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Stability, domination, and irredundance in W_{AR} graphs

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Abstract

A graph is well-covered if it has no isolated vertices and all the maximal independent sets have the same cardinality. If furthermore this cardinality is exactly half the number of vertices, the graph is called very well covered. Sankaranarayana in [5] presented a certain subclass of well covered graphs (called W_{AR}) and gave a characterization of this class which generalized the characterization of very well covered graphs given by Favaron [2]. The purpose of this article is to generalize to this new subclass some results concerning the stability, domination, and irredundance parameters proved for very well covered graphs in [2].

1. Introduction

Let $G = (V, E)$ be a simple connected graph. A *clique* is a complete subgraph of G, and throughout this paper, we will denote by the same symbol the clique and its corresponding vertex set. We will denote by $G[K]$ the subgraph induced by the subset K of the vertex set V, and by $d(v)$ the degree of the vertex v. An *independent* set S is a set of nonadjacent vertices. The minimum (resp. maximum) cardinality of a maximal independent set is denoted by $i(G)$ (resp. $\alpha(G)$). A set D of vertices of G is *dominating* if every vertex of $V - D$ has at least one neighbor in D. The minimum (resp. maximum) cardinality of a dominating set is denoted by $\gamma(G)$ (resp. $\Gamma(G)$). A set I of vertices of G is *irredundant* if every vertex x of I which is not isolated in I has at least one I-private neighbor x', that is a vertex of $V - I$ which is adjacent to x but to no other vertex of I. The minimum (resp. maximum) cardinality of a maximal irredundant set is denoted by $ir(G)$ (resp. $IR(G)$). A vertex *annihilates* (or is an *annihilator* of) a vertex x of an irredundant I (not isolated in I) if it dominates the whole I-private neighborhood of x . We mention the well known chain of inequalities between these parameters :

$$ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G).$$

A simple connected graph is said to be *well-covered* if $i(G) = \alpha(G)$, that is every maximal independent set is maximum (see [4] for a survey on well-covered graphs). This concept, which was first introduced by Plummer in 1970 [3], is of interest since the independence number problem, which is NP-complete for general graphs, can be solved efficiently for this family. We say that a well-covered graph of even order is *very well covered* if every maximal independent set in it contains exactly half the vertices in the graph.

A simple connected graph is said to be *complete k-partite* if its vertex set can be partitioned into k disjoint independent sets, or *parts*, such that each vertex is adjacent to every other vertex that is not in the same part. It is said to be *complete k_n-partite* if furthermore all parts have the same number n of vertices.

Let $\mathcal{C} = \{C_1, C_2, \cdots, C_p\}$ be a clique partition of a graph G. For every $v \in V$ we denote by $N_{C_i}(v)$ the set $N(v) \cap C_i$ and by $d_{C_i}(v)$ its cardinality, by $C(v)$ the clique of $\mathcal{C}$ that v belongs to. We say that a clique partition $\mathcal{C}$ is a Q-clique partition if $\mathcal{C}$ satisfies the following property :

Property Q:
a) $\forall v \in V, \forall i \in \{1, 2, \cdots, p\}, \quad d_{C_i}(v) = 0$ or $d_{C_i}(v) = |C_i| - 1$.
b) $\forall v \in V, \quad (w \in C(v), u \in N(v) - N(w)) \Rightarrow (u$ dominates $N(w) - N(v))$.

The first condition states that if a vertex in G has a neighbor in some clique of the clique partition, then it is adjacent to all but one vertex in that clique. Note that all cliques contain at least two vertices, since G is connected. The second one states that for every two vertices in a clique, their non-common neighbors are adjacent.

Proposition 1.1. :
A graph admiting a Q-clique partition is well-covered, and each Q-clique partition consists of $i(G) = \alpha(G)$ cliques and all of them are maximal.

Proof : Let S be a maximal stable set and $\mathcal{C} = \{C_1, C_2, \cdots, C_p\}$ be a clique partition. Each clique C_i contains at most one vertex of S, hence $|S| \leq p$. Suppose that $C_1 \cap S = \emptyset$. Then, as S dominates C_1 and by Property Qa), there exist at least two vertices x_2 and x_3 in S (we can suppose $x_i \in C_i$) and there exist y_2 and y_3 in C_1 such that for $i = 1, 2$, x_i dominates y_i but not y_j $(j \neq i)$. Then by Property Qb), x_2x_3 is an edge, which contradicts the stability of S. Thus each clique C_i contains exactly one vertex of S and therefore $p = i(G) = \alpha(G)$. Each clique C_i is maximal, otherwise there exist C_j and a vertex x in $V - C_j$ such that x dominates C_j, hence $d_{C_j}(x) = |C_j|$, which contradicts Property Qa).□

Sankaranarayana in [6] introduced a hierarchy of four new subclasses of well-covered graphs. Among them is the class W_{AR}, mentioned in the abstract and defined below in Definition 1.2 by one of its characterizations given in [5]. This generalizes the characterization of very well covered graphs observed in [2] : a graph is very well covered if and only if it admits a Q-clique partition with each clique of cardinality two.

Definition 1.2. : (Sankaranarayana)[5]
G belongs to W_{AR} if G admits a Q-clique partition.

In very well covered graphs, the chain of inequalities between the six studied parameters splits into two chains of equalities as shown by :

Theorem 1.3. : (Favaron)[2]
If G is very well covered then $i = \alpha = \Gamma = IR$ and $ir = \gamma$.

The aim of this paper is to generalize to W_{AR} graphs the two previous chains of equalities dealing with the six parameters concerning stability, domination, and irredundance, and to relate them to the structure into Q-clique partitions.

2. Equivalence relation

In this section let G belong to W_{AR} and let $\mathcal{C}$ be a Q-clique partition of G. Like Sankaranarayana we define the following equivalence relation :

Definition 2.1. : (Sankaranarayana)[5]
We say that $u \equiv v$ if either $u = v$ or $|C(u)| = |C(v)|$ and $\forall (x, y) \in C(u) \times C(v)$, xv and yu are edges if and only if $x \neq u$ and $y \neq v$.

Note that two vertices of the same clique cannot be equivalent and that by Property Q, this relation is effectively an equivalence. We will denote by U the equivalence class of u and by $C(U)$ the corresponding clique class, that is, $C(U)$ is made up of the cliques $C(u)$ corresponding to each vertex $u \in U$ together with the edges between the cliques. This equivalence relation is very helpful to describe the structure of W_{AR} graphs :

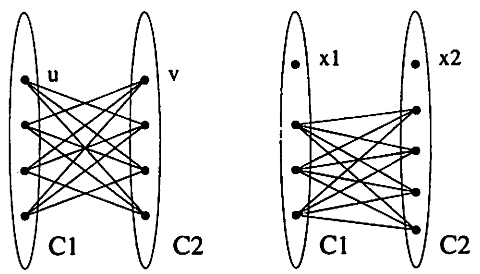
Proposition 2.2. : (Sankaranarayana)[5]
a) The equivalence classes partition V into independent sets.
b) Each clique class is complete k_n-partite, with each part forming an equivalence class, and the clique classes form a partition of G.

Now we mention some results concerning the relation $\equiv$ and some definitions that will be useful afterwards.

Lemma 2.3. :
If $u \equiv v$ and $ux \in E$, then $vx \in E$.

That is, two equivalent vertices have the same neighborhood.

Proof : This is clear if $x \in C(u) \cup C(v)$. Now suppose $x \notin C(u) \cup C(v)$. As $ux \in E$, by Property $Qa)$ there exists w in $C(u)$ such that wx is not an edge. As $u \equiv v$ we have $wv \in E$ (indeed v is adjacent to every vertex in $C(u) - \{u\}$) and Property $Qb)$ implies $vx \in E$.□



$C_1 \equiv C_2$ $(C_1 - x_1)\perp(C_2 - x_2)$

Figure 1 : C_1 and C_2 are two cliques.

Definition 2.4. : (See Figure 1)
Let C_1 and C_2 be two cliques in $\mathcal{C}$. We say that :
- $C_1 \equiv C_2$ if $G\ [C_1 \cup C_2\ ]$ is complete k_2-partite.
- $(C_1 - x_1)\perp(C_2 - x_2)$ if there exists $(x_1, x_2) \in C_1 \times C_2$ such that $d_{C_2}(x_1) = d_{C_1}(x_2) = 0$ and such that $G\ [(C_1 - \{x_1\}) \cup (C_2 - \{x_2\})\ ]$ is a clique.

Remark 2.5. : (See Figure 1)
a) $u \equiv v$ if and only if $C(u) \equiv C(v)$ with $\{u, v\}$ forming one of the parts.
b) If $C_1 \equiv C_2$ every $(y_1, z_1, y_2, z_2) \in {C_1}^2 \times {C_2}^2$ such that $y_1 \equiv y_2$ and $z_1 \equiv z_2$ satisfies $G\ [y_1, z_1, y_2, z_2\ ] \simeq C_4$.

c) If $(C_1 - x_1) \perp (C_2 - x_2)$ every $(y_1, z_1, y_2, z_2) \in {C_1}^2 \times {C_2}^2$ satisfies $G\ [y_1, z_1, y_2, z_2\ ] \simeq P_4, K_4$ or Z_1 (a triangle with another vertex adjacent to exactly one of the three vertices of the triangle).

Definition 2.6. : Let C be a Q-clique partition of G.
A *link* is an induced subgraph $G\ [y_1, z_1, z_2, y_2\ ]$ isomorphic to C_4 such that $y_1 z_1$ and $y_2 z_2$ are edges of two distinct cliques of C.

The following result shows how the edges are positionned between two cliques of a Q-clique partition, whether they are in the same clique class or not.

Lemma 2.7. : (See Figure 1)
If there exists an edge between two cliques C_1 and C_2 in C then :
Either $C_1 \equiv C_2$ and $G\ [C_1 \cup C_2\ ]$ contains a link.
Or $(C_1 - x_1) \perp (C_2 - x_2)$ and $G\ [C_1 \cup C_2\ ]$ is linkless.

Proof : Suppose for instance that $|C_1| \le |C_2|$. We will consider two cases :
Case 1 : *If there exists $x_1 \in C_1$ such that $d_{C_2}(x_1) = 0$* : Let $y_1 y_2$ be an edge with $y_i \in C_i$ $(i = 1, 2)$. Then by Property Qa) let us call x_2 the vertex of C_2 such that $N_{C_2}(y_1) = C_2 - \{x_2\}$. Suppose that there exists $z_1 \in C_1$ such that $z_1 x_2 \in E$. Let $z_2 \in C_2 - \{x_2\}$. We know that $y_1 z_2$ is an edge, hence by Property Qa), z_2 is adjacent to every vertex of $C_1 - \{x_1\}$ (because of $d_{C_2}(x_1) = 0$) and in particular to z_1. Therefore z_1 is adjacent to $C_2 - \{x_2\}$ and to x_2, a contradiction. Thus $d_{C_1}(x_2) = 0$. Then, by Property Qa), there exists an edge between each vertex of $C_1 - \{x_1\}$ and each vertex of $C_2 - \{x_2\}$.
Case 2 : *If for all $x \in C_1$ $d_{C_2}(x) > 0$* : Let $C_1 = \{x_1, x_2, \cdots, x_l\}$. By Property Qa), for each x_i there exists one and only one y_i in C_2 such that $x_i y_i \notin E$. Moreover if $|C_1| < |C_2|$ there exists $y_{l+1} \in C_2$ such that y_{l+1} is adjacent to each vertex of C_1, a contradiction. Hence $G\ [C_1 \cup C_2\ ]$ is complete k_2-partite.
To conclude, by Remark 2.5, it is clear that $C_1 \equiv C_2$ if and only if $G\ [C_1 \cup C_2\ ]$ contains a link.□

We now express the translation of Property Qb) in terms of whole cliques.

Lemma 2.8. : (See Figure 2)
Suppose that C_1, C_2 and C_3 are three cliques in $\mathcal{C}$ and suppose there exists (x_1, y_2, z_2, x_3) in $C_1 \times {C_2}^2 \times C_3$ such that $(C_1 - x_1)\perp(C_2 - y_2)$ and $(C_2 - z_2)\perp(C_3 - x_3)$.
If $y_2 \neq z_2$ then $(C_1 - x_1)\perp(C_3 - x_3)$.

Proof : We know that $N_{C_3}(y_2) = C_3 - \{x_3\}$ and $d_{C_1}(y_2) = 0$. Similarly $N_{C_1}(z_2) = C_1 - \{x_1\}$ and $d_{C_3}(z_2) = 0$. We now apply Property $Qb)$ to C_2 ($v = y_2$ and $w = z_2$) and get that $G\,[C_1 - \{x_1\}, C_3 - \{x_3\}\,]$ is a clique, and thus $(C_1 - x_1)\perp(C_3 - x_3)$ by Lemma 2.7 .□

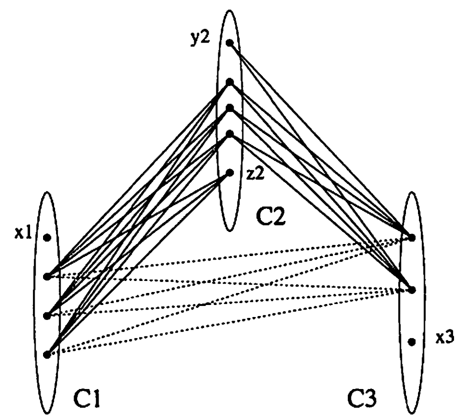


Figure 2 : C_1, C_2 and C_3 are three cliques.

3. Reduction

Let G belong to W_{AR} and let $\mathcal{C}$ be a Q-clique partition of G. One can associate with the graph G the quotient graph $\tilde{G}$ obtained by replacing each class U of G by one vertex called U, and where two vertices are joined in $\tilde{G}$ if there exists in G an edge between the corresponding two classes. This is equivalent by Lemma 2.3 to saying that the subgraph of G induced by the two classes is complete bipartite (for an example of reduction see Figure 3).

Theorem 3.1. :
$\tilde{G}$ belongs to W_{AR} and is the same for every choice of a Q-clique partition of G.

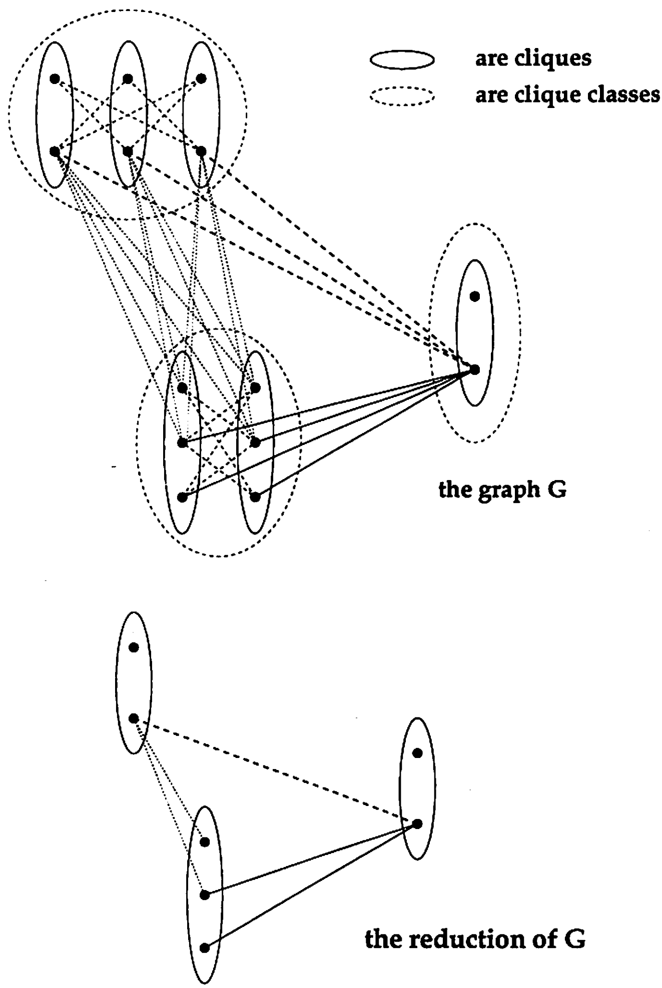


Figure 3 : G and its associated $\tilde{G}$.

Proof :
It is clear that $\tilde{G}$ belongs to W_{AR} since every two vertices in the same class have the same neighbors. Let $\mathcal{C} = \{C_1, C_2, \cdots, C_p\}$ and $\mathcal{C}^* = \{C_1^*, C_2^*, \cdots, C_p^*\}$ be two Q-clique partitions of G (by Proposition 1.1, $p = i(G) = \alpha(G)$). Every $C^* \in \mathcal{C}^*$ is entirely included in a clique class in relation to $\mathcal{C}$. Indeed, otherwise suppose that there exists $C^* \in \mathcal{C}^*$ satisfying $C_1 \cap C^* \neq \emptyset$ and $C_2 \cap C^* \neq \emptyset$ with C_1 and C_2 not in the same clique class in relation to $\mathcal{C}$. For $i = 1, 2$ let y_i be a vertex of $C_i \cap C^*$ and as $C_1 \not\equiv C_2$ there exist $x_i \in C_i$ $(i = 1, 2)$ such that $(C_1 - x_1) \perp (C_2 - x_2)$ (see Lemma 2.7). The vertex x_1

is not in C^* because $x_1y_2 \notin E$ (C^* is a clique), and the same for x_2. Then by Property Qb) applied to C^* (with $v = y_1$ and $w = y_2$), we must have $x_1x_2 \in E$, a contradiction.
Let $\{C_1, C_2, \cdots, C_l\}$ be a clique class in relation to $\mathcal{C}$ constituted by $|C_1|$ equivalence classes. As C^* is a clique, C^* does not contain two vertices u and v such that u is equivalent to v in relation to $\mathcal{C}$, hence $|C^*| \leq |C_1|$. If $|C^*| < |C_1|$, let U be a class such that $U \cap C^* = \emptyset$ and let u be an element in U; then u dominates C^*, because in a clique class $uv \notin E$ if and only if $u \equiv v$, which contradicts the maximality of C^* (see Proposition 1.1). Thus $|C^*| = |C_1|$ and C^* contains exactly one vertex by class in the clique class. Therefore all the cliques of $\mathcal{C}^*$ located in the clique class in relation to $\mathcal{C}$ are equivalents and the clique classes in relation to $\mathcal{C}$ and in relation to $\mathcal{C}^*$ are the same.□

Remark 3.2. :
To resume, if we consider the classes in relation to a fixed Q-clique partition, each clique of another Q-clique partition contains exactly one vertex by class in some clique class, and we can obtain all the Q-clique partitions in this way.

Definition 3.3. :
We say that $\tilde{G}$ is the *W_{AR}-irreducible graph* associated to G and we say that G is *W_{AR}-irreducible* if $\tilde{G} = G$.

Conversely, in a graph G which has a Q-clique partition $\mathcal{C}$, let us consider the following operation which gives a new graph denoted $G(C, S)$: if $C \in \mathcal{C}$ and $C = \{u_1, u_2, \cdots, u_k\}$ replace every vertex u_i by a copy S_i of an independent set S, join every pair of vertices in S_i and, corresponding to each edge u_ix in G, add edges s_ix in $G(C, S)$ for all s_i in S_i. Then $G(C, S)$ belongs to W_{AR} and one can get all W_{AR} graphs from irreducible ones by applying the above operation to different cliques. Let us now study a characterization of W_{AR}-irreducible graphs in order to recognize them.

Theorem 3.4. :
The following are equivalent for a graph G.
a) G is W_{AR}-irreducible.
b) G has a linkless Q-clique partition.
c) G has an unique Q-clique partition.

Proof :
a) $\Leftrightarrow$ b) : G is W_{AR}-irreducible if and only if there is no C_1 and C_2 in $\mathcal{C}$ such as $C_1 \equiv C_2$, that is, if and only if G is linkless (See Lemma 2.7)

a) ⇔ c) : See Remark 3.2 □

We now suppose that G is W_{AR}-irreducible, and it is important to note that, as G is irreducible, for every $(C_1, C_2) \in \mathcal{C}^2$ such that there exists an edge between C_1 and C_2, then $(C_1 - x_1) \perp (C_2 - x_2)$.

<u>Definition 3.5.</u> : Let G be a W_{AR}-irreducible graph.
A vertex v is *simplicial* if $G\ [N(v)\ ]$ is a clique in a Q-clique partition, a clique containing a simplicial vertex is also said to be *simplicial*, and the number of simplicial cliques in G is denoted by $q(G)$.

<u>Proposition 3.6.</u> : Let G be a W_{AR}-irreducible graph.
1) A vertex v is simplicial if and only if $N(v) = C(v) - \{v\}$, that is, if and only if $d(v) = |C(v)| - 1$.
2) If G is not a clique, then each clique of the clique partition contains at most one simplicial vertex.

<u>Proof</u> :
1) Clearly $C(v) - \{v\} \subset N(v)$ and by Property $Qa)$ there cannot exist a vertex of $N(v)$ out of $C(v)$, otherwise $G\ [N(v)\ ]$ could not be a clique.
2) By Property Qa).□

<u>Proposition 3.7.</u> : Let G be a W_{AR}-irreducible graph.
Every vertex is simplicial or is adjacent to every vertex except the simplicial vertex of some simplicial clique.

<u>Proof</u> : Suppose that z_1 is not simplicial. Then z_1 is adjacent to some vertices in another clique C_2. Let C_1 be $C(z_1)$. As G is W_{AR}-irreducible, by Lemma 2.7, for some $x_1 \in C_1$ and $x_2 \in C_2$ we have $(C_1 - x_1) \perp (C_2 - x_2)$. We can suppose that there exists $y_3 \in N(x_2) - C_2$, otherwise by Proposition 3.6 x_2 is simplicial and note that z_1 is adjacent to $C_2 - \{x_2\}$. Let $C_3 = C(y_3)$, and by Lemma 2.7 we have $(C_2 - y_2) \perp (C_3 - x_3)$ with $y_2 \in C_2 - \{x_2\}$ and $x_3 \in C_3 - \{y_3\}$. Then by Lemma 2.8 we have $(C_1 - x_1) \perp (C_3 - x_3)$. Finally, $N(x_2) \subset N(z_1)$ and because of $x_1 x_2 \notin E$ we have $d(x_2) < d(z_1)$. Let us rename x_2 as z_2, and note that if $C_2 = C(z_2)$ then z_1 dominates $C_2 - \{z_2\}$. We can iterate the process by replacing z_1 by z_2. The process stops since there is no strictly descending infinite chain of degree. Remark that if z_1, z_2 and z_3 are three consecutive terms of the process, z_1 dominates $C_2 - \{z_2\}$ and z_2 dominates $C_3 - \{z_3\}$, hence by Property $Qb)$ applied to $w = z_2$, $u = z_1$ and v some vertex of $C_2 - \{z_2\}$, z_1 dominates $C_3 - \{z_3\}$. Thus, we can suppose that z_3 is the last vertex of the process, which must be simplicial and the associated clique suits.□

4. Independent, Dominating, and irredundant sets

We are now ready to prove the main result of this article which gives, according to the structure due to the Q-clique partition of G, the relation between the six parameters concerning stability, domination, and irredundance.

Proposition 4.1. :
Let G be a W_{AR} and let Y_j, $1 \leq j \leq q(G)$, be the simplicial vertices in $\tilde{G}$. Then every subgraph $\tilde{A}$ of $\tilde{G}$ containing one vertex Z_j of each $C(Y_j) - \{Y_j\}$ is a dominating set of $\tilde{G}$ and every subgraph A of G containing one vertex z_j of each class Z_j is a dominating set of G. Hence, there exists a dominating set in G which contains exactly one vertex of each simplicial clique of the associated W_{AR}-irreducible graph $\tilde{G}$ of G.

Proof : Let t be a vertex of G with equivalence class T in G. From Proposition 3.7, the vertex T of $\tilde{G}$ has a neighbor Z_j in $\tilde{G}$. Moreover by Lemma 2.3, t is adjacent in G to all the vertices in the class Z_j and in particular to the only vertex z_j of $A \cap Z_j$.□

Remark 4.2.:Since two equivalent vertices have the same neighborhood:
1) If an irredundant set I of G contains a vertex x, either x is isolated in I and if I is maximal, it contains all the vertices of the class X of x, or x has a neighbor adjacent to no other vertex of I, and I does not contain any other vertex of X.
2)a) If I is an irredundant set of G, then the set $\tilde{I}$ defined by $\tilde{I} = \bigcup_{x \in I} X$ is irredundant in $\tilde{G}$.
b) If $\tilde{I}$ is an irredundant set of $\tilde{G}$, let us call $\mathcal{Z}$ the set of the vertices of $\tilde{I}$ isolated in $\tilde{I}$ and let $\mathcal{Y}$ be $\tilde{I} - \mathcal{Z}$. Then the set I of G defined by $I = (\bigcup_{Z \in \mathcal{Z}} Z) \cup (\bigcup_{Y \in \mathcal{Y}} \{y/\text{for a choice of } y \text{ in } Y\})$ is irredundant in G.
This choice (see 2)b)) induces an injection from the set of the irredundant sets of $\tilde{G}$ into the set of the irredundant sets of G which is compatible with the inclusion. Therefore this injection maps the maximal irredundant sets of $\tilde{G}$ into the maximal irredundant sets of G.

Proposition 4.3. :
If G belongs to W_{AR}, then $ir(G) \geq q(G)$.

Proof : If G is a clique, then $ir(G) = q(G) = 1$. Assume henceforth that G is not a clique, and let us consider a minimum maximal irredundant I of G. Then $\tilde{I}$ is a maximal irredundant of $\tilde{G}$ (see Remark 4.2) such that

$|\tilde{I}| \leq |I|$, hence $ir(\tilde{G}) \leq ir(G)$.
Since $ir(\tilde{G}) \leq ir(G)$ we can suppose that G is W_{AR}-irreducible and then it suffices to show that every maximal irredundant set I contains at least q vertices. When G is not a clique we will find an injective function f from the q simplicial vertices into I. There are two cases to consider.
Case 1 : *x is simplicial and satisfies $C(x) \cap I \neq \emptyset$* : let u be a vertex in $C(x) \cap I$ and define $f(x)$ by u. Note that by Proposition $3.6_{2)}$, $f(x) = f(z)$ cannot stand in this case for another simplicial vertex z and note that $f(x)$ is located in a simplicial clique.
Case 2 : *y is simplicial and satisfies $C(y) \cap I = \emptyset$* : by maximality $I \cup \{y\}$ is not irredundant and as y is isolated in $I \cup \{y\}$ (since that $N(y) = C(y) - \{y\}$), y annihilates at least one vertex a in I.
We will prove that $C(a)$ is not simplicial. Since the I-private neighborhood of a is contained in $C(y)$, more precisely is exactly $C(y) - \{y\}$ by Property Qa), a is not simplicial. As G is connected, we have $C(a) \neq \{a\}$.
If $C(a) \cap I = \{a\}$, every vertex b of $C(a) - \{a\}$ dominates some vertex c of $I - \{a\}$, since the I-private neighborhood of a is contained in $C(y)$ and $b \notin C(y)$. $C(a) \cap I = \{a\}$ and $c \in I$ imply $c \notin C(a)$. Hence b, which is adjacent to c, is not simplicial, and thus $C(a)$ is not simplicial.
If $C(a) \cap I \neq \{a\}$, then a vertex b of $C(a) \cap (I - \{a\})$ admits an I-private neighbor $b' \notin C(a) \cup C(y)$ since $C(a) \cup (C(y) - \{y\}) \subset N(a)$ and y is simplicial. As G is W_{AR}-irreducible, the cliques $C(y), C(a), C(b')$ form the configuration described in Lemma 2.8 with $y_2 = a$ and $z_2 = b$, and thus $C(a)$ is not simplicial.
We put $f(y) = a$. Since $f(y)$ is located in a nonsimplicial clique, $f(y) = f(x)$ cannot hold if x belongs to case 1. Suppose furthermore that t is simplicial such that $C(t) \cap I = \emptyset$ and $f(t) = f(y) = a$: then $C(y) = C(t)$ is the clique of the Q-clique partition including the I-private neighborhood of a. Then, by Proposition $3.6_{2)}$, $y = t$. Thus f is injective and the proposition is proven.□

Theorem 4.4. :
Let G belong to W_{AR} and let $p(G)$ be the number of cliques and $q(G)$ be the number of simplicial cliques in any Q-clique partition, then $i(G) = \alpha(G) = \Gamma(G) = IR(G) = p(G)$ and $ir(G) = \gamma(G) = q(G)$.

Proof : By Propositions 4.1 and 4.3, it is clear that if G is not a clique, then $ir(G) = \gamma(G) = q(G)$, and if G is a clique, then $ir(G) = \gamma(G) = q(G) = 1$. Let I be a maximum irredundant set and $\mathcal{C} = \{C_1, C_2, \cdots, C_p\}$ be a Q-clique partition of G. Each C_i contains at most two vertices of I : indeed, suppose $\{x_1, x_2, x_3\} \subset C_i \cap I$ and let y_1 be a I-private neighbor of x_1 included in C_j $(j \neq i)$. Then y_1x_2 and y_1x_3 are not edges, which contradicts

Property Qa) since $y_1 x_1 \in E$.
Let $F_i = \{C \in \mathcal{C} / |C \cap I| = i\}$, $0 \leq i \leq 2$.

1) Suppose that G is W_{AR}-irreducible.
Consider H the graph obtained from G in the following manner : the vertices of H are the cliques of $F_0 \cup F_2$ and the edges of H are the couple $(C_0, C_2) \in F_0 \times F_2$ such that C_0 contains at least one I-private neighbor of some vertex in $C_2 \cap I$. It is clear that H is bipartite with bipartition (F_0, F_2).

Claim 1 : $\forall C_2 \in F_2 \quad d_H(C_2) \geq 2$.
Let x_1 and x_2 be the vertices of $C_2 \cap I$ and for $i = 1, 2$, let x'_i be one I-private neighbor of x_i. If $C(x'_1) = C(x'_2)$ then $G\ [x_1, x_2, x'_1, x'_2\ ]$ is a link, which contradicts G is W_{AR}-irreducible (see Theorem 3.4).

Claim 2 : $\forall C_0 \in F_0 \quad d_H(C_0) \leq 2$.
Suppose that x'_1, x'_2, and x'_3, which are respectively three I-private neighbors of three vertices of x_1, x_2, and x_3 of I, are located in the same clique C_i of $\mathcal{C}$. Clearly x_1 cannot be in C_i and as $x_1 x'_1 \in E$ then by Property Qa) x_1 dominates x'_2 or x'_3, a contradiction.

As H is bipartite and if $X_2 \subset F_2$ we have :

$$\sum_{C_0 \in N(X_2)} d_H(C_0) \geq e(X_2, N(X_2)) = \sum_{C_2 \in X_2} d_H(C_2)$$

where $e(X_2, N(X_2))$ denotes the number of edges between X_2 and $N(X_2)$.
Then by Claims 1 and 2 :

$$2|N(X_2)| \geq \sum_{C_0 \in N(X_2)} d_H(C_0) \quad \text{and} \quad \sum_{C_2 \in X_2} d_H(C_2) \geq 2|X_2|$$

Therefore $|N(X_2)| \geq |X_2|$ and we are now ready to apply the following famous theorem :

Theorem (Hall 1935)[1] Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if $|N(S)| \geq |S|$ for all $S \subset X$.

Thus there exists a matching that saturates every vertex in F_2 and which induces an injection ϕ from F_2 into F_0.

2) If G is W_{AR} not irreducible.
Let $\tilde{G}$ be the associated W_{AR}-irreducible graph and consider the set L_i of the cliques of $\tilde{G}$ which correspond to the cliques of G which are in F_i.

Claim 3 : There exists a bijection s from F_2 into L_2.
Let C_2 be in F_2. Let x_1 and x_2 be the vertices of $C_2 \cap I$. Note that, for $i = 1, 2$ X_i is the class of x_i. Then the clique class $C = C(X_1) = C(X_2)$ is such that $C \cap I$ is reduced to $\{x_1, x_2\}$, by Remark $4.2_{1)}$ and because the clique C of $\tilde{G}$ contains at most two vertices X_1 and X_2 of the maximal irredundant set $\tilde{I}$ defined as in Remark 4.2. Thus $C \cap \tilde{I} = \{X_1, X_2\}$ and $C \in L_2$. Then define $s(C_2)$ by C.

Claim 4 : There exists an injection t from L_0 into F_0.
For every clique $\tilde{C}_0$ in L_0 define $t(\tilde{C}_0)$ by C_0 with C_0 being one clique in G corresponding to the clique $\tilde{C}_0$ in $\tilde{G}$.

By 1) there exists an injection Φ from L_2 into L_0. Then, define ϕ by $t \circ \Phi \circ s$ which induces an injection from F_2 into F_0.
Thus $|F_2| \leq |F_0|$ and $IR = |I| = 2|F_2| + |F_1| \leq |F_0| + |F_1| + |F_2| \leq p$. Moreover as $i(G) = p$ (Proposition 1.1) we have the following chain of equalities $i(G) = \alpha(G) = \Gamma(G) = IR(G) = p$.□

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