Connectivity of cycle matroids and bicircular matroids

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ABSTRACT. A unified approach to prove former connectivity results of Tutte, Cunningham, Inukai and Weinberg, Oxley and Wagner.

1 Introduction

We assume familiarity with elementary matroid theory and graph theory. Graphs in this note are finite and have no isolated vertices. The terminology used in this note for matroids and graphs will in general follow Oxley [5] and Bondy and Murty [1], respectively. A *cycle* in a graph G is a 2-regular connected graph. The term *circuit* is reserved for matroids.

This note considers the relationship between the n-connection of a matroid on the edge set of a graph G and the n-connection of the graph G. Such a problem was first studied by Tutte in [9], where Tutte characterized graphs G with Tutte n-connected cycle matroid (to be defined in Section 2) in terms of partitions of the edge set $E(G)$ with certain properties. Simpler proof was later found by Cunningham [2]. Pursuing the relationship between vertex connectivity of a graph G and the corresponding concept in matroids, Cunningham [2], Inukai and Weinberg [3], and Oxley [6] independently discovered the notion of vertical connectivity of a matroid,

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as well as its dual concept (the cyclical connectivity), and characterized graphs G whose cycle matroid $M(G)$ is vertically n-connected (cyclically n-connected, respectively), in a way similar to the Tutte's characterization.

Following the same track, Wagner [10] studied the relationship between the Tutte n-connection of the bicircular matroid (to be defined in Section 2) of a graph G and the Tutte n-connection of G, and found a similar characterization.

The main purposes of this paper are: (1) to investigate properties that are common in different types of n-connection in both the cycle matroid of a nontrivial graph G and the bucircular matroid of G; and (2) to complete the obviously undone job: characterizations of graphs whose bicircular matroids are vertically n-connected and cyclically n-connected, respectively.

The definitions of various type of connections will be given in Section 2. The exact statements of the abovementioned characterizations will be given in Section 4. Some connectivity properties of the cycle matroid and the bicircular matroid of a graph G will be investigated in Section 3. In Section 4, we shall apply the results obtained in Section 3 to present alternative proofs of the abovementioned characterizations, and to prove the characterizations of graphs whose bicircular matroids are vertically n-connected and cyclically n-connected, respectively.

2 Definitions

We will be concerned with partitions of E into two sets, X and $E-X$; thus both X and $E-X$ are assumed non-empty. Let M be a matroid on $E=E(M)$ with rank function r. The *connectivity function*, $k(\cdot)$ of M is

$$k(X)=r(X)+r(E-X)-r(E), \quad \text{for any } X\subseteq E.$$

Let l be an integer. A partition $\{X,E-X\}$ is a *Tutte l-separation* of M if

$$k(X)<l \text{ and } \min\{|X|,|E-X|\}\geq l, \tag{1}$$

a *vertical l-separation* of M if

$$k(X)<l \text{ and } \min\{r(X),r(E-X)\}\geq l, \tag{2}$$

and a *cyclical l-separation* of M if

$$k(X)<l \text{ and each of } M|X \text{ and } M|(E-X) \text{ has a circuit.} \tag{3}$$

An l-separation $\{X,E-X\}$ is *exact* if $k(X)=l-1$.

For a positive integer n, the matroid M is *Tutte n-connected* if for all l, $1\leq l<n$, M has no Tutte l-separation. We define a matroid M to

be *vertically n-connected* and *cyclically n-connected* similarly. A Tutte 2-connected matroid is also called a *connected* matroid.

Let G be a finite connected graph. The *set of vertices of attachment* of the subgraph H in G, is

$$A_G(H) = V(H) \cap V(G[E(G) - E(H)]).$$

Let l be an integer. A partition $\{X, E(G) - X\}$ is a *Tutte l-separation* of G if

$$|A_G(G[X])| \leq l \text{ and } \min\{|X|, |E - X|\} \geq l, \tag{4}$$

a *vertical l-separation* of G if

$$|A_G(G[X])| \leq l \text{ and } \min\{|V(G[X])|, |V(G[E(G) - X])|\} \geq l + 1, \tag{5}$$

and a *cyclical l-separation* of G if

$$|A_G(G[X])| \leq l \text{ and each of } G[X] \text{ and } G[E(G) - X] \text{ has a cycle.} \tag{6}$$

For a positive integer n, the graph G is *Tutte n-connected* if for all l, $1 \leq l < n$, M has no Tutte l-separation. We define a graph G to be *vertically n-connected* and *cyclically n-connected* similarly.

Let G be a graph with $E(G) \neq \emptyset$. The *cycle matroid* of G, denoted $M(G)$, is the matroid on $E(G)$ whose collection of circuits consists of all the cycles of G.

Let G be a graph with $E(G) \neq \emptyset$. Let $D_1(G)$ denote the set of vertices of degree 1 in G. A *bicycle* of G is a connected subgraph H with $|E(H)| = |V(H)| + 1$ and with $D_1(H) = \emptyset$. The *bicircular matroid* of G, denoted $B(G)$, is a matroid on $E(G)$ whose collection of circuits consists of all the bicycles of G. Bicircular matroids were first discovered by Simões-Pereira [7], and have been studied extensively. (See Simões-Pereira [8] for an overview.)

3 Some Properties of $M(G)$ and $B(G)$

Let G be a graph. A component H of G is *acyclic* if H is a tree; otherwise H is *cyclic*. The number of components of G is $\omega(G)$, and the number of acyclic components of G is $\omega_a(G)$. Note that for $X \subseteq E(G)$, the rank function $r(\cdot)$ in the cycle matroid $M(G)$ and the bicircular matroid $B(G)$ can be expressed as follows:

$$r(X) = \begin{cases} |V(G[X])| - \omega(G[X]) & \text{if } M = M(G) \\ |V(G[X])| - \omega_a(G[X]) & \text{if } M = B(G). \end{cases}$$

An edge $e \in E(G)$ is an *end edge* if e is incident with a vertex in $D_1(G)$. If $\{X, E(G)-X\}$ is a partition of $E(G)$, then define $o(X) = \omega(G[X]) + \omega(G[E(G)-X])$. The main result of this section is Theorem 3.8, which shows a property commonly shared by both the cycle matroid $M(G)$ and the bicircular matroid $B(G)$ of a nontrivial graph G. First, we establish some lemmas.

Lemma 3.1. *Let M be a matroid with rank function r. Each of the following holds:*

(i) *If $k(X) = r(X)$ for some $X \subset E(M)$, then $E - X$ contains a basis of M.*

(ii) *If $k(X) = |X|$ for some $X \subset E(M)$, then X is independent and $E - X$ contains a basis of M.*

Proof: (i) The definition of $k(X)$ and the equality $k(X) = r(X)$ give $r(E-X) = r(E)$. Therefore $E-X$ contains a basis of M.

(ii) If $k(X) = r(X) + r(E-X) - r(E) = |X|$, then $r(X) = |X|$ and $r(E-X) = r(E)$. Therefore X is independent and $E-X$ contains a basis of M. □

Lemma 3.2. *Each of the following holds:*

(i) (Proposition 4.1.4 of [5]) *The matroid M is connected if and only if, for every pair of distinct elements of $E(M)$, there is a circuit in M containing both.*

(ii) (Proposition 4.1.8 of [5]) *Let G be a loopless graph without isolated vertices and with $|E(G)| \geq 3$. Then $M(G)$ is connected if and only if G is 2-connected.*

Lemma 3.3. (Proposition 4 of [4]) *Let G be a graph with $|E(G)| \geq 2$ and without isolated vertices, and let $B(G)$ be the bicircular matroid of G. Then $B(G)$ is connected if and only if each of the following holds:*

(i) *G is connected,*

(ii) *G is not a cycle, and*

(iii) *G has no vertices of degree 1.*

Lemma 3.4. *Let M be a matroid with rank function r, and let $\{X, E(M)-X\}$ be a partition of $E(M)$. If there exist nonempty subsets $X_1, X_2, \cdots X_c$ of X such that $r(X) = \sum_{i=1}^{c} r(X_i)$, then each of the following holds.*

(i) $r(X) = r(X_i) + r(X - X_i)$, *for each* i *with* $1 \leq i \leq c$.

(ii) $k(X_i) \leq k(X)$.

Proof: (i) is obvious. By the submodularity of the rank function, $r(E - X_i) \leq r(E - X) + r(X - X_i)$. Therefore by (i),

$$\begin{aligned} k(X_i) &= r(X_i) + r(E - X_i) - r(E) \\ &\leq r(X_i) + r(X - X_i) + r(E - X) - r(E) = k(X). \end{aligned}$$

□

The following lemma is an immediate consequence of connectivity.

Lemma 3.5. *Let* G *be a connected graph with* $E = E(G)$ *and let* $\{X, E - X\}$ *be a partition of* E *with* $X \neq \emptyset$ *and* $E - X \neq \emptyset$. *If* $X_1 \subset X$ *induces a component* $G[X_1]$ *of* $G[X]$, *then*

$$V(G[X_1]) \cap V(G[E - X]) \neq \emptyset, \text{ and } V(G[X - X_1]) \cap V(G[E - X]) \neq \emptyset.$$

Lemma 3.6. *Let* G *be a connected graph with* $|E(G)| \geq 3$ *and let* M *be either the cycle matroid or the bicircular matroid of* G. *Let* $E = E(G)$ *and let* $\{X, E - X\}$ *be a partition of* $E(G)$. *Suppose that* $X_1, X_2, \cdots, X_c$ *are nonempty subsets of* X *such that* $X = \cup_{i=1}^{c} X_i$, *where* $c = \omega(G[X])$, *and such that* $r(X) = \sum_{i=1}^{c} r(X_i)$. *For a proper subset* $N \subset \{1, 2, \cdots, c\}$, *let* $X_N = \cup_{i \in N} X_i$.

(i) *Suppose that* M *is Tutte (vertically, resp.)* n*-connected, and that* $\{X, E - X\}$ *is a Tutte (vertical, resp.)* n*-separation of* M. *If* $c \geq 2$ *and if for some* $N \subset \{1, 2, \cdots, c\}$, $r(X_N) \geq n$, *then* $\{X_N, E - X_N\}$ *is a Tutte (vertical, resp.)* n*-separation of* M *with* $o(X_N) < o(X)$.

(ii) *Suppose that* M *is Tutte cyclically* n*-connected, and that* $\{X, E - X\}$ *is a cyclical* n*-separation of* M. *If* $c \geq 2$, *if for some* $N \subset \{1, 2, \cdots, c\}$, $r(X_N) \geq n$, *and if* X_N *contains a circuit, then* $\{X_N, E - X_N\}$ *is a cyclical* n*-separation of* M *with* $o(X_N) < o(X)$.

(iii) *Suppose that* M *is Tutte (cyclically, resp.)* n*-connected, and that* $\{X, E - X\}$ *is a Tutte (cyclical, resp.)* n*-separation of* M. *If* $c \geq 2$ *and if for some* $N \subset \{1, 2, \cdots, c\}$, $r(X_N) < |X_N|$, *then* $\{X_N, E - X_N\}$ *is a Tutte (vertical, or cyclical, resp.)* n*-separation of* M *with* $o(X_N) < o(X)$.

(iv) *Suppose that* M *is vertically* n*-connected, and that* $\{X, E - X\}$ *is a vertical* n*-separation of* M. *If* $c \geq 2$ *and if for some* $N \subset \{1, 2, \cdots, c\}$, $k(X_N) < r(X_N))$, *then* $\{X_N, E - X_N\}$ *is a vertical* n*-separation of* M *with* $o(X_N) < o(X)$.

(v) *Suppose that M is Tutte (vertically, resp.) n-connected, and that $\{X, E-X\}$ is an exact Tutte (vertical, resp.) n-separation of M. If $V(G)-(G[E-X]) \neq \emptyset$, then there is Tutte (vertical, resp.) n-separation $\{X', E-X'\}$ of M such that $o(X')=2$.*

(vi) *Suppose that M is Tutte (vertically, resp.) n-connected, and that $\{X, E-X\}$ is an exact Tutte (vertical, resp.) n-separation of M. If both X and $E-X$ are independent in M, then there is a Tutte (vertical, resp.) n-separation $\{X', E-X'\}$ of M such that $o(X')=2$.*

Proof: (i) and (ii). Assume $r(X_N) \geq n$. Then $|X_N| \geq r(X_N) \geq n$. Note that if $\{X, E-X\}$ is a Tutte n-separation, then $|E-X_N| \geq |E-X| \geq n$; if $\{X, E-X\}$ is a vertical n-separation, then $r(E-X_N) \geq r(E-X) \geq n$; if $\{X, E-X\}$ is a cyclical n-separation, then $M|(E-X_N)$, containing $M|(E-X)$ as a restriction, has a circuit. Thus (i) and (ii) follow by Lemma 3.4.

(iii) and (iv). By (i) and (ii), for Tutte or cyclical connection, it suffices to show that $r(X_N) \geq n$. If not, we assume that there is an X_i with $r(X_N) < |X_N|$ and $r(X_N) < n$. Note that $r(X_N) \geq k(X_N)$. By Lemma 3.4, $k(X_N) \leq k(X) \leq n-1$. Since $|E-X_N| \geq |E-X| \geq n > r(X_N)$ (since $r(X_N) < |X_N|$ implies that X_N contains a circuit, resp.), $\{X_N, E-X_N\}$ would be a Tutte (cyclical, resp.) $r(X_N)$-separation, contrary to the assumption that M is Tutte (cyclical, resp.) n-connected. Thus $r(X_N) \geq n$, and so (ii) follows from (i). For vertical connection, if there is some X_i with $k(X_i) < r(X_i)$, then by (i), $r(X_i) < n$, and so $r(E-X_i) \geq r(E-X) \geq n > r(X_i)$. Therefore $\{X_i, E-X_i\}$ is a vertical $r(X_i)$-separation of M, contrary to the assumption that M is vertically n-connected.

(v) and (vi). Assume $\omega(G[E-X]) \leq \omega(G[X])$ and $c \geq 2$. Among all Tutte (vertical, resp.) n-separations $\{X, E-X\}$ such that $V' = V(G) - V(G[E-X]) \neq \emptyset$ for (v), or such that both X and $E-X]$ are independent for (vi), choose one $\{X, E-X\}$ such that $o(X)$ is minimized.

By (i) - (iv), we may assume that $r(X_N) = |X_N| < n$ (in the Tutte connection case) and $k(X_N) = r(X_N)$ (in the vertical connection case), $\forall \emptyset \neq N \subset \{1, 2, \cdots, c\}$. Therefore we observe:

(A) $\forall N$ with $\emptyset \neq N \subset \{1, 2, \cdots, c\}$, $k(X_N) = r(X_N)$ and $r(E-X_N) = r(E)$.

If $k(X_N) < |X_N| < n$, then since $|E-X_N| > |E-X| \geq n > |X_N|$, $\{X_N, E-X_N\}$ is a Tutte $|X_N|$-separation, contrary to the assumption that M is n-connected. Therefore $k(X_N) = |X_N|$, and so (A) follows from Lemma 3.1. This proves (A).

Assume (v). Pick $v \in V'$. By (A) with $N = \{1, \cdots, c\} - \{i\}$, $v \in V(G[X_i])$, for each i. Thus there exist distinct i and j such that $v \in$

$V(G[X_i]) \cap V(G[X_j])$, contrary to the assumption that $G[X_i]$'s are the components of $G[X]$.

Now assume (vi). Label the components of $G[X]$ so that $|X_1| \geq |X_2| \geq \cdots \geq |X_c|$. We further choose $\{X, E-X\}$, subject to minimizing $o(X)$, such that $|X_1|$ is maximized. We shall show that $c = 1$ and so (vi) follows from (v) (with X in (vi) replacing $E - X$ in (v)). Suppose $c \geq 2$. By (A), there is an edge $e' \in X - X_1$ such that $r((E-X) \cup e') = r(E-X) + 1$. Since G is connected, there is an edge $e'' \in E - X$ incident with exactly one vertex in $G[X_1]$. Let $X' = (X - e') \cup e''$. Then $r(X) = r(X') = |X'|$ and $r(E-X) = r(E-X') = |E-X'|$, and so $\{X', E-X'\}$ is also an n-separation with both X' and $E - X'$ independent in M, contrary to the choice of $\{X, E-X\}$. Hence $c = 1$. □

Lemma 3.7. *Let $M = M(G)$ or $M = B(G)$, and let $\{X, E-X\}$ be a Tutte (vertical, resp.) n-separation of M. Suppose that some component of $G[X]$ has an end edge e and that some component of $G[E-X]$ has a cut edge e' such that in G, e' is not incident with the isolate vertex in $G[X-e]$. Let $X' = (X-e) \cup e'$. Then $\{X', E-X'\}$ is a Tutte (vertical, resp.) n'-separation of M such that $n' \leq n$ and $V(G) - V(G[X']) \neq \emptyset$.*

Proof: Note that $r(X') \leq r(X)$ and $r(E-X') \leq r(E-X)$, and so $k(X') \leq k(X)$. Note also that the vertex of degree one incident with e in X becomes an isolated vertex in $G[X']$. Thus $\{X', E-X'\}$ is a Tutte (vertical, resp.) n'-separation with $n' = k(X') + 1 \leq n$ and $V(G) - V(G[X']) \neq \emptyset$. □

Note that such a pair (e, e') in Lemma 3.7 can always be found if both $G[X]$ and $G[E-X]$ have a component which is a tree of at least 3 vertices.

Theorem 3.8. *Let $n \geq 1$ be an integer, let G be a connected graph with $|E(G)| \geq 2$, and let M be either the cycle matroid or the bicircular matroid of G. Let $E = E(G)$. If one of the following holds:*

(i) *M is Tutte n-connected and $\{X, E-X\}$ is a Tutte n-separation of M such that*

$$o(X) = \min\{o(X') : \{X', E-X'\} \text{ is a Tutte } n\text{-separation of } M\}, \text{ or} \quad (7)$$

(ii) *M is vertically n-connected and $\{X, E-X\}$ is a vertical n-separation of M such that*

$$o(X) = \min\{o(X') : \{X', E-X'\} \text{ is a vertical } n\text{-separation of } M\}, \text{ or} \quad (8)$$

(iii) *M is cyclical n-connected and $\{X, E-X\}$ is a cyclical n-separation of M such that*

$$o(X) = \min\{o(X') : \{X', E-X'\} \text{ is a cyclical } n\text{-separation of } M\}, \quad (9)$$

then both $G[X]$ and $G[E-X]$ are connected.

Proof:: We argue by contradiction. Assume that G is a counterexample. We may assume that $|E(G)| \geq 3$ since otherwise Theorem 3.8 holds trivially.

We may also assume that $n \geq 2$. For if $n = 1$, then $k(X) = 0$ for some $X \subset E$ implies that $G[X]$ is a union of blocks of G, and so Theorem 3.8 follows trivially.

We assume that there is an n-separation $\{X, E-X\}$ of $M(G)$ satisfying one of the conditions of Theorem 3.8 but $o(X) \geq 3$. We may assume that $\omega(G[E-X]) \leq \omega(G[X])$ and $c = \omega(G[X]) \geq 2$. Let $X_1, X_2, \cdots, X_c$ are nonempty subsets of X such that $X = \cup_{i=1}^{c} X_i$ and such that each $G[X_i]$ is a component of $G[X]$. Let $d = \omega(G[E-X])$ and $Y_1, Y_2, \cdots, Y_d$ are nonempty subsets of $E-X$ such that $E-X = \cup_{i=1}^{d} Y_i$ and such that each $G[Y_i]$ is a component of $G[E-X]$.

If $\{X, E-X\}$ is an exact cyclical n-separation, then there must be an X_i with $r(X_i) < |X_i|$, and so by Lemma 3.6(iii), $c = 1$, a contradiction. This proves Theorem 3.8(iii).

Assume then that $\{X, E-X\}$ is an exact Tutte n-separation. Apply Lemma 3.6(iii) to both X and $E-X$, we may assume either $\omega(G[E-X]) = 1$, or both X and $E-X$ are independent in M. Note that $k(X) = n-1$ implies $r(E) > r(E-X)$. Hence by Lemma 3.6 (v) and (vi), and by $\omega(G[E-X]) = 1$, we conclude that $M = B(G)$ and $G[E-X]$ is a spanning tree of G. If $G[X]$ has an acyclic component, then by Lemma 3.7, there is an n-separation $\{X', E-X'\}$ with $V(G) - V(G[X']) \neq \emptyset$, and so Theorem 3.8(i) follows from Lemma 3.6(v). Hence every component of $G[X]$ is cyclic. But then $r(X) = \sum_{i=1}^{c} |V(G[X_i])| = |V(G)| = r(E)$, and so $n-1 = k(X) = r(X) + r(E-X) - r(E) = r(E-X) > n$, a contradiction. This proves Theorem 3.8(i).

Hence we assume that $\{X, E-X\}$ is an exact vertical n-separation. By Lemma 3.6, we observe:

(A) $r(X_N) < n$, $\forall N$ with $\emptyset \neq N \subset \{1, 2, \cdots, c\}$; and $r(Y_N) < n$, $\forall N$ with $\emptyset \neq N \subset \{1, 2, \cdots, d\}$.

(B) $k(X_N) = r(X_N)$ and $r(E - X_N) = r(E)$, $\forall N$ with $\emptyset \neq N \subset \{1, 2, \cdots, c\}$; and $k(Y_N) = r(Y_N)$ and $r(E - Y_N) = r(E)$, $\forall N$ with $\emptyset \neq N \subset \{1, 2, \cdots, d\}$.

(C) $V(G) = V(G[E-X]) = V(G[X])$.

Suppose $M = B(G)$. By (A), (B), and Lemma 3.7, either G has a pair of parallel edges $\{e_1, e_2\}$ such that $Y_{i'} = \{e_1\}$ and $X_{i''} = \{e_2\}$ for some i', i'', whence M has a vertical $(n-1)$-separation $\{X - e_2, E - (X - e_2)\}$, contrary to the assumption that M is vertically n-conncted; or we may assume that every $G[Y_i]$ is cyclic, whence by (C), $r(E-X) = \sum_{i=1}^{d} |V(G[Y_i])| =$

$|V(G)| = r(E)$, and so $n-1 = k(X) = r(X)+r(E-X)-r(E) = r(X) > n$, a contradiction. This proves Theorem 3.8(ii) when $M = B(G)$.

Suppose $M = B(G)$. For vertices $u, v \in V(G)$, define

$$\kappa(u, v) = \text{maximum number of internly disjoint } (u, v)\text{-paths in } G,$$

and let $k = \min\{\kappa(u, v) : u, v \in V(G)\}$. Since M has a vertical n-separation, G is not spanned by a complete subgraph. Hence one can find nonadjacent $u, v \in V(G)$ such that $\kappa(u, v) = k$. By Menger's Theorem, there is a subset $V' \subset V(G) - \{u, v\}$ such that G has connected subgraphs G_1 and G_2 with $G = G_1 \cup G_2$, with $u \in V(G_1)$ and $v \in V(G_2)$ and with $|V(G_1) \cap V(G_2)| = k$. Therefore, $\{E(G_1), E(G_2)\}$ is a vertically k-separation of M with $o(E(G_1)) = 2$. Clearly $k \geq n$.

To complete the proof of Theorem 3.8(ii) when $M = M(G)$, it suffices to show that $k = n$. However, this is proved by Cunningham in [2]. □

4 Connectivity in cycle matroids and bicircular matroids

Throughout this section, G denotes a nontrivial connected graph. If H is a subgraph of G, then $\bar{H} = G[E(G) - E(H)]$.

Proposition 4.1. (Cunningham, Proposition 1 of [2]) *If $X \subset E(G)$, then in $M(G)$, $k(X) \leq |A_G(G[X])| - 1$, where equality holds if and only if both $G[X]$ and $G[E(G) - X]$ are connected.*

Theorem 4.2. (Tutte, Theorem 3.5 of [9]) *Let G be a nontrivial connected graph. Then $M(G)$ is Tutte n-connected if and only if G is Tutte n-connected.*

Theorem 4.3. (Cunningham, Theorem 2 of [2]) *Let G be a nontrivial connected graph. Then $M(G)$ is cyclically n-connected if and only if G is cyclically n-connected.*

Theorem 4.4. (Cunningham, Theorem 1 of [2], Inukai and Weinberg, Theorems 1 and 2 of [3], and Oxley, Theorem 2 of [6]) *Let G be a nontrivial connected graph. Then $M(G)$ is vertically n-connected if and only if G is vertically n-connected.*

Proofs of Theorems 4.2 and 4.3: Let $E = E(G)$. Assume that $M(G)$ is Tutte n-connected but G is not Tutte n-connected. Then G has a Tutte l-separation $\{X, E-X\}$, for some $1 \leq l < n$. By Proposition 4.1, $\{X, E-X\}$ is a Tutte l-separation of $M(G)$, contrary to the assumption that $M(G)$ is Tutte n-connected. Hence G must be Tutte n-connected.

Assume that G is Tutte n-connected but $M(G)$ is not Tutte l-connected. Let l be an integer with $1 \leq l < n$ such that $M(G)$ is Tutte l-connected but not Tutte $(l+1)$-connected. Then by Theorem 3.8(i), there is a Tutte

l-separation $\{X, E-X\}$ of $M(G)$ such that both $G[X]$ and $G[E(G)-X]$ are connected. It follows by Proposition 4.1 that $l-1 \geq k(X) = |A_G(G[X])|-1$. Thus $l \geq |A_G(G[X])|$, and so by (4), $\{X, E-X\}$ is a Tutte l-separation of G. Therefore G is not Tutte n-connected. This proves Theorem 4.2.

The proof for Theorems 4.3 is similar, using Theorem 3.8(iii) in place of Theorem 3.8(i) in the argument.

Theorem 4.4 can also be proved by a similar argument using Theorem 3.8(ii). However, as our proof for Theorem 3.8(ii) when $M = M(G)$ uses the same idea in Cunningham's proof for Theorem 4.4, this should not be regarded as a different proof.

Let $l \geq 1$ be an integer. Define $\mathcal{F}(l;G)$ to be the collection of partitions $\{E(H), E(G) - E(H)\}$, where H is a subgraph of G such that both H and $\bar{H}$ are connected, and such that

$$|A_G(H)| = \begin{cases} l-1 & \text{if } \omega_a(H) = \omega_a(\bar{H}) = 0 \\ l & \text{if } \omega_a(H) + \omega_a(\bar{H}) = 1 \text{ or } \omega_a(G) = 1 \\ l+1 & \text{if } \omega_a(H) = \omega_a(\bar{H}) = 1 \text{ and } \omega_a(G) = 0 \end{cases}$$

Let G be a connected graph and let $E = E(G)$. Let $X \subset E$ with $\{X, E-X\} \in \mathcal{F}(l;G)$. The partition $\{X, E-X\}$ is a *l-biseparation* if

$$\min\{|X|, |E-X|\} \geq l,$$

a *vertical l-biseparation* if

$$\min\{|V(G[X])| - \omega_a(G[X]), |V(G[E-X])| - \omega_a(G[E-X])\} \geq l,$$

and a *cyclical l-biseparation* if

both $G[X]$ and $G[E-X]$ have a bicycle.

The graph G is *n-biconnected* if G has no l-biseparation for any $1 \leq l < n$. We define a graph G to be *vertically n-biconnected* and *cyclically n-biconnected* similarly.

Examples: Fix $i \in \{1,2\}$. Let H_i be the graph with $V(H_i) = \{v_j^i : 1 \leq j \leq 4\}$ and $E(H_i) = \{v_j^i v_{j'}^i : 1 \leq j < j' \leq 4\} - \{v_1^i v_2^i\}$. (That is, H_i is isomorphic to K_4 minus an edge.) Let G be obtained from the disjoint union from H_1 and H_2 by adding four more edges $\{v_j^1 v_j^2 : 1 \leq j \leq 4\}$. Let $X_1 \subset E(G)$ be the three edges incident with a vertex of degree 3 in G; let $X_2 = E(H_1)$. Then it can be seen that $\{X_1, E(G) - X_1\}$ is a vertical 3-biseparation, and that $\{X_2, E(G) - X_2\}$ is both a 5-biseparation and a cyclical 5-biseparation of G. It can be verified that G is 5-biconnected, vertically 3-biconnected, and cyclically 5-biconnected.

Proposition 4.5. *Let G be a nontrivial connected graph. If $X \subset E(G)$, and if $H = G[X]$, then in $B(G)$,*

$$k(X) \leq |A_G([X])| - \begin{cases} 0 & \text{if } \omega_a(H) = \omega_a(\bar{H}) = 0 \\ 1 & \text{if } \min\{\omega_a(H), \omega_a(\bar{H})\} = 0 \text{ and } \{\omega_a(H), \omega_a(\bar{H})\} \geq 1, \\ & \text{or if } \omega_a(G) = 1 \\ 2 & \text{if } \min\{\omega_a(H), \omega_a(\bar{H})\} \geq 1 \text{ and } \omega_a(G) = 0 \end{cases}$$

where equality holds when both $G[X]$ and $G[E(G) - X]$ are connected.

Proof: Let $E = E(G)$, and let r denote the rank function of $B(G)$. By the definition of bicircular matroids, for any $X \subseteq E$, $r(X) = |V(G[X])| - \omega_a(G[X])$, and so

$$\begin{aligned} k(X) &= r(X) + r(E - X) - r(E) \\ &= |V(H)| + |V(\bar{H})| - |V(G)| - \omega_a(H) - \omega_a(\bar{H}) + \omega_a(G) \\ &= |A_G(H)| - \omega_a(H) - \omega_a(\bar{H}) + \omega_a(G). \end{aligned}$$

This completes the proof. □

Theorem 4.6. (Wagner, [10]) *Let G be a connected graph. Then $B(G)$ is Tutte n-connected if and only if G is n-biconnected.*

The proof for Theorem 4.6 will be similar to that for Theorem 4.7, and so it will be omitted.

Theorem 4.7. *Let G be a connected graph. Each of the following holds.*

(i) *$B(G)$ is vertically n-connected if and only if G is vertically n-biconnected.*

(ii) *$B(G)$ is cyclically n-connected if and only if G is cyclically n-biconnected.*

Proof: Let $E = E(G)$. Suppose that $B(G)$ is vertically (cyclically, resp.) n-connected but G is not vertically (cyclically, resp.) n-connected. Then G has a vertical (cyclical, resp.) l-biseparation $\{X, E - X\}$ with $1 \leq l < n$. Then by Proposition 4.5, $\{X, E - X\}$ is a vertical (cyclical, resp.) l-separation of $B(G)$, contrary to the assumption that $B(G)$ is not vertical (cyclical, resp.) n-connected.

Suppose that G is vertically (cyclically, resp.) n-connected but $B(G)$ is not vertically (cyclically, resp.) n-connected. Then there is an integer l, $1 \leq l < n$, such that $B(G)$ is vertically (cyclically, resp.) l-connected, but not vertically (cyclically, resp.) $(l+1)$-connected. Then by Theorem 3.8(ii) and (iii), there is a vertical (cyclical, resp.) l-separation $\{X, E - X\}$ of $B(G)$ such that both $G[X]$ and $G[E - X]$ are connected. By Proposition 4.5, $\{X, E - X\} \in \mathcal{F}(l; G)$, and so $\{X, E - X\}$ is a vertical (cyclical, resp.) l-biseparation, contrary to the assumption that G is vertically (cyclically, resp.) n-biconnected. This proves Theorem 4.7. □

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