Polychrome labelings of trees and cycles

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Abstract

This paper deals with a new kind of graph labeling similar to the well known harmonious, graceful, and elegant labelings. A polychrome labeling of a simple and connected graph $G = (V, E)$ by an abelian group A is a bijective map from V onto A such that the induced edge labeling $f^*(vw) = f(v) + f(w)$, $vw \in E$, is injective. Polychrome labelings of a path and a cycle by a large class of abelian groups are designed, and the connection to the above mentioned labelings is shown. In addition, the author presents a conjecture which is similar to a famous conjecture of G. Ringel about graceful trees (see [9]).

1 Introduction

A *labeling* of a simple connected graph $G = (V, E)$ is an injective mapping f from V on a finite set M. If $e = |E|$, then each labeling f from V on $\{0, \ldots, e\}$ induces an edge labeling f_g from E on the set $\{1, \ldots, e\}$ by $f_g(vw) = |f(v) - f(w)|$ for $vw \in E$. If f_g is surjective, f is called *graceful*. This labeling was defined by Rosa [11] first, but the terminology is due to Golomb [3]. Moreover, we introduce two additive versions of graceful labelings.

If G is not a tree, Graham and Sloane [5] call a labeling f from V on $\mathbb{Z}_e$ *harmonious*, if the map $f_h : E \to \mathbb{Z}_e$ defined by $f_h(vw) = f(v) + f(w)$ (mod e) for $vw \in E$ is bijective. If G is a tree, a map f from V onto $\mathbb{Z}_e$ is *harmonious*, if f_h (defined as above) is bijective. We find an exact additive version by the elegant labelings.

A labeling f from V on $\mathbb{Z}_{e+1}$ with induced edge labeling $f_{el} : E \to \mathbb{Z}_{e+1}$ defined by $f_{el}(vw) = f(v) + f(w)$ (mod $e + 1$), $vw \in E$ is said to be *elegant* (see [1]), if $f_{el}(E) = \mathbb{Z}_{e+1} \setminus \{0\}$. A graph is called *graceful (harmonious, elegant)*, if it has a graceful (harmonious, elegant) labeling.

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The previous labelings above use the cyclic group. We introduce a new kind of labeling by extending the considered group to an abelian group. Let $G = (V, E)$ be a simple connected graph and A be an abelian group. A labeling f of G by A is a bijective map from V onto A. Let $f^* : E \to A$ be defined by $f^*(vw) = f(v) + f(w)$ for $vw \in E$. We call f and G *polychrome*, if all edge labels are different. If f is a polychrome labeling of a tree T, there is one vertex whose label does not occur as an edge label. This vertex is called the *root* of f and T.

We consider abelian groups of order n labeling the path P_n polychrome. Then we study labelings of a cycle in order to find more groups labeling a path polychrome. In the last section we investigate trees with polychrome labelings by cyclic groups and show the relationship to a special class of graceful trees.

2 Polychrome labelings of $\mathbf{P_n}$

Let $P_n = (V, E)$ be a path with vertex set $V = \{v_0, \ldots, v_{n-1}\}$ and edge set $E = \{v_{i-1}v_i \mid i < n\}$. Then we denote this by $P_n = v_0 \ldots v_{n-1}$. We start our investigations with the cyclic groups, the basic components of abelian groups.

2.1 Proposition. *Let n be an odd number. Then there exists a polychrome labeling of P_n by $\mathbb{Z}_n$.*

Proof. Let $P_n = v_0 \ldots v_{n-1}$. We define the map f from V onto $\mathbb{Z}_n$ by $f(v_i) = i$. Since n is odd, the induced edge labeling f^* is injective. Thus f is polychrome with root v_{n-1}. □

2.2 Proposition. *For all positive integers n the path P_n has a polychrome labeling by $\mathbb{Z}_n$.*

Proof. Let $P_n = v_0 \ldots v_{n-1}$ and f be the map from V onto $\mathbb{Z}_n$ defined by

$$f(v_i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even,} \\ \lfloor \frac{n-1}{2} \rfloor + \frac{i+1}{2} & \text{if } i \text{ is odd} \end{cases} \qquad (\text{for } i = 0, \ldots, n-1).$$

Evidently, f is a labeling. Let $e_i = v_{i-1}v_i$ for $1 \leq i < n$. Then we obtain for the edge labels $f^*(e_i) = \lfloor \frac{n-1}{2} \rfloor + i$ for $i = 1, \ldots, n-1$. Thus f is a polychrome labeling with root of colour $\lfloor \frac{n-1}{2} \rfloor$. □

The following theorem is a special case of Result 3.3 (see Walker [12] Th. 3).

2.3 Theorem. *Let A be an abelian group of odd order n. Then there exists a polychrome labeling of P_n by A.*

In order to prove the next theorem we need to introduce the representation of an integer.

2.4 Lemma. *Let $b_1, \ldots, b_d > 1$ be integers and $b_0 = 1$. Then the map $f_d : \mathbb{Z}_{b_1} \times \cdots \times \mathbb{Z}_{b_d} \to \mathbb{Z}_{b_1 \cdots b_d}$ with*

$$f_d(a_0, \ldots, a_{d-1}) = \sum_{i=0}^{d-1} \left(a_i \cdot \prod_{j=0}^{i} b_j \right)$$

is bijective.

Proof. Using induction on d one can show that

$$f_d(a_0, \ldots, a_{d-1}) < \prod_{j=0}^{d} b_j \qquad \text{for } d \in \mathbb{N} \text{ and } a_j \in \mathbb{Z}_{b_{j+1}}. \qquad (*)$$

We shall prove by induction on d that f_d is bijective.
Clearly, f_1 is bijective.
Suppose now that $d > 1$ and the assertion is true for $d - 1$.
Let $(a_0, \ldots, a_{d-1}), (a'_0, \ldots, a'_{d-1}) \in \mathbb{Z}_{b_1} \times \cdots \times \mathbb{Z}_{b_d}$ such that

$$f_d(a_0, \ldots, a_{d-1}) = f_d(a'_0, \ldots, a'_{d-1}).$$

From $f_d(a_0, \ldots, a_{d-1}) = f_{d-1}(a_0, \ldots, a_{d-2}) + a_{d-1} \cdot \prod_{j=0}^{d-1} b_j$ we obtain

$$f_{d-1}(a_0, \ldots, a_{d-2}) - f_{d-1}(a'_0, \ldots, a'_{d-2}) = (a'_{d-1} - a_{d-1}) \cdot \prod_{j=0}^{d-1} b_j.$$

By $(*)$, this implies $a_{d-1} = a'_{d-1}$ and $f_{d-1}(a_0, \ldots, a_{d-2}) = f_{d-1}(a'_0, \ldots, a'_{d-2})$. By induction, it follows that $(a_0, \ldots, a_{d-1}) = (a'_0, \ldots, a'_{d-1})$. Hence f_d is bijective. □

If $x \in \mathbb{Z}_{b_1 \cdots b_k}$ with representation $x = a_0 + a_1 b_1 + a_2 b_1 b_2 + \ldots + a_{k-1} \cdot \prod_{i=1}^{k-1} b_i$, we define $\text{repr}_{(b_1, \ldots, b_k)}(x) := (a_0, \ldots, a_{k-1})$. If the values for $b_1, \ldots, b_k$ are given by the context, we write $\text{repr}(x)$ as well.

2.5 Theorem. *Let $A = \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_d}$ with $d \geq 2$, k_1 even and k_i odd for $i > 1$. Then P_n, $n = |A|$, has a polychrome labeling by A.*

Proof. Let $P_n = v_0 \ldots v_{n-1}$ and $\overline{n} = (k_1, \ldots, k_d)$. If $x \in \mathbb{Z}_n$ with $(x_1, \ldots, x_d) = \mathrm{repr}_{\overline{n}}(x)$, we define $f : V \to A$ by

$$f(v_x) = \begin{cases} \left(\frac{x}{2}\,(\mathrm{mod}\ k_1),\ x_2,\ \ldots,\ x_d\right) & \text{if } x \text{ is even,} \\ \left(\frac{x-1}{2} + \frac{k_1}{2}\,(\mathrm{mod}\ k_1),\ x_2,\ \ldots,\ x_d\right) & \text{if } x \text{ is odd.} \end{cases}$$

We prove that f is bijective.

Let x, $y \in \mathbb{Z}_n$ and $(x_1, \ldots, x_d) = \mathrm{repr}_{\overline{n}}(x)$, $(y_1, \ldots, y_d) = \mathrm{repr}_{\overline{n}}(y)$. Let $f(v_x) = f(v_y)$. By Th. 2.4, we obtain $x_i = y_i$, $i > 1$. Thus $|x - y| < k_1$.

If x and y have the same parity, we conclude $k_1 \mid \frac{x-y}{2}$. Since $|x - y| < k_1$, we get $x = y$. If x and y have different parity, we assume w.l.o.g. that x is even and y is odd. Then $k_1 \mid \frac{y-x-1+k_1}{2}$ follows. Together with $-k_1 \leq y - x - 1 \leq k_1 - 2$ we obtain $x = y$. Thus f is bijective and a labeling of P_n.

Let $e_x = v_x v_{x+1}$ for $0 \leq x \leq n-2$. We obtain $f^*(e_x) = f(v_x) + f(v_{x+1})$. If $f(v_x) = (x_1, \ldots, x_d)$, we define

$$s_x := \max\{i \in \mathbb{N} \mid i \leq d,\ a_j = k_j - 1 \text{ for } 1 \leq j < i\}.$$

Moreover, let $t_x = (t_{x,1}, \ldots, t_{x,d}) \in A$, where

$$t_{x,i} = \begin{cases} 0 & \text{if } i > s_x, \\ 1 & \text{if } 1 \leq i \leq s_x \end{cases} \qquad (\text{for } i = 1, \ldots, d).$$

We suppose that $f(v_x) = (x_1, \ldots, x_d)$, $f(v_{x+1}) = (y_1, \ldots, y_d)$ with $x < n-1$ and conclude $y_i = x_i + t_{x,i}$ for $i > 1$. Moreover, $x_1 + y_1 = \frac{k_1}{2} + x\,(\mathrm{mod}\ k_1)$. Thus

$$f^*(e_x) = \left(x + \frac{k_1}{2}\,(\mathrm{mod}\ k_1), 2x_2 + t_{x,2}, \ldots, 2x_d + t_{x,d}\right).$$

Let e_x and e_y be equally labeled egdes such that $\mathrm{repr}_{\overline{n}}(x) = (x_1, \ldots, x_d)$ and $\mathrm{repr}_{\overline{n}}(y) = (y_1, \ldots, y_d)$. We obtain

$$2\,x_i + t_{x,i} = 2\,y_i + t_{y,i},\ i > 1 \tag{1}$$

and

$$\frac{k_1}{2} + x\,(\mathrm{mod}\ k_1) = \frac{k_1}{2} + y\,(\mathrm{mod}\ k_1). \tag{2}$$

Assume $s_x < s_y$. If $s_x > 1$, then $x_{s_x} < k_{s_x} - 1 = y_{s_x}$. But, by (1), we conclude $2x_{s_x} + 1 = 2y_{s_x} + 1$, which implies $x_{s_x} = y_{s_x}$. This is a contradiction. If $s_x = 1$, we get $x \pmod{k_1} \neq k_1 - 1 = y \pmod{k_1}$. This contradicts (2).

Thus $s_x \geq s_y$. Similarly, we get $s_x \leq s_y$. Therefore $s_x = s_y$ and $t_x = t_y$. Since the numbers k_i are odd for $i > 1$, (1) implies $x_i = y_i$, $i > 1$. By (2), we have $x \pmod{k_1} = y \pmod{k_1}$. Finally, we obtain $x = y$, and P_n has a polychrome labeling by the group A. □

Hence there is a polychrome labeling of a path by all abelian groups with cyclic Sylow 2-group.

Now we prove a non-existence result due to M. Maamoun and H. Meyniel [9], which is a counterexample to a conjecture of G. Hahn.

2.6 Lemma. *If $d > 1$, then in $(\mathbb{Z}_2)^d$ the equation $\sum_{x \in (\mathbb{Z}_2)^d} x = 0$ holds.*

Proof. Let $d > 1$ and $(s_1, \ldots, s_d) = \sum_{x \in (\mathbb{Z}_2)^d} x$, then $s_i = 2^{d-1} \pmod 2$, $i = 1, \ldots, d$. Since $d > 1$, it follows that $s_i = 0$. □

2.7 Result. *If $d > 1$, then there is no polychrome labeling of P_{2^d} by $(\mathbb{Z}_2)^d$.*

Proof. Let $d > 1$ and $P_{2^d} = v_0 \ldots v_{2^d - 1}$. We assume that there exists a polychrome labeling f of P_{2^d} by $(\mathbb{Z}_2)^d$. Since each element of $(\mathbb{Z}_2)^d$ is of order two, the root of f is of colour 0. On the one hand the sum of edge labels equals the sum of vertex labels; on the other hand

$$\sum_{i=1}^{2^d - 1} f^*(v_{i-1}v_i) = 2 \cdot \sum_{i=0}^{2^d - 1} f(v_i) - (f(v_0) + f(v_{2^d - 1})) .$$

The Lemma above implies $\sum_{i=0}^{2^d - 1} f(v_i) = 0$. We obtain $f(v_0) + f(v_{2^d - 1}) = 0$. Thus $f(v_0) = f(v_{2^d - 1})$. This is a contradiction. □

The groups with elementary abelian Sylow 2-group different from $\mathbb{Z}_2$ play a special role, as we shall see in the next section as well.

3 Polychrome labelings of the cycle C_n

In this section we study cycles and thereby get results about labeling a path.

Let C_n ($n > 2$) consist of vertices $v_0, \dots, v_{n-1}$ and edges $v_0v_1, \dots, v_{n-2}v_{n-1}$, $v_{n-1}v_0$. We denote this by $C_n = v_0 \dots v_{n-1}$.

We begin our investigations – similar to the last section – with cyclic groups. By definition, a labeling of the cycle C_n is harmonious if and only if it is polychrome by $\mathbb{Z}_n$. Therefore we can use Th. 14 from [5] as follows.

3.1 Result. *The cycle C_n has a polychrome labeling by $\mathbb{Z}_n$ if and only if n is odd.*

The next four theorems characterize those abelian groups that admit a polychrome labeling of a cycle.

3.2 Theorem. *Let A be an abelian group of odd order n and f be a polychrome labeling of P_n by A. Then f is a polychrome labeling of C_n.*

Proof. Let $d, n_1, \dots, n_d \in \mathbb{N}$ such that $A = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d}$ and $n = \prod_{i=1}^{d} n_i$. Let $P_n = v_0 \dots v_{n-1}$. We shall prove that $f(v_0) + f(v_{n-1}) = f(r)$, where r is the root of P_n.

Let $(s_1, \dots, s_d) \in A$ such that $(s_1, \dots, s_d) = \sum_{i=0}^{n-1} f(v_i)$. Then

$$s_i = \frac{n}{n_i} \cdot \sum_{l=0}^{n_i-1} l \,(\text{mod } n_i) = \frac{n}{n_i} \cdot \frac{n_i(n_i-1)}{2} (\text{mod } n_i) = n \cdot \frac{n_i - 1}{2} (\text{mod } n_i)$$

for $i = 1, \dots, d$. Since the numbers n_i are odd, we obtain $s_i = 0$, $1 \le i \le d$. Thus $(s_1, \dots, s_d) = 0$.

Define $e_i = v_{i-1}v_i$, $i < n$. Then

$$\sum_{i=1}^{n-1} f^*(e_i) = \sum_{i=0}^{n-1} f(v_i) - f(r) = -f(r)$$

and

$$\sum_{i=1}^{n-1} f^*(e_i) = 2 \cdot \sum_{i=0}^{n-1} f(v_i) - f(v_0) - f(v_{n-1}) = -f(v_0) - f(v_{n-1}).$$

This implies $f(r) = f(v_0) + f(v_{n-1})$. Accordingly, $v_0 \dots v_{n-1}$ is a polychrome cycle.

□

The next result is due to Walker ([12] Th. 3) who considered another kind of labelings called 'standard colourings'. In our terminology his result reads as follows.

3.3 Result. *Let A be an abelian group of odd order n. Then C_n has a polychrome labeling by A.*

3.4 Theorem. *Let A be an abelian group of odd order m, and let l be a positive even integer. Then there is no polychrome labeling of $C_{m\cdot l}$ by $A \times \mathbb{Z}_l$.*

Proof. Let $d, n_1, \ldots, n_d \in \mathbb{N}$ such that $A = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d}$ and $m = \prod_{i=1}^{d} n_i$. Let $n = m \cdot l$. We assume that there is a polychrome labeling of C_n by $A \times \mathbb{Z}_l$. Let $(s_1, \ldots, s_{d+1}) \in A \times \mathbb{Z}_l$ such that $(s_1, \ldots, s_{d+1}) = \sum_{x \in A \times \mathbb{Z}_l} x$. We obtain

$$\begin{aligned} s_{d+1} &= \frac{n}{l} \cdot \sum_{i=0}^{l-1} i \pmod{l} = \frac{n}{l} \cdot \frac{l(l-1)}{2} \pmod{l} \\ &= \frac{n}{2} \cdot (l-1) \pmod{l} = \frac{n}{2} \pmod{l}. \end{aligned}$$

Thus $\sum_{x \in A \times \mathbb{Z}_l} x \neq 0$.

Since each vertex is incident with two edges, on the one hand the sum of edge labels equals $2 \cdot \sum_{x \in A \times \mathbb{Z}_l} x$. On the other hand the sum of edge labels equals the sum of vertex labels. We conclude $\sum_{x \in A \times \mathbb{Z}_l} x = 2 \cdot \sum_{x \in A \times \mathbb{Z}_l} x$. This implies $\sum_{x \in A \times \mathbb{Z}_l} x = 0$, a contradiction. □

3.5 Theorem. *Let $A = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d}$, $d \geq 2$, be an abelian group such that at least two n_i are even. If $n = |A|$, then each polychrome labeling of P_n by A is a polychrome labeling of C_n by A, too.*

Proof. Let $n = \prod_{i=1}^{d} n_i$ and f be a polychrome labeling of $P_n = v_0 \ldots v_{n-1}$ with root r. We establish $f(r) = f(v_0) + f(v_{n-1})$.

Let $(s_1, \ldots, s_d) \in A$ such that $(s_1, \ldots, s_d) = \sum_{x \in A} x = \sum_{i=0}^{n-1} f(v_i)$. Then

$$\begin{aligned} s_j &= \frac{n}{n_j} \cdot \sum_{i=0}^{n_j - 1} i \pmod{n_j} = \frac{n}{n_j} \cdot \frac{n_j(n_j-1)}{2} \pmod{n_j} \\ &= n \cdot \frac{n_j - 1}{2} \pmod{n_j} = \frac{n}{2} \pmod{n_j} \qquad (\text{for } 1 \leq j \leq d). \end{aligned}$$

Since at least two n_i are even, it follows that $s_j = 0$ for $1 \leq j \leq d$.
As in proof of Th. 3.2 we obtain $f(r) = f(v_0) + f(v_{n-1})$. Hence one can close $P_n = v_0 \dots v_{n-1}$ to a polychrome cycle. □

Notice that the preceding theorem does not imply the existence of a polychrome cycle by the considered groups.

We are now ready to compose small cycles to bigger ones.

3.6 Theorem. *Let A and B be abelian groups of relatively prime orders m and n. If C_m and C_n have polychrome labelings by A and B, resp., then there exists a polychrome labeling of $C_{m \cdot n}$ by $A \times B$.*

Proof. Let f and f' be polychrome labelings of $C_m = v_0 \dots v_{m-1}$ and $C_n = v'_0 \dots v'_{n-1}$ by A and B, resp. Let $C_{m \cdot n} = w_0 \dots w_{mn-1}$. We define the map $g : \{w_0, \dots, w_{mn-1}\} \to A \times B$ by

$$g(w_i) = \Big(f(v_{i \pmod m}), f'(v'_{i \pmod n})\Big) \qquad (\text{for } i = 0, \dots, mn-1).$$

Since m and n are relatively prime, and f and f' are bijective, g also is bijective by the chinese remainder theorem. Hence it is a labeling of $C_{m \cdot n}$.

Let the $e_i = w_i w_{i+1 \pmod{mn}}$, $0 \leq i < mn$, be the edges of $C_{m \cdot n}$. We obtain for $i < mn$

$$g^*(e_i) = \Big(f^*(v_{i \pmod m}\, v_{i+1 \pmod m}), f'^*(v'_{i \pmod n}\, v'_{i+1 \pmod n})\Big).$$

With the same argument one can see that g^* is bijective, too. Thus g is a polychrome labeling of $C_{m \cdot n}$ by $A \times B$. □

Remark. The converse of Th. 3.6 is not true. While $\mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ permit polychrome labelings of C_3 and C_{12}, resp., there is no polychrome labeling of P_4 (thus also C_4) by $\mathbb{Z}_2 \times \mathbb{Z}_2$ (cf. Th. 2.7).

Consider an abelian group A and its Sylow 2-group S, $k = |S|$. If there is a polychrome labeling of C_k by S, we obtain a polychrome labeling of C_n, $n = |A|$, by A (cf. Res. 3.3, Th. 3.2 and Th. 3.6). If S is elementary abelian, we cannot refer to the last theorem. However, we get a generalization of Th. 3.6.

3.7 Theorem. *Let A and B be abelian groups of order m and n. If n is odd, and if there are polychrome labelings of C_m and C_n by A and B, resp., then there exists a polychrome labeling of $C_{m \cdot n}$ by $A \times B$.*

Proof. Let f and f' are polychrome labelings of $C_m = v_0 \dots v_{m-1}$ and $C_n = v'_0 \dots v'_{n-1}$ by A and B, resp. Let V be the vertex set of $C_{m \cdot n} = w_0 \dots w_{mn-1}$. We define the map g from V onto $A \times B$ by

$$g(w_i) = \left(f(v_{i \,(\text{mod } m)}), f'(v'_{\lfloor \frac{i}{m} \rfloor})\right) \qquad (\text{for } i = 0, \dots, mn-1).$$

Then g is bijective by construction.

Let $e_i = w_i w_{i+1 \,(\text{mod } mn)}$, $i = 0, \dots, mn-1$, be the edges of $C_{m \cdot n}$ and $\overline{e}_i = v_i v_{i+1 \,(\text{mod } m)}$ for $i = 0, \dots, m-1$. Then we obtain for $i = 0, \dots, mn-1$

$$\begin{aligned} g^*(e_i) &= g(w_i) + g(w_{i+1 (\text{mod } mn)}) \\ &= \left(f(v_{i \,(\text{mod } m)}), f'(v'_{\lfloor \frac{i}{m} \rfloor})\right) + \left(f(v_{i+1 \,(\text{mod } m)}), f'(v'_{\lfloor \frac{i+1 \,(\text{mod } mn)}{m} \rfloor})\right) \\ &= \left(f^*(\overline{e}_{i \,(\text{mod } m)}), f'(v'_{\lfloor \frac{i}{m} \rfloor}) + f'(v'_{\lfloor \frac{i+1}{m} \rfloor \,(\text{mod } n)})\right). \qquad (*) \end{aligned}$$

We prove that g^* is bijective.

If $g^*(e_i) = g^*(e_j)$, then $f^*(\overline{e}_{i \,(\text{mod } m)}) = f^*(\overline{e}_{j \,(\text{mod } m)})$ holds. Since f is polychrome, we conclude $i \,(\text{mod } m) = j \,(\text{mod } m)$.

If $i \,(\text{mod } m) \neq m-1$, we obtain $\lfloor \frac{i}{m} \rfloor = \lfloor \frac{i+1}{m} \rfloor$ and $\lfloor \frac{j}{m} \rfloor = \lfloor \frac{j+1}{m} \rfloor$. In conjunction with $(*)$ follows that $2f'(v'_{\lfloor \frac{i}{m} \rfloor}) = 2f'(v'_{\lfloor \frac{j}{m} \rfloor})$. Since n is odd, this implies $f'(v'_{\lfloor \frac{i}{m} \rfloor}) = f'(v'_{\lfloor \frac{j}{m} \rfloor})$. Furthermore, f' is polychrome. Thus we get $\lfloor \frac{i}{m} \rfloor = \lfloor \frac{j}{m} \rfloor$ and $|i-j| < m$. Since $i \,(\text{mod } m) = j \,(\text{mod } m)$, we obtain $i = j$.

If $i \,(\text{mod } m) = m-1$, then $\lfloor \frac{i+1}{m} \rfloor \,(\text{mod } n) = \lfloor \frac{i}{m} \rfloor + 1 \,(\text{mod } n)$ and $\lfloor \frac{j+1}{m} \rfloor \,(\text{mod } n) = \lfloor \frac{j}{m} \rfloor + 1 \,(\text{mod } n)$. We define $k = \lfloor \frac{i}{m} \rfloor$ and $l = \lfloor \frac{j}{m} \rfloor$. Together with $(*)$ this implies

$$f'(v'_k) + f'(v'_{k+1 \,(\text{mod } n)}) = f'(v'_l) + f'(v'_{l+1 \,(\text{mod } n)}).$$

Since f' is polychrome and $v'_i v'_{i+1 \,(\text{mod } n)}$ for $i = 0, \dots, n-1$ is an edge of C_n, we obtain $k = l$. Using $i \,(\text{mod } m) = j \,(\text{mod } m)$ we get $i = j$. Thus g is a polychrome labeling of $C_{m \cdot n}$ by $A \times B$. □

To decide for every abelian group A, whether a polychrome labeling of a path or a cycle by A exists or not, it suffices to show that there are polychrome labelings of a cycle by the following groups:

- $(\mathbb{Z}_2)^k \times \mathbb{Z}_n$ for $k > 1$ and all odd prime powers n.
- The non-cyclic and non-elementary abelian 2-groups.

One result concerning such groups is the following.

3.8 Proposition. *There exists a polychrome labeling of C_{8n} by $\mathbb{Z}_{4n} \times \mathbb{Z}_2$.*

Proof. Consider $G = (V, E)$ with $V = \{v_0, \ldots, v_{8n-1}\}$ and $E = \{e_0, \ldots, e_{8n-1}\}$. We define the vertex-edge-incidences for $i = 0, \ldots, 2n-1$ by

$$\begin{aligned} e_{2n+2i \pmod{4n}} &= v_i v_{2n+i} \\ e_{2n+2i-1 \pmod{4n}} &= v_{4n+i} v_{4n+2n+i-1} \\ e_{4n+(2i-1 \pmod{4n})} &= v_i v_{4n+(i-1 \pmod{4n})} \\ e_{4n+2i} &= v_{2n+i} v_{4n+2n+i}. \end{aligned}$$

We obtain the paths $v_i v_{2n+i} v_{6n+i} v_{4n+i+1} v_{i+2}$, $i < 2n-2$, $v_{2n-2} v_{4n-2} v_{8n-2} v_{6n-1} v_{4n}$, $v_{2n-1} v_{4n-1} v_{8n-1} v_0$ and $e_{4n+1} = v_1 v_{4n}$ in G. It is a routine matter to check that G contains the cycle

$$\begin{aligned} C = {} & v_0 v_{2n} v_{6n} v_{4n+1} v_2 v_{2n+2} v_{6n+2} v_{4n+3} v_4 \\ & \ldots v_{2n-2} v_{4n-2} v_{8n-2} v_{6n-1} v_{4n} v_1 v_{2n+1} v_{6n+1} v_{4n+2} v_3 v_{2n+3} v_{6n+3} v_{4n+4} v_5 \\ & \ldots v_{2n-1} v_{4n-1} v_{8n-1} \end{aligned}$$

of length $8n$. Hence G is equal to C.

We define $f : V \to \mathbb{Z}_{4n} \times \mathbb{Z}_2$ by $f(v_i) = repr_{(4n,2)}(i)$ for $i = 0, \ldots, 8n-1$. By Th. 2.4, f is a labeling of G. By definition of G, we obtain $f^*(e_i) = repr_{(4n,2)}(i)$ for $i = 0, \ldots, 8n-1$. Hence f is a polychrome labeling of G and thus of C_{8n} by $\mathbb{Z}_{4n} \times \mathbb{Z}_2$. □

As a direct conclusion we obtain

3.9 Theorem. *Suppose that B is an abelian group of odd order m. Then there is a polychrome labeling of $C_{2^{k+1} \cdot m}$, $k > 1$, by $\mathbb{Z}_2 \times \mathbb{Z}_{2^k} \times B$.*

Remark. There is a polychrome labeling of $P_{2^{k+1} \cdot m}$ for $k > 1$ by $\mathbb{Z}_2 \times \mathbb{Z}_{2^k} \times B$ with root of colour x for all $x \in \mathbb{Z}_2 \times \mathbb{Z}_{2^k} \times B$.

With the help of a computer we obtained polychrome labelings of a path by all abelian groups – which are not elementary abelian 2-groups – of cardinality 30 at most. Using Th. 3.7 we find polychrome labelings of a cycle by more groups. For instance, we get polychrome labelings of a cycle (and a path) by all abelian groups with $\mathbb{Z}_2 \times \mathbb{Z}_2$ as Sylow 2-group and $\mathbb{Z}_p \times S$ as Sylow p-group for $p \in \{3, 5, 7\}$ and a p-group S.

4 The special case $\mathbb{Z}_n$

In this section we call a tree T on n vertices *polychrome* if and only if there is a polychrome labeling of T by $\mathbb{Z}_n$.

First we notice that the elegant trees are exactly the trees on n vertices which have a polychrome labeling by $\mathbb{Z}_n$ with root of colour 0. We shall now study the close connection between a special class of graceful labelings called 'balanced labelings' which are also known as 'interlaced labelings' or 'α-labelings' and the polychrome labelings.

A labeling f of a graph $G = (V, E)$ is called *balanced*, if it is graceful, and if there is a $x_f \in \mathbb{N}$ such that $f(v) \leq x_f < f(w)$ or $f(w) \leq x_f < f(v)$ holds for all edges $vw \in E$.

The labeling used in the next theorem is due to [4] Prop. 3.

4.1 Proposition. *Suppose that T is a tree with balanced labeling. Then T is polychrome.*

Proof. Let $T = (V, E)$ be a tree on e edges with balanced labeling $f : V \to \{0, \ldots, e\}$ and x_f as defined above. We consider $h : V \to \mathbb{Z}_{e+1}$, where

$$h(v) = \begin{cases} f(v) & \text{if } f(v) \leq x_f, \\ e + 1 - f(v) + x_f & \text{if } f(v) > x_f. \end{cases}$$

Obviously h is a labeling. Let $vw \in E$ and $f(v) < f(w)$. By definition, $f(v) \leq x_f < f(w)$ holds. So we obtain

$$h^*(vw) = e + 1 + x_f - f_g(vw) \,(\text{mod } e + 1).$$

Hence T is polychrome with root of colour x_f. □

Conversely, we have

4.2 Proposition. *Suppose $T = (V, E)$ is a tree with polychrome labeling f. If x is the colour of the root, and if $f(v) \leq x < f(w)$ or $f(w) \leq x < f(v)$ for all edges $vw \in E$, then T has a balanced labeling.*

Proof. Let $n = |V|$. We define $h : V \to \{0, \ldots, n-1\}$ by

$$h(v) = \begin{cases} f(v) & \text{if } f(v) \leq x, \\ n + x - f(v) & \text{if } f(v) > x. \end{cases}$$

Clearly, h is bijective. Let $vw \in E$ and w.l.o.g $f(v) \leq x$. Then $h(v) \leq x < h(w)$ and $h_g(vw) = |h(v) - h(w)| = n + x - f(v) - f(w)$.

Since $f(v) \leq x < f(w)$ or $f(w) \leq x < f(v)$ for all edges $vw \in E$, we obtain, conceiving $f(v)$ and $f(w)$ as integers,

$$\{f(v) + f(w) | \, vw \in E\} = \{x+1, \ldots, x+n-1\}.$$

Hence $h_g(E) = \{1, \ldots, n-1\}$, and h is a balanced labeling of T with $x_h = x$. □

During the last 30 years graceful graphs, especially graceful trees, have received extensive attention. Some of the research results can be transferred onto polychrome trees. In particular, caterpillars (the trees with the property that the removal of its endnodes leaves a path), several classes of product trees and all trees with 10 vertices at most are polychrome. Some few theorems can be obtained by having a careful look at [1], [2], [6], [7], [8] and [11].

We conclude with a conjecture which is similar to the famous conjecture of G. Ringel [10] that all trees are graceful.

Conjecture. Every tree on n vertices has a polychrome labeling by $\mathbb{Z}_n$.

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