A Generalization of Maximal k-Multiple-Free Sets of Integers

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ABSTRACT. Let k and b be integers and $k > 1$. A set S of integers is called (k,b) linear-free (or (k,b)-LF for short) if $x \in S$ implies $kx + b \notin S$. Let $F(n,k,b) = \max\{|A|: A$ is (k,b)-LF and $A \subseteq [1,n]\}$, where $[1,n]$ denotes all integers between 1 and n. A subset A of $[1,n]$ with $|A| = F(n,k,b)$ is called a maximal (k,b)-LF subset of $[1,n]$. In this paper a recurrance relation for $F(n,k,b)$ is obtained and a method to construct a maximal (k,b)-LF subset of $[1,n]$ is given.

Let k be an integer greater than one. A set S of integers is called k-multiple-free (or k-MF for short) if $x \in S$ implies $kx \notin S$. Let

$$f_k(n) = \max\{|A|: A \subseteq [1,n] \text{ is } k\text{-multiple-free}\}.$$

For integers c and d, let $[c,d] = \{x \mid x \text{ is an integer and } c \le x \le d\}$, $[c,\infty) = \{x \mid x \text{ is an integer and } x \ge c\}$. A subset A of $[1,n]$ with $|A| = f_k(n)$ is called a k-MF subset of $[1,n]$.

In [1] E.T.H. Wang investigated 2-MF subsets of $[1,n]$ which is called double-free subsets and gave a recurrance relation and a formula for $f_2(n)$. In [2] Leung and Wei obtained a recurrance and a formula for $f_k(n)$.

Naturally the concept of the relation of multiple-free can be generalized to a relation of multiple and translation-free or linear-free relation. A set S of integers is called (k,b) linear-free (or (k,b)-LF for short) if $x \in S$ implies $kx + b \notin S$. Clearly, if $b = 0$, S is k-MF; if $b = 0$, $k = 2$, S is double-free.

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Let $F(n,k,b) = \max\{|A| : A \subseteq [1,n] \text{ is } (k,b)\text{-LF}\}$. A subset A of $[1,n]$ with $|A| = F(n,k,b)$ is called a maximal (k,b)-LF subset of $[1,n]$.

In this paper applying a method for constructing a maximal (k,b)-LF subset of $[1,n]$, we obtain a recurrance relation for $F(n,k,b)$.

We first prove the following lemma.

Lemma 1. *There exists a maximal (k,b)-LF subset of $[1,n]$, A, such that*

$$[\left\lfloor \frac{n-b}{k} \right\rfloor + 1, n] \subseteq A$$

where $\lfloor \frac{n-b}{k} \rfloor + 1 \leq n$.

Proof: Let B be a maximal (k,b)-LF subset of $[1,n]$. If $[\lfloor \frac{n-b}{k} \rfloor + 1, n] \subseteq B$, then we are done. Thus we may assume that there is an $x \in [\lfloor \frac{n-b}{k} \rfloor + 1, n]$, but $x \notin B$. Observe that $kx + b > n$ and hence, $kx + b \notin B$.

If $x \not\equiv b \pmod{k}$, then $x \neq ka + b$ for any $a \in B$. Thus $\tilde{B} = B \cup \{x\}$ is a (k,b)-LF set. But $|\tilde{B}| = |B| + 1 > |B|$, a contradiction to the maximality of B. Hence we have $x \equiv b \pmod{k}$.

In this case, we have $x = k^c a + b\sum_{i=0}^{c-1} k^i$ for some $c \geq 1$ and $a \not\equiv b \pmod{k}$. Consider the set $\tilde{B} = B \setminus \{k^{c-1}a + b\sum_{i=0}^{c-2} k^i\} \cup \{x\}$. $\tilde{B}$ is a (k,b)-LF set and $|\tilde{B}| \geq |B|$. Thus, $\tilde{B}$ is a maximal (k,b)-LF subset of $[1,n]$ with $x \in \tilde{B}$.

Repeating the above procedure, we will eventually obtain a maximal (k,b)-LF subset of $[1,n]$ with $[\lfloor \frac{n-b}{k} \rfloor + 1, n]$ being its subset. □

According to the above lemma, we can give a recurrance relation as follows.

Theorem 1. *For any positive integers n, k and integer b with $k-1+b \neq 0$, we have*

$$F(n,k,b) = n - \left\lfloor \frac{n-b}{k} \right\rfloor + F(\left\lfloor \frac{\lfloor \frac{n-b}{k} \rfloor - b}{k} \right\rfloor; k, b)$$

and $F(n,k,b) = n$, if $n < k + b$.

Proof: Let A be a maximal (k,b)-LF subset of $[1,n]$. By lemma 1, we may assume that $[\lfloor \frac{n-b}{k} \rfloor + 1, n] \subseteq A$. Since

$$k(\left\lfloor \frac{\lfloor \frac{n-b}{k} \rfloor - b}{k} \right\rfloor + 1) + b \geq \left\lfloor \frac{n-b}{k} \right\rfloor + 1 \text{ and } k\left\lfloor \frac{n-b}{k} \right\rfloor + b \leq n,$$

we know that $\left[\left\lfloor \frac{\lfloor \frac{n-b}{k} \rfloor - b}{k} \right\rfloor + 1, \lfloor \frac{n-b}{k} \rfloor\right] \cap A = \emptyset$, and hence $A - [\lfloor \frac{n-b}{k} \rfloor + 1, n]$ is a maximal (k,b)-LF subset of $\left[1, \left\lfloor \frac{\lfloor \frac{n-b}{k} \rfloor - b}{k} \right\rfloor\right]$. Therefore we have

$$|A| = |[\left\lfloor \frac{n-b}{k} \right\rfloor + 1, n]| + F(\left\lfloor \frac{\lfloor \frac{n-b}{k} \rfloor - b}{k} \right\rfloor; k, b)$$
$$= n - \left\lfloor \frac{n-b}{k} \right\rfloor + F(\left\lfloor \frac{\lfloor \frac{n-b}{k} \rfloor - b}{k} \right\rfloor; k, b).$$

□

For the purpose of stating the next theorem, we introduce the following notations. Define

$$I(n,k,b;0) = n,$$
$$I(n,k,b;i) = \left\lfloor \frac{I(n,k,n;i-1) - b}{k} \right\rfloor \quad (i \geq 1),$$
$$\hat{A} = \bigcup_{i \geq 0} [I(n,k,b;2i+1)+1, I(n,k,b;2i)] \cap [1,\infty).$$

Theorem 2. *For any positive integers n, k ($k > 1$) and nonnegative integer b, $\hat{A}$ is a maximal(k,b)-LF subset of* $[1,n]$

Proof: The proof is by induction on n. By definition, the maximal (k,b)-LF subset of $[1,n]$ is $\{1\}$ which can also be expressed as $[\lfloor \frac{1-b}{k} \rfloor + 1, 1] \cap [1,\infty)$. Thus the basis of the induction holds. Assume that the induction hypothesis holds for all positive intger less than n, where $n > 1$. We shall prove that it also holds for n.

Let A be a maximal (k,b)-LF subset of $[1,n]$. By lemma 1, we may assume that $[I(n,k,b;1)+1, I(n,k,b;0)] \subseteq A$. Moreover, from the proof of theorem 1, we may assume that

$$A = [I(n,k,b;1)+1, I(n,k,b;0)] \cup A_1,$$

where A_1 is a maximal (k,b)-LF subset of $[1, I(n,k,b;2)]$. By the induction hypothesis, we may assume that

$$A_1 = \bigcup_{i \geq 0} [I(I(n,k,b;2),k,b;2i+1)+1, I(I(n,k,b;2),k,b;2i)] \cap [1,\infty).$$

Note that $I(I(n,k,b;2),k,b;0) = I(n,k,b;2)$. Using the definition of $I(n,k,b;i)$, we conclude by induction that $I(I(n,k,b;2),k,b;i) = I(n,k,b;i+2)$ for integers $i \geq 0$. So we have

$$A_1 = \bigcup_{i \geq 0} [I(n,k,b;2i+3)+1, I(n,k,b;2i+2)] \cap [1,\infty)$$
$$= \bigcup_{i \geq 1} [I(n,k,b;2i+1)+1, I(n,k,b;2i)] \cap [1,\infty).$$

Hence we obtain

$$A = [I(n,k,b;1)+1, I(n,k,b;0)] \cup A_1$$
$$= \bigcup_{i \geq 0} [I(n,k,b;2i+1)+1, I(n,k,b;2i)] \cap [1,\infty) = \hat{A}.$$

Let the base k expansion of n be

$$n = \sum_{i=0}^{m} a_i k^i$$

where $0 \le a_i < k$ for each $0 \le i < m$, and $0 < a_m < k$.

Let $I'(n,k,b;i) = \max\{I(n,k,b;i),0\}$. We have

Theorem 3. *$F(n,k,b) = \sum_{i\ge 0}(I'(n,k,b;2i) - I'(n,k,b;2i+1))$. In particular, if $\max_{0\le i\le m} a_i + 1 - k \le b \le \min_{0\le i\le m} a_i$, then*

$$F(n,k,b) = \frac{1}{k+1}(kn + \sum_{i\ge 0}(-1)^i a_i)$$

Proof: By theorem 2, we have

$$F(n,k,b) = |\hat{A}| = \sum_{i\ge 0}(I'(n,k,b;2i) - I'(n,k,b;2i+1)).$$

If $\max_{0\le i\le m} a_i + 1 - k \le b \le \min_{o\le i\le m} a_i$, then by induction it is easy to show that $I(n,k,b;i) = \lfloor \frac{n}{k^i} \rfloor$ for integers $i \ge 0$. We have

$$I(n,k,b;2i) - I(n,k,b;2i+1) = \left\lfloor \frac{n}{k^{2i}} \right\rfloor - \left\lfloor \frac{n}{k^{2i+1}} \right\rfloor.$$

Hence

$$\begin{aligned} F(n,k,b) &= \sum_{i\ge 0}(\left\lfloor \frac{n}{k^{2i}} \right\rfloor - \left\lfloor \frac{n}{k^{2i+1}} \right\rfloor) \\ &= \frac{1}{k+1}(kn + \sum_{i\ge 0}(-1)^i a_i). \end{aligned}$$

This completes the proof of the theorem. □

If $b = 0$, by theorem 3 we have

Corollary (Theorem 3[2]). $f_k(n) = \frac{1}{k+1}(kn + \sum_{i\ge 0}(-1)^i a_i)$.

Example 1: Let $n = 75$, $k = 5$, and $b = 3$. $I(75,5,3;0) = 75$, $I(75,5,3;1) = \lfloor \frac{75-3}{5} \rfloor = 14$, $I(75,5,3;2) = \lfloor \frac{14-3}{5} \rfloor = 2$, $I(75,5,3;i) = 0$ for $i \ge 4$. By theorem 3, $F(75,5,3) = (75-14) + (2-0) = 63$.

Example 2: Let $n = 63 = 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0$, $k = 2$ and $b = 1$. Here $a_i = 1$ for $0 \le i \le 5$ and $a_i = 0$ for $i > 5$. By theorem 3, $F(63,2,1) = \frac{1}{3}(2 \times 63 + \sum_{i\ge 0}^{5}(-1)^i) = \frac{1}{3} \times 2 \times 63 = 42$.

References

[1] E.T.H. Wang, On double-free sets of integers, *Ars Combinatoria* **28** (1989), 97–100.

[2] J.Y-T. Leung and W-D. Wei, Maximal k-multiple-free sets of integers, *Ars Combinatoria* **38** (1994), 113–117.