Some New Characterizations of Matroids

S. Mishra *

Abstract

Three new characterizations of matroids are presented.
Keywords: greedy algorithm, matroids, convexity

1. Introduction

Let E be a nonempty finite set and let $\mathcal{I}$ be a nonempty set of subsets of E. Then the system $\sum = (E, \mathcal{I})$ is an independence system (IS for short) if it satisfies the conditions (a) $\phi \in \mathcal{I}$, (b) if $F_1 \in \mathcal{I}$ and $F_2 \subseteq F_1$ then $F_2 \in \mathcal{I}$. In the sequel, we use the symbol $\sum$ to represent an IS $\sum = (E, \mathcal{I})$. A set in the family $\mathcal{I}$ is called independent and a maximal independent set in $\mathcal{I}$ is called a base. We will denote the collection of all bases of $\sum$ by $\mathcal{B}$. We define rank of a set $S \subseteq E$ in an IS $\sum$ as $r(S) = \max \{|T| : T \in \mathcal{I}, T \subseteq S\}$. Rank of an IS is the cardinality of the largest base. A subset of E is called dependent if it is not independent, and a minimal dependent set is called a circuit. An IS $\sum$ is called a matroid if it satisfies (c) for all $I, J \in \mathcal{I}$, if $|J| > |I|$, then there exists an element $e \in J - I$ such that $I \cup \{e\} \in \mathcal{I}$.

Given an IS $\sum = (E, \mathcal{I})$ where $E = \{e_1, e_2, \ldots, e_n\}$, and a cost function $C : E \longrightarrow \mathcal{R}_+$ we define the combinatorial optimization problem $P(C)$ as follows:

$$P(C) \qquad \text{Maximize} \quad C(F) := \sum_{e \in F} C(e) \tag{1}$$

$$\text{Subject to} \quad F \in \mathcal{I}. \tag{2}$$

With the problem P(C) we can associate the following integer linear programming problem ILP(C):

$$\text{Maximize} \quad CX \tag{3}$$

$$\text{Subject to} \quad AX \leq \mathbf{r} \tag{4}$$

$$X \geq 0 \tag{5}$$

$$X \text{ binary} \tag{6}$$

*Stat/Math. Unit, Indian Statistical Institute, 203 B. T. Road, Calcutta 700 035, India.

where $X \in \mathcal{R}^n$, $C \in \mathcal{R}^n$ as in P(C) and $A_{(2^n-1)\times n}$ is a $(0,1)$ matrix whose rows are the characteristic vectors of the $2^n - 1$ nonempty subsets of E so that each inequality of (4) is of the form $\sum_{e\in S} X_e \leq r(S)$. ILP(C) can be transformed to the following form by adding $2^n - 1$ slack variables.

$$
\begin{array}{llll}
ILP(C)^{=} & \text{Maximize} & CX & (7) \\
 & \text{Subject to} & AX + IY = \mathbf{r} & (8) \\
 & & X \geq 0, Y \geq 0 & (9) \\
 & & X \text{ binary} & (10)
\end{array}
$$

Let $\mathbf{B}$ be any basis for (8) with $\mathbf{B}^{-1}\mathbf{r} \geq \mathbf{0}$ (i.e. a feasible basis) and in the corresponding solution, let the variables belonging to X be binary. Let

$$\mathcal{K}_o(\mathbf{B}) = \{C \in \mathcal{R}^n_+ : \mathbf{B} \text{ is optimal for } ILP(C)^{=}\}. \quad (11)$$

Recall that a subset $S \subseteq \mathcal{R}^n$ is a convex set if for any $x^1, x^2, \ldots, x^k \in S$, $\sum_{i=1}^k \alpha_i x^i \in S$, whenever $\alpha_i \in \mathcal{R}_+$ and $\sum_{i=1}^k \alpha_i = 1$. S is a cone if $x \in S$ implies $\alpha x \in S$ $\forall \alpha \in \mathcal{R}_+$. It is well known that, if $\sum$ is a matroid then all extreme points of $\{X \in \mathcal{R}^n : X \geq 0, AX \leq \mathbf{r}\}$ have integer components and are in one-to-one correspondence with the characteristic vectors of the independent subsets of the matroid (Edmonds [70]). Hence ILP(C) can be considered as a linear programming problem by dropping constraints (6), whenever $\sum$ is a matroid. From the theory of linear programming (Dantzig [63]) it follows that $\mathcal{K}_o(\mathbf{B})$ is a covex cone for each basis $\mathbf{B}$ or, equivalently, for each independent set $B \in \mathcal{I}$, whenever $\sum$ is a matroid. We are intrested in the converse statement i.e., "given an IS $\sum$, suppose $\mathcal{K}_o(\mathbf{B})$ is a convex cone for each feasible basis $\mathbf{B}$ of the corresponding ILP(C)$^{=}$, then can we say that $\sum$ is a matroid?" While attempting to answer this question we got new characterizations of matroids, which we present in this paper.

There are several well-known characterizations of matroids (Rado [42] and Mac Lane [38] etc.). In particular, characterization of matroids in terms of greedy algorithm is due to Gale [68], Edmonds [71], Kruskal [56], Rado [57] and Welsh [68]. For a brief survey of matroids we refer to Kung [86] and Welsh [76]. In this paper we provide three new characterizations of matroids in terms of convex sets.

Before we give our characterizations let us recall some of the characterizations of matroids which we shall use in our work and whose proofs can be found in Lawler [76], Nemhauser [88] and Welsh [68]. Let $\{e_1, e_2, ..., e_n\}$ be a permutation of E such that $i < j \Rightarrow C(e_i) \geq C(e_j)$. Let $E_i = \{e_1, e_2, ..., e_i\}$ for $1 \leq i \leq n$. A subset $F \subseteq E$ is called a greedy solution of $P(C)$ if, for $1 \leq i \leq n, F \cap E_i$ is a maximal independent subset of E_i. It is easy to see by induction that a greedy solution is just a set F yielded by the following greedy algorithm.

Greedy Algorithm
begin
  order the elements of E such that $C(e_1) \geq C(e_2) \geq \ldots \geq C(e_n)$;
  $F := \phi$;
  for $i := 1$ to n do
    if $F \cup \{e_i\} \in \mathcal{I}$ then $F := F \cup \{e_i\}$;
end. /* of Greedy Algorithm */

For an IS $\sum$, a greedy solution for P(C) is called a greedy optimal solution if it is an optimal solution for P(C).

Theorem 1 *An IS $\sum$ is a matroid iff every greedy solution is an optimal solution for $P(C)$ for all $C \in \mathcal{R}^n_+$.*

Theorem 2 *A nonempty collection $\mathcal{B}$ of subsets of E is the set of bases of a matroid on E iff it satisfies the base exchange axiom: if $B_1, B_2 \in \mathcal{B}$ and $e_1 \in B_1 - B_2$, there exists $e_2 \in B_2 - B_1$ such that $(B_1 \cup e_2) - e_1 \in \mathcal{B}$.*

Theorem 3 *An IS $\sum$ is a matroid iff, for any nonnegative cost of the elements in E, a lexicographically maximum set in $\mathcal{I}$ has maximum cost.*

Theorem 4 *An IS $\sum$ is a matroid iff, whenever $F \in \mathcal{I}$ and $F \cup e \notin \mathcal{I}$, the set $F \cup e$ contains exactly one circuit.*

Corollary 4.1 If $\sum$ is a nonmatroid IS, then there exists a base B and $e \in E - B$ such that $(B \cup e) - e' \notin \mathcal{I}$ for all $e' \in B$.
Proof: Follows from Theorems 2 and 4. ■

2. New characterizations

Let $\sum = (E, \mathcal{I})$ be an IS and let B be a base. Define the sets $\mathcal{K}_o(B)$ and $\mathcal{K}(B)$ as:

$$\mathcal{K}(B) = \{C \in \mathcal{R}^n_+ : B \text{ is greedy optimal for } P(C)\} \qquad (12)$$

$$\mathcal{K}_o(B) = \{C \in \mathcal{R}^n_+ : B \text{ is optimal for } P(C)\} \qquad (13)$$

Clearly $\mathcal{K}(B)$ and $\mathcal{K}_o(B)$ are cones and $\mathcal{K}_o(B)$ is a convex cone.

Lemma 1 *If the IS $\sum$ is a matroid then $\mathcal{K}(B)$ is convex for all bases B.*

Proof: Let B be any base. We provr the lemma by showing $\mathcal{K}(B) = \mathcal{K}_o(B)$. Obviously $\mathcal{K}(B) \subseteq \mathcal{K}_o(B)$. If possible, let $C \in \mathcal{K}_o(B)$ and $C \notin \mathcal{K}(B)$, i.e. B is optimal for $P(C)$ but not a greedy base. Thus there exists another base B_1 which is greedy with respect to the cost vector C. Since $\sum$ is a matroid, all greedy bases are lexicographically maximum with respect to the cost ordering C of the elements of E and of maximum cost in $\mathcal{I}$ (by Theorems 1 and 3). Let rank of $\sum$ be p, $B = \{a_1, a_2, \ldots, a_p\}$ with $C(a_1) \geq C(a_2) \geq \ldots \geq C(a_p)$ and $B_1 = \{b_1, b_2, \ldots, b_p\}$ with $C(b_1) \geq C(b_2) \geq \ldots \geq C(b_p)$. From the definition of a lexicographically maximum base it follows that $C(b_i) \geq C(a_i)$ for $1 \leq i \leq p$. Since B is not greedy, it must be strictly lexicographically smaller than B_1. Hence $\sum_{i=1}^{p} C(b_i) > \sum_{i=1}^{p} C(a_i)$. Hence B is not optimal for $P(C)$, which is a contradiction. Hence $\mathcal{K}_o(B) = \mathcal{K}(B)$. ∎

Lemma 2 *Let $\sum$ be an IS for which $\mathcal{K}(B)$ is convex for all bases B. Then $\sum$ is a matroid.*

Proof: If possible, assume that $\sum$ is not a matroid. By Corollary 4.1, there exists a base B_1 and $y \in E - B_1$ such that $(B_1 \cup y) - e \notin \mathcal{I}$ for all $e \in B_1$. Let $S = B_1 \cup y$ and B_y be a maximal independent subset of S containing y. Then $B_y \neq S$ since S is dependent and if $|B_y| = |S| - 1$, we get a contradiction to the property just stated. Thus $|B_y| \leq |S| - 2$, hence there exist two distinct elements p and q in $S - B_y$. Note that $B_y \cup p \notin \mathcal{I}$ and $B_y \cup q \notin \mathcal{I}$. Now consider the cost vector C^1 defined by

$$C^1(e) = \begin{cases} 1, & \text{for } e \in B_y - y \\ 1 - \epsilon, & \text{for } e = y \\ 1, & \text{for } e = p \\ \frac{1}{2}, & \text{for } e = q \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

where $0 < \epsilon < \frac{1}{4}$. Let C^2 be the cost function obtained from C^1 by interchanging the values of 1 and $\frac{1}{2}$ at p and q. One can easily show that B_1 is greedy solution for C^1 as well as for C^2.

Now we shall prove that B_1 is optimal for $P(C^1)$. Clearly $C^1(B_1) = C^1(B_y) + \frac{1}{2} + \epsilon$, and $C^1(S) = C^1(B_1) + 1 - \epsilon$. Suppose B_1 is not optimal for $P(C^1)$ and let $B_2 \in \mathcal{I}$ be a base with $C^1(B_2) > C^1(B_1)$. Let $I = B_2 \cap S$. Thus $C^1(I) = C^1(B_2)$ as $C^1(e) = 0$ for all $e \in E - S$. Hence

$$C^1(I) > C^1(B_1). \tag{15}$$

Since S is the set of elements with nonzero costs, $C^1(S)$ is an upper bound for $C^1(I)$. Since S is dependent set $I \subset S$. We claim that $B_y \cup p \subset I$. Suppose $y \notin I$, then $C^1(I) \leq C^1(S - y) = C^1(B_1)$, which contradicts (15).

Suppose $x \notin I$ for any $x \in S - y$ and $x \neq q$, then $C^1(I) \leq C^1(S - x) = C^1(S) - 1 \leq C^1(B_1)$, again this contradicts (15). Hence our claim is true. Since $B_Y \cup p$ is dependent and $B_Y \cup p \subseteq I \subseteq B_2$, B_2 is dependent, which is a contradiction. Hence B_1 is optimal for $PC(C^1)$. Similarly it can be proved that B_1 is also optimal for $PC(C^2)$.

$C^1, C^2 \in \mathcal{K}(B_1)$ as B_1 is greedy optimal for both $PC(C^1)$ and $PC(C^2)$. Let $C = \frac{1}{2}C^1 + \frac{1}{2}C^2$. For the cost vector C, let $\overline{B}_y$ be any greedy solution. Any greedy solution for $PC(C)$ clearly contains B_y and cannot contain p and q. Thus B_1 is not a greedy solution for $PC(C)$ and $c \notin \mathcal{K}(B_1)$. Hence C is not a member of $\mathcal{K}(B_1)$, i.e. $\mathcal{K}(B_1)$ is not convex. ■

Combining Lemmas 1 and 2 we have Theorem 5.

Theorem 5 *An IS $\sum$ is a matroid iff $\mathcal{K}(B)$ is convex for all B.*

Let

$$\mathcal{K} = \bigcup_{B \in \mathcal{B}} \mathcal{K}(B) \tag{16}$$

Theorem 6 *An IS $\sum$ is a matroid iff $\mathcal{K}$ is convex.*

Proof: Soppose $\sum$ is not a matroid. Construct B_1, y, C^1, C^2 and C as in the proof of Lemma 2. Then with respect to cost function C, any greedy solution has cost $|B_y| - \epsilon$ whereas the base B_1 has cost $|B_y| + 0.5$. Thus no greedy solution for $PC(C)$ is optimal for $PC(C)$ and so there is no base B such that $C \in \mathcal{K}(B)$. Thus $C \notin \mathcal{K}$ and $\mathcal{K}$ is not convex. On the other hand, if $\sum$ is a matroid then a greedy solution is optimal for all $C \in \mathcal{R}^n_+$ (by Theorem 1), hence $\mathcal{K} = \mathcal{R}^n_+$. Thus $\mathcal{K}$ is convex. ■

Theorem 7 *An IS $\sum$ is a matroid iff $\mathcal{K} = \mathcal{R}^n_+$.*

Proof: This follows easily from the proof of the previous theorem. ■

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