \Delta$. Since $\gamma_c = p - \Delta$, T has exactly Δ pendant vertices so that $\deg_T(v) = 1$ or 2 for every vertex v different from u. Since $3 \leq \Delta \leq p-3$, $|N(u)| \geq 3$ and $|V \setminus N[u]| \geq 2$. Hence there exists a pendant vertex v_1 of T such that v_1 is not adjacent to u. Let w_1 be the vertex in T which is adjacent to v_1. If $v_1v \in E(G)$ for some internal vertex v of T with $v \neq u, w_1$ then $T + v_1v - v_1w_1$ is a spanning tree of G with $\Delta+1$ pendant vertices which is a contradiction. Hence there exists a pendant vertex v_2 such that $v_1v_2 \in E(G)$. If v_1 is the only pendant vertex of T which is not adjacent to u, then we can replace T by $T + v_1v_2 - v_1w_1$ and thus we may assume that there exist at least two pendant vertices v_1, v_2 in T such that neither is adjacent to u. Since v_1 and v_2 are not adjacent to any internal vertex of T and have degree Δ or $\Delta - 1$, there exists a pendant vertex v_3 such that v_3v_1, $v_3v_2 \in E(G)$. Let w_2 be the vertex in T which is adjacent to v_2. Now $T - w_1v_1 - w_2v_2 + v_3v_1 + v_3v_2$ is a spanning tree of G with $\Delta + 1$ pendant vertices which is a contradiction. Hence $\gamma_c = 1$ or 2 or $p - 2$. If $\gamma_c = 1$, then $q = p(p-1)/2$ and $G = K_p$. If $\gamma_c = 2$, $q = \lfloor p(p-2)/2 \rfloor$ and $G = K_p - Q$ where Q is a minimum edge cover of K_p. If $\gamma_c = p-2$, then $q = p$ and $G = C_p$. ∎

Theorem 2.2. *Let G be a connected (p,q)-graph with total domination number γ_t. Then $q \leq \lfloor p(p-\gamma_t)/2 \rfloor$ and equality holds if and only if $G = C_5$ or C_6 or $K_p - Q$ where $p \geq 4$ and Q is a minimum edge cover of K_p.*

Proof: Since $2q \leq p\Delta$ and $\gamma_t \leq p - \Delta$ we obtain $q \leq \lfloor p(p-\gamma_t)/2 \rfloor$. Now let G be any connected (p,q)-graph with $q = \lfloor p(p-\gamma_t)/2 \rfloor$. Then $\gamma_t = \gamma_c = p - \Delta$. Hence it follows from Theorem 2.1 that $G = C_p$ or K_p or $K_p - Q$ where Q is a minimum edge cover of K_p. Further $\gamma_t = \gamma_c$ and hence $G = C_5$ or C_6 or $K_p - Q$ where Q is a minimum edge cover of K_p and $p \geq 4$. The rest of the proof is obvious. ∎

Theorem 2.3. *Let G be a (p,q)-graph without isolated vertices and with total domination number γ_t. Then $q \leq \lfloor p(p-\gamma_t+1)/2 \rfloor$ and equality holds if and only if $G = K_p$ or mK_2 with $m \geq 2$.*

Proof: Since $2q \leq p\Delta$ and $\gamma_t \leq p-\Delta+1$ we obtain $q \leq \lfloor p(p-\gamma_t+1)/2 \rfloor$. For $G = K_p$, $\gamma_t = 2$ and $q = p(p-1)/2$. For $G = mK_2$, $\gamma_t = p$ and $q = p/2$. In both cases $q = \lfloor p(p-\gamma_t+1)/2 \rfloor$. Now suppose G is a (p,q)-graph without isolated vertices, for which $q = \lfloor p(p-\gamma_t+1)/2 \rfloor$. Then $\gamma_t = p - \Delta + 1$ and $q = \lfloor p\Delta/2 \rfloor$.

Let G_i with p_i vertices and $\gamma_t(G_i) = \gamma_i$ for $i = 1, 2, \ldots, m$ be the components of G. Then

$$p - \Delta + 1 = \gamma_t = \sum_i \gamma_i \leq \sum_i (p_i - \Delta + 1) = p - m\Delta + m.$$

Therefore $\Delta - 1 \geq m(\Delta - 1)$ which implies that either $m = 1$ or $\Delta = 1$. i.e., either G is connected in which case, by Theorem 1.3, it follows that $\Delta = p-1$ so that $q = p(p-1)/2$ implying that $G = K_p$ or $G = mK_2$. ∎

Theorem 2.4. *Let G be a connected (p,q)-graph with clique domination number γ_k. Then $q \geq \binom{\gamma_k+1}{2}$ and equality holds if and only if p is even and $G = K^+_{p/2}$.*

Proof: Let S be a clique dominating set of cardinality γ_k. Then $|S| \leq |V \setminus S|$ so that $\gamma_k \leq p - \gamma_k$ and hence $q \geq \binom{\gamma_k}{2} + p - \gamma_k \geq \binom{\gamma_k}{2} + \gamma_k = \binom{\gamma_k+1}{2}$. Moreover if equality holds then $\gamma_k = p - \gamma_k$ and every vertex in $V \setminus S$ is adjacent to exactly one vertex in S. Hence p is even and $G = K^+_{p/2}$. The rest of the proof is obvious. ∎

Corollary 2.5. *Let G be a connected (p,q)-graph with clique domination number γ_k and p odd. Then $q \geq \binom{\gamma_k+1}{2} + 1$.*

Theorem 2.6. *Let G be a connected (p,q)-graph with clique domination number γ_k. Then $q = \binom{\gamma_k+1}{2} + 1$ if and only if G is isomorphic to the graph obtained from $K^+_{\gamma_k}$ by either joining two pendant vertices or adding one vertex and joining it to exactly one vertex of degree greater than 1.*

Proof: Suppose $q = \binom{\gamma_k + 1}{2} + 1$ and let S be a minimum clique dominating set. Then $\langle S \rangle$ is isomorphic to the complete graph on γ_k vertices. By minimality of S, for each $x \in S$, there corresponds one vertex x' which is joined to x but not to any other vertex of S; therefore, $|V \setminus S| \geq \gamma_k$; further, since the number of edges not in $\langle S \rangle$ is $\binom{\gamma_k + 1}{2} + 1 - \binom{\gamma_k}{2} = \gamma_k + 1$ and each vertex in $V \setminus S$ is joined to some vertex in S, we have $|V \setminus S| \leq \gamma_k + 1$. Thus $|V \setminus S| = \gamma_k$ or $\gamma_k + 1$. If $|V \setminus S| = \gamma_k$, then G is isomorphic to the graph obtained from $K^+_{\gamma_k}$ by joining two pendant vertices. If $|V \setminus S| = \gamma_k + 1$, then G is isomorphic to the graph obtained from $K^+_{\gamma_k}$ by adding one vertex and joining it to exactly one vertex of degree > 1. The converse is obvious. ∎

Theorem 2.7. *Let G be a (p, q)-graph with edge domination number γ'. Then $q \leq \gamma'(2p - 2\gamma' - 1)$ and equality holds if and only if $G = K_{2\gamma'} + \overline{K}_{p-2\gamma'}$.*

Proof: Let S be a minimum edge dominating set of G so that $|S| = \gamma'$. Let S_1 be the set of all vertices which are incident with at least one edge of S. Let $S_2 = V \setminus S_1$. Clearly $\langle S_1 \rangle$ contains at most $\binom{2\gamma'}{2}$ edges and every other edge of G has one end in S_1 and other end in S_2. Hence

$$q \leq \binom{2\gamma'}{2} + 2\gamma'(p - 2\gamma') = \gamma'(2p - 2\gamma' - 1).$$

Further $q = \gamma'(2p - 2\gamma' - 1)$ if and only if $\langle S_1 \rangle = K_{2\gamma'}$ and every vertex of S_2 is adjacent to every vertex of S_1 or equivalently, $G = K_2\gamma' + \overline{K}_{p-2\gamma'}$.
∎

Theorem 2.8. *Let G be a connected (p, q)-graph with given connected edge domination number γ'_c. Then $q \leq \lfloor \gamma'_c(2p - \gamma'_c - 3)/2 \rfloor + p - 1$ and equality holds if and only if $G = K_m + \overline{K}_{p-m}$ where $m = \gamma'_c + 1$.*

Proof: Let S be a minimum connected edge dominating set of G. Let S_1 be the set of vertices which are incident with at least one edge of S and let $S_2 = V \setminus S_1$. Now S_1 contains exactly m vertices and hence $\langle S_1 \rangle$ has at most $\binom{m}{2}$ edges. Further every other edge of G has one end in S_1 and the other end in S_2 so that

$$q \leq \binom{m}{2} + m(p-m) = [\gamma_c'(2p - \gamma_c' - 3)/2] + p - 1 \qquad \cdots (*)$$

If equality holds in $(*)$, then $\langle S_1 \rangle = K_m$ and every vertex of S_2 is adjacent to every vertex of S_1 and hence $G = K_m + \overline{K}_{p-m}$. The rest of the proof is obvious. ∎

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