Graph Labelings in Boolean Lattices

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ABSTRACT. A graph G is said to be embeddable in a set X if there exists a mapping f from $E(G)$ to the set $\mathcal{P}(X)$ of all subsets of X such that if we define a mapping g from $V(G)$ to $\mathcal{P}(X)$ by letting $g(x)$ be the union of $f(e)$ as e ranges over all edges incident with x, then g is injective. We show that for each integer $k \geq 2$, every graph of order at most 2^k all of whose components have order at least 3 is embeddable in a set of cardinality k.

1 Introduction

By a graph, we mean a finite simple undirected graph with no loops and no multiple edges. For a finite set X, we let $\mathcal{P}(X)$ denote the set of all subsets of X.

Let G be a graph. Let X be a finite set, and let g be an injective mapping from the vertex set $V(G)$ of G to $\mathcal{P}(X)$. If there exists a mapping f from the edge set $E(G)$ of G to $\mathcal{P}(X)$ such that

$$g(x) = \bigcup_{\substack{e \in E(G) \\ e \text{ is incident with } x}} f(e)$$

for all $x \in V(G)$, then we say that G is embeddable in X, and call g an embedding of G into X. It follows that g is an embedding if and only if

$$g(x) \subseteq \bigcup_{\substack{z \in V(G) \\ z \text{ is adjacent to } x}} g(z) \tag{1}$$

for all $x \in V(G)$.

(see [1; Lemma 2]). It is easy to see that if G is embeddable in a set of cardinality k, then $|V(G)| \leq 2^k$, G has no component of order 2, and G has at most one component of order 1.

For an integer $k \geq 2$, let $\phi(k)$ denote the maximum of those integers n for which every graph G of order at most n such that all components of G have order at least 3 is embeddable in a set of cardinality k. In [1], Aigner and Triesch proved that $\phi(k) \geq 2^{k-1}$ for all $k \geq 2$, and conjectured that $\phi(k) = 2^k$. In this paper, we settle this conjecture affirmatively by proving:

Main Theorem: *Let k be a nonnegative integer. Let G be a graph of order at most 2^k, and suppose that G has no component of order 2 and that G has at most one component of order 1. Then G is embeddable in a set of cardinality k.*

We prove the Main Theorem by induction and, to make the induction argument work, we modify the statement of the theorem. The modified statement is somewhat technical, and will thus be stated in Section 2.

We conclude this section with related results. Let G be a graph, let X be a finite set, and let g be an injective mapping from $V(G)$ to $\mathcal{P}(X)$. Suppose that there exists a mapping f from $E(G)$ to $\mathcal{P}(X)$ such that

$$g(x) = \ominus_{\substack{e \in E(G) \\ e \text{ is incident with } x}} f(e)$$

for all $x \in V(G)$, where $\ominus$ denotes symmetric difference. In this situation, we say that G is realizable in X. For an integer $k \geq 2$, let $\psi(k)$ denote the maximum of those integers n for which every graph G of order at most n such that all components of G have order at least 3 is realizable in a set of cardinality k. It is proved in [1] that $\psi(k) \geq 2^{k-2}$ for all $k \geq 2$, and it is proved in [5] that $\lim_{k\to\infty} \psi(k)/2^k = 1$, and the following theorem has recently proved in [2] and [3] independently:

Theorem A. *Let $k \geq 2$ be an integer. Let G be a graph with $|V(G)| \leq 2^k$ and $|V(G)| \neq 2^k - 2$, and suppose that every component of G has order at least 3. Then G is realizable in a set of cardinality k.*

2 Statement of Proposition

We start with definitions. Let G be a graph. For $x \in V(G)$, we let $N(x)$ denote the set of vertices adjacent to x in G, and let $\deg(x)$ denote the degree of x; thus $\deg(x) = |N(x)|$. When G is a forest, a vertex x with $\deg(x) \leq 1$ is called a leaf of G.

In view of (1), in proving the Main Theorem, we may assume that G is edge-minimal under the condition that G satisfies the assumptions of the theorem. Then it easily follows that each component T of G is a tree such that T has a vertex with the property that every component of $T-u$ has order at most 2. Following [1], we call such a tree an octopus.

Let T be an octopus, and let u be a vertex of T such that every component of $T-u$ has order at most 2. Let t denote the number of components of $T-u$ having order 2, and let s denote the number of components of $T-u$ having order 1. Let $A=\{x \in N(u) \mid \deg(x)=2\}$. Then $|A| = |V(T)-\{u\}-N(u)| = t$, and $\deg(u)=t+s$. Let $a,b \in V(T)$, and suppose that one of the following three conditions is satisfied:

$t+s \geq 2$, $a,b \in N(u)$, $a \neq b$, and $|\{a,b\} \cap A| \geq \min\{2,t\}$;

$t+s=1$ (so $2 \leq |V(T)| \leq 3$), and $a=b$ is the unique vertex in $N(u)$; or

$|V(T)|=1$ (so $a=b=u$).

In this situation, $(T;u;a,b)$ is called a rooted octopus of type (t,s), u is called the primary root, and a and b are called the secondary roots. By abuse of terminology, we also say that T is a rooted octopus of type (t,s). Let X be a finite set. An injective mapping g from $V(T)$ to $\mathcal{P}(X)$ is called a strong embedding of T into X if it satisfies the following three conditions:

for each $x \in N(u)$, $g(x) \subseteq g(u)$;

for each $x \in A$, $g(x) \supseteq g(z)$, where $N(x)=\{u,z\}$; and

if $t+s \geq 2$, then $g(u)=g(a) \cup g(b)$

(thus when $|V(T)|=1$, any mapping from $V(T)$ to $\mathcal{P}(X)$ is a strong mapping). It follows from (1) that

$$\text{if } t+s \geq 2, \text{ every strong embedding of } T \text{ is an embedding of } T. \quad (2)$$

Also it is obvious that

$$\text{in the case where } |V(T)|=1, \text{ a strong embedding } g \text{ of } T \text{ is an embedding of } T \text{ if and only if } g(u)=\phi. \quad (3)$$

If G is a forest each of whose components is a rooted octopus, we call G a rooted forest. When G is a rooted forest and X is a finite set, an injective mapping g from $V(G)$ to $\mathcal{P}(X)$ is called a strong embedding if the restriction of g to each component of G is a strong embedding. In view of (2) and (3), the Main Theorem follows from the following proposition:

Proposition. *Let k be a nonnegative integer, and let X be a set of cardinality k. Let G be a rooted forest of order 2^k, and let c be a leaf of G. Let R be the component of G containing c, and let w be the primary* *root of R, and let a, b be the secondary roots of R. Suppose that G has at most one component of type (1,0) and, if G has a component of type (1,0), then R is the unique component of type (1,0). Suppose further that one of the following five conditions is satisfied:*

(i) *R is of type* (t,s) *with* $t \geq 3$, $c \in V(R) - \{w\} - N(w)$, *and* $c \notin N(a) \cup N(b)$;

(ii) *R is of type* (t,s) *with* $1 \leq t \leq 2$, *and* $c \in V(R) - \{w\} - N(w)$;

(iii) *R is of type* (t,s) *with* $s \geq 1$ *and* $t+s \geq 3$, $c \in N(w)$, *and* $c \neq a, b$;

(iv) $|V(R)| = 2$, *and* $c = a = b$; *or*

(v) $|V(R)| = 1$.

Then there exists a strong embedding g of G into X such that $g(c) = \phi$.

3 Proof of Proposition

Let k, X, G, c be as in the proposition. We proceed by induction on the lexicographic order of the pair (k, h), where h denotes the number of leaves of G. The proposition clearly holds for $k = 0$. Thus we may assume $k \geq 1$.

Assume first that there exists a component T of G such that T is of type (t, s) with $s \geq 2$ and $t + s \geq 3$. Let u be the primary root of T, and write $N_T(u) = \{a_1, \ldots, a_{t+s}\}$. At the cost of relabeling, we may assume that $\deg(a_i) = 2$ for all $1 \leq i \leq t$ and $\deg(a_i) = 1$ for all $t+1 \leq i \leq t+s$, and that a_1 and a_2 are the secondary roots of T. For each $1 \leq i \leq t$, write $N(a_i) = \{u, b_i\}$. In the case where $c \in V(T)$, we may assume we have either $t \neq 0$ and $c = b_t$, or $c = a_{t+s}$. Define a rooted forest H by letting $V(H) = V(G)$ and $E(H) = (E(G) - \{ua_{t+s}\}) \cup \{a_{t+1}a_{t+s}\}$, and letting each component have the same primary and secondary roots in H as in G (note that $V(T)$ remains to be the vertex set of a component in H). Further, let $d = c$ or a_{t+s}, according to whether $c \notin V(T)$ or $c \in V(T)$. Then H and d satisfy the assumptions of the proposition, and the number of leaves of H is one less than that of G. Hence by the induction hypothesis, there exists a strong embedding f of H into X such that $f(d) = \phi$. Define a mapping g from $V(G)$ to $\mathcal{P}(x)$ as follows: if $c \neq b_t$ (so $c = a_{t+s}$ or $c \notin V(T)$), simply let $g = f$; if $c = b_t$, let $g(a_{t+s}) = f(b_t)$ and $g(b_y) = f(a_{t+s})$, and let $g(x) = f(x)$ for each $x \in V(G) - \{a_{t+s}, b_t\}$. Then g is a strong embedding of G with $g(c) = \phi$.

Thus we may assume each component of G is of type $(t, 0)$, $(t, 1)$ or $(0, 2)$. Let p denote the number of components of even order, and let $2q$ denote the number of components of odd order (note that $|V(G)| = 2^k$ is even). Let $T_1, \ldots, T_p$ be the components of even order, and let $S_1, \ldots, S_{2q}$ be the components of odd order. We now proceed to define a rooted forest R of order 2^{k-1} and a leaf d of H. For this purpose, we construct a rooted octopus P_j of order $|V(T_j)|/2$ for each $1 \leq j \leq p$, and construct a rooted octopus Q_j of order $(|V(S_j)| + |V(S_{q+j})|)/2$ for each $1 \leq j \leq q$.

We first let $1 \leq j \leq p$, and define $P = P_j$ as follows. For simplicity, let $T = T_j$, and let u be the primary root of T. Let $|V(T)| = 2t + 2$, and write $N(u) = \{a_1, \ldots, a_t, a_{t+1}\}$. We may assume that $\deg(a_i) = 2$ for all $1 \leq i \leq t$ and $\deg(a_{t+1}) = 1$. In the case where $t \geq 1$, we may further assume that a_1 and a_2 are the secondary roots of T. Write $N(a_i) = \{u, c_i\}$ for each $1 \leq i \leq t$. In the case where $c \in V(T)$, we may assume we have either $t \neq 0$ and $c = c_t$, or $t \neq 1$ and $c = a_{t+1}$. Assume first that $t \leq 1$ or $c \notin V(T)$. In this case, define $P = P_j$ by $V(P) = \{u, a_1, \ldots, a_t\}$ and $E(P) = \{ua_1, \ldots, ua_t\}$, and let u be the primary root of P. If $t \geq 2$, let P have the same secondary roots as T; if $t = 1$, let a_1 be the secondary root of P; if $t = 0$, let u be the secondary root of P. Further if $c \in V(T)$ (so $t \leq 1$), let $d = a_1$ or u, according to whether $t = 1$ or $t = 0$. Assume now that $t \geq 2$ and $c \in V(T)$. In this case, define P by $V(P) = \{u, a_1, \ldots, a_t\}$ and $E(P) = \{ua_1, ua_2, \ldots, ua_{t-1}, a_1a_t\}$, and let u be the primary root of P. If $t \geq 3$, let P have the same secondary roots as T; if $t = 2$, let a_1 be the secondary root of P. Further let $d = a_t$.

We now let $1 \leq j \leq q$, and define $Q = Q_j$ as follows. Let $T = S_j$ and $S = S_{q+j}$. Let u and v be the primary roots of T and S, respectively, and let $t = \deg(u)$ and $s = \deg(v)$. Write $N(u) = \{a_1, \ldots, a_t\}$. In the case where $t \geq 2$, we may assume a_1 and a_2 are the secondary roots of T. Write $N(v) = \{b_1, \ldots, b_s\}$. In the case where $s \geq 2$, we may assume b_1 and b_2 are the secondary roots of S. We consider three cases separately.

Case 1. Neither T nor S is of type $(0, 2)$.

We may assume $t \geq s$ and, if $t = s$ and $c \in V(T) \cup V(S)$, we may further assume $c \in V(S)$. Write $N(a_i) = \{u, c_i\}$ for each $1 \leq i \leq t$. In the case where $c \in V(T)$, we have $t \neq 0$ by the assumption we have just made, and we may therefore assume $c = c_t$. Write $N(b_i) = \{v, d_i\}$ for each $1 \leq i \leq s$. In the case where $c \in V(S)$, we may assume we have either $s \neq 0$ and $c = d_s$, or $s = 0$ and $c = v$. Assume first that $s \geq 1$ or $c \notin V(T) \cup V(S)$. In this case, define $Q = Q_j$ by $V(Q) = \{u, a_1, \ldots, a_t, c_1, \ldots, c_s\}$ and $E(Q) = \{ua_1, \ldots, ua_t, a_1c_1, \ldots, a_sc_s\}$, and let Q have the same roots as T. Assume further that $c \in V(T) \cup V(S)$ (so $s \geq 1$). In this case, let $d = a_t$ or c_s according to whether $c = c_t$ or d_s. Note that if $t = s$, then $c \in V(S)$ (so $c = d_s$ and $d = c_s$) by the assumption we made at the beginning of Case 1. Note also that if $s = 1$, then ($t \geq 2$ and) $c \in V(S)$ (so $c = d_1$ and $d = c_1$) by the assumption stated in the fourth sentence of the statement of the proposition. Assume now that $s = 0$ and $c \in V(T) \cup V(S)$. In this case, let $V(Q) = \{u, a_1, \ldots, a_t\}$. If $t \geq 2$, let $E(Q) = \{ua_1, \ldots, ua_{t-1}, a_1a_t\}$; if $t = 1$, let $E(Q) = \{ua_1\}$; if $t = 0$, let $E(Q) = \phi$. Let u be the primary root of Q. If $t \neq 2$, let Q have the same secondary roots as T; if $t = 2$, let a_1 be the secondary root of Q. Further let $d = a_t$ or u, according to whether $t \geq 1$ or $t = 0$.

Case 2. Precisely one of T and S is of type $(0, 2)$.

We may assume S is of type $(0, 2)$, and T is not. Then $c \notin V(S)$ by the assumptions of the proposition. Let the c_i be as in Case 1. Assume first that $t \geq 1$. In this case, let $V(Q) = \{u, a_1, \ldots, a_t, c_1\}$ and $E(Q) = \{ua_1, \ldots, ua_t, a_1c_1\}$, and let Q have the same roots as T. Further if $c \in V(T)$ (so $c = c_t$), let $d = c_1$. Assume now that $t = 0$. In this case, let $V(Q) = \{v, b_1\}$ and $E(Q) = \{vb_1\}$, and let v and b_1 be the primary and the secondary roots of Q, respectively. Further if $c \in V(T)$ (so $c = u$), let $d = b_1$.

Case 3. Both T and S are of type $(0, 2)$.

In this case, simply let $Q = T$, and let Q have the same roots as T.

Now with the P_j $(1 \leq j \leq p)$ and the Q_j $(1 \leq j \leq q)$ and d as above, let $H = (\cup_{1 \leq j \leq p} P_j) \cup (\cup_{1 \leq j \leq q} Q_j)$. Then H and d satisfy the assumptions of the proposition for $k - 1$. Fix $r \in X$. By the induction hypothesis, there exists a strong embedding f of H into $X - \{r\}$ such that $f(d) = \phi$.

Let again $1 \leq j \leq p$. We define a mapping f_j from $V(T_j)$ to $\mathcal{P}(X)$ as follows. Let T, t, u, a_i, b_i be as in the definition of P_j. If $t \leq 1$ or $c \notin V(T)$, let $f_j(u) = f(u) \cup \{r\}$ and $f_j(a_{t+1}) = f(u)$, and let $f_j(a_i) = f(a_i) \cup \{r\}$ and $f_j(c_i) = f(a_i)$ for each $1 \leq i \leq t$. If $t \geq 2$ and $c \in V(T)$, let $f_j(u) = f(u) \cup \{r\}$, $f_j(a_1) = f(u)$, $f_j(c_1) = f(a_1)$, $f_j(a_t) = f(a_1) \cup \{r\}$, $f_j(c_t) = f(a_t)$ or $f(a_t) \cup \{r\}$ according to whether $c = c_t$ or $c = a_{t+1}$, and $f_j(a_{t+1}) = f(a_t) \cup \{r\}$ or $f(a_t)$ according to whether $c = c_t$ or $c = a_{t+1}$, and let $f_j(a_i) = f(a_i) \cup \{r\}$ and $f_j(c_i) = f(a_i)$ for each $2 \leq i \leq t - 1$. Then f_j is a strong embedding of T and, in the case where $c \in T$, we have $f_j(c) = \phi$.

Let now $1 \leq j \leq q$. We define a mapping g_j from $V(T_j)$ to $\mathcal{P}(X)$ and a mapping h_j from $V(T_{j+q})$ to $\mathcal{P}(X)$ as follows. We again consider three cases separately. In each case, we let T, S, t, s, u, v, a_i, b_i, c_i, d_i be as in the definition of Q_j.

Case 1. Neither T nor S is of type $(0, 2)$.

If $s \geq 1$ or $t \leq 1$ or $c \notin V(T) \cup V(S)$, let $g_j(u) = f(u) \cup \{r\}$ and $h_j(v) = f(u)$, let $g_j(a_i) = f(a_i) \cup \{r\}$, $g_j(c_i) = f(c_i) \cup \{r\}$, $h_j(b_i) = f(a_i)$ and $h_j(d_i) = f(c_i)$ for each $1 \leq i \leq s$, and let $g_j(a_i) = f(a_i) \cup \{r\}$ and $g_j(c_i) = f(a_i)$ for each $s + 1 \leq i \leq t$. If $s = 0$, $t \geq 2$ and $c \in V(T) \cup V(S)$, let $g_j(u) = f(u) \cup \{r\}$, $g_j(a_1) = f(u)$, $g_j(c_1) = f(a_1)$, $g_j(a_t) = f(a_1) \cup \{r\}$, $g_j(c_t) = f(a_t)$ or $f(a_t) \cup \{r\}$, according to whether $c = c_t$ or $c = v$, and $h_j(v) = f(a_t) \cup \{r\}$ or $f(a_t)$ according to whether $c = c_t$ or $c = v$, and let $g_j(a_i) = f(a_i) \cup \{r\}$ and $g_j(c_i) = f(a_i)$ for each $2 \leq i \leq t-1$. Then g_j and h_j are strong embeddings of T and S and, in the case where $c \in V(T) \cup V(S)$, we have $g_j(c) = \phi$ or $h_j(c) = \phi$, according to whether $c \in V(T)$ or $c \in V(S)$.

Case 2. S is of type $(0, 2)$, but T is not.

Assume first that $t \geq 1$. In this case, let $h_j(v) = f(a_1) \cup \{r\}$, $h_j(b_1) = f(a_1)$, $h_j(b_2) = f(c_1) \cup \{r\}$, $g_j(u) = f(u) \cup \{r\}$ and $g_j(a_1) = f(u)$, and let $g_j(a_i) = f(a_i) \cup \{r\}$ for each $2 \leq i \leq t$. Further if $t = 1$ (so $c = c_1 \in V(T)$) or $c \notin V(T) \cup V(S)$, let $g_j(c_1) = f(c_1)$, and let $g_j(c_i) = f(a_i)$ for each $2 \leq i \leq t$; if $t \geq 2$ and $c \in V(T) \cup V(S)$ (so $c \in V(T)$ and $c = c_t$), let $g_j(c_1) = f(a_t)$ and $g_j(c_t) = f(c_1)$, and let $g_j(c_i) = f(a_i)$ for each $2 \leq i \leq t-1$. Assume now that $t = 0$. In this case, let $h_j(v) = f(v) \cup \{r\}$, $h_j(b_1) = f(v)$, $h_j(b_2) = f(b_1) \cup \{r\}$ and $g_j(u) = f(b_1)$. In either case, g_j and h_j are strong embeddings of T and S and, in the case where $c \in V(T)$, we have $g_j(c) = \phi$.

Case 3. Both T and S are of type $(0, 2)$.

Let $g_j(u) = f(u) \cup \{r\}$, $g_j(a_1) = f(a_1) \cup \{r\}$, $g_j(a_2) = f(a_2) \cup \{r\}$, $h_j(v) = f(u)$, $h_j(b_1) = f(a_1)$ and $h_j(b_2) = f(a_2)$. Then g_j and h_j are strong embeddings of T and S.

Now we define a mapping g from $V(G)$ to $\mathcal{P}(X)$ by putting together all f_j, g_j and h_j; that is to say, define g to be the mapping such that the restriction to $V(T_j)$ coincides with f_j for each $1 \leq j \leq p$, and such that the restrictions to $V(S_j)$ and $V(S_{q+j})$ coincides with g_j and h_j for each $1 \leq j \leq q$. Then g is a strong embedding of G with $g(c) = \phi$.

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