The smallest covering code of length 8 and radius 2 has 12 words

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Abstract

We prove that the smallest covering code of length 8 and covering radius 2 has exactly 12 words. The proof is based on partial classification of even weight codewords, followed by a search for small sets of odd codewords covering the part of the space that has not been covered by the even subcode.

1 Introduction

Covering codes have got a great deal of attention in the last decade, see [1, 2, 3] and references therein. One of the main challenges in the theory of covering codes is to determine minimal possible size of codes of given length and radius. This problem turns out to be difficult even for small lengths.

Let $(n, K)R$ stand for a covering code of length n, cardinality K and of covering radius R. For $n \leq 8$, the smallest possible K was not known only in the case $n = 8$ and $R = 2$. The best covering of length 8 and radius 2, known so far, is the code $(8, 12)2$, constructed using the amalgamated direct sum of the piecewise constant $(6, 12)1$ code [4, 5] and the trivial $(3, 2)1$ code [4]:

$$00000111, 00001000, 00010000, 01100000, 10100000, 11000000,$$

$$00111111, 01011111, 10011111, 11101111, 11110111, 11111000$$

In [6] it was proved that such a covering cannot have less than 11 words. In the present correspondence we confirm that 11 is not achievable.

It is clear that a brute force attack on the problem, by checking all possibilities of choosing 11 words, is far from being tractable. However, to find a combination of odd weight words that cover a given set of even weight vectors is quite easy. So, we use the following scheme. Assuming existence of a code of size 11, we prove that there should be a translation of the code with four or five even weight codewords. Further, we classify the possibilities for choosing four even codewords, reducing the problem to considering 28 possible combinations. For each one of the combinations it is a routine check that the space not covered by the four chosen words cannot be covered by seven odd weight words, or six odd words and one even word.

2 The proof

Let $d(\mathbf{x}, \mathbf{y})$ stand for the Hamming distance between vectors $\mathbf{x}, \mathbf{y} \in \mathcal{F}$, the set of binary vectors of length 8. We say that a word $\mathbf{c} \in \mathcal{F}$ covers a vector $\mathbf{x} \in \mathcal{F}$ if $d(\mathbf{c}, \mathbf{x}) \leq 2$. An $(8, K)2$ covering code is a set of words $\mathbf{c}_1, \ldots, \mathbf{c}_K \in \mathcal{F}$, such that

$$\forall \mathbf{x} \in \mathcal{F}\ \exists i \in \{1, \ldots, K\} : \ \mathbf{c}_i \text{ covers } \mathbf{x}.$$

Assume an $(8, 11)2$ code C exists. Denote by $\mathcal{E}$ and $\mathcal{O}$ the subsets of even and odd vectors of $\mathcal{F}$. Let $\mathcal{F}_i \subset \mathcal{F}$, be the subset of vectors of weight i. Denote by $\mathcal{F}_i^{(a)}$ the subset of $\mathcal{F}_i$ consisting of the vectors having weight a in the first four coordinates. For instance, $|\mathcal{F}| = 256, |\mathcal{E}| = |\mathcal{O}| = 128, |\mathcal{F}_4| = 70, |\mathcal{F}_4^{(2)}| = 36$.

We use a simple algorithm to check if there exists a collection of k odd words that cover $\mathcal{V} \subseteq \mathcal{E}$. To do this we first assume an order on the elements of $\mathcal{V}$. We proceed in k steps, choosing on each step a new word. Let prior to the i-th step, $(i-1)$ odd words have been chosen. We find the first vector in the list vector not covered by the already chosen words. There are 8 possible odd words which cover this vector. Appending one of them to the list of chosen words we pass to the next step. This we do for every one out of 8 possible words. The algorithm terminates after k steps. We will write $A_k(\mathcal{V}) = 1$ if a covering is possible, and 0 otherwise. The maximal number of collections of k words to be checked is 8^k, and for $k \leq 7$ it takes at most several minutes to compute it. Notice, that as a result we obtain all possible odd coverings of $\mathcal{V}$ of size k or less.

Now we are in a position to present the proof. Although most of the statements below have been proved analytically, we prefer to refer to results of the algorithm whenever it allows avoiding cumbersome analysis.

Let C_e and C_o stand for the even and odd codewords respectively. W.l.o.g. assume that $|C_e| < |C_o|$, otherwise we always may shift the code by an odd vector. A codeword from C_e covers at most 17 vectors of $\mathcal{F}_4$, and a codeword from C_o covers at most 5 vectors of $\mathcal{F}_4$. Since $1\cdot 17+10\cdot 5 < 70$, we get that $|C_e| \geq 2$, and we may assume that the word (00000000) belongs to C. Since this zero word does not cover any vector of $\mathcal{F}_4$, we need at least two more even codewords to cover $\mathcal{F}_4$, thus giving $|C_e| \geq 3$.

Lemma 1 $|C_e| \geq 4$.

Proof. For every vector $\mathbf{v} \in \mathcal{F}$, $\mathbf{v} = (v_1, \dots, v_8)$, we define its characteristic vector of size 4 as

$$\mathbf{f_v} = (|v_1 - v_2|, |v_3 - v_4|, |v_5 - v_6|, |v_7 - v_8|),$$

i.e. in the positon i the vector $\mathbf{f}$ has 0 if $v_{2i-1} = v_{2i}$, and 1 otherwise. Clearly, we have 16 different vectors corresponding to a specific characteristic vector. Moreover, $\mathcal{F}$ can be partitioned to 16 subsets of vectors having different characteristic vectors. We use notation $\varphi(\mathbf{f})$ for the set of vectors having $\mathbf{f}$ as the characteristic vector. Let $\bar{\mathbf{f}}$ be the binary complement of $\mathbf{f}$. We say that the set $\varphi(\bar{\mathbf{f}})$ is complementary to $\varphi(\mathbf{f})$. If $\mathbf{f}$ is even, then $\varphi(\mathbf{f}) \subset \mathcal{E}$ and is called an even class. If $\mathbf{f}$ is odd, then $\varphi(\mathbf{f}) \subset \mathcal{O}$ and is called an odd class. Notice, that every codeword from C_o covers exactly two vectors in one of two complementary even classes, and none in its complementary class.

Consider an even class $\varphi(\mathbf{f})$. The class contains 16 words, and we can partition them to two subclasses, say $\varphi_0(\mathbf{f})$ and $\varphi_1(\mathbf{f})$, such that the distance between any two different vectors in the same subclass is 4 or 8. To see that it is possible, we just do it for $\varphi(0000)$, namely, $\varphi_0(0000)$ consists of all the vectors from $\varphi(0000)$ of weight 0, 4 or 8, and $\varphi_1(0000)$ contains the rest 8 vectors. Other classes $\varphi(\mathbf{f})$ can be seen as shifts of $\varphi(0000)$, and the partition can be done also for them. If a codeword from C_o covers two vectors in an even class $\varphi(\mathbf{f})$, then one of the covered vectors is in $\varphi_0(\mathbf{f})$ and the other one is in $\varphi_1(\mathbf{f})$.

Assume that $|C_e| = 3$. Among three even words we always have at least two being at distance at most 4.

Case 1: There are two even codewords, say $\mathbf{c}_1$ and $\mathbf{c}_2$, being at distance exactly 4.

W.l.o.g. we may assume that $\mathbf{c}_1 = 00000000$ and $\mathbf{c}_2 = 11110000$. Consider the subset $\mathcal{F}_4^{(2)}$ of vectors from $\mathcal{F}_4$ having exactly two ones in the first 4 coordinates. Clearly, $|\mathcal{F}_4^{(2)}| = 36$, and $\mathbf{c}_1$ and $\mathbf{c}_2$ do not cover any vector in $\mathcal{F}_4^{(2)}$. An even word covers at most 9 vectors of $\mathcal{F}_4^{(2)}$, and an odd word covers at most 3 vectors of $\mathcal{F}_4^{(2)}$. Hence, there are at most $1\cdot 9+8\cdot 3 = 33 < 36$, vectors of $\mathcal{F}_4^{(2)}$ that can be covered, a contradiction.

Case 2: There are no two even codewords being at distance 4.

Then there are two codewords at distance 2, say $\mathbf{c}_1 = 00000000$ and $\mathbf{c}_2 = 11000000$. Consider the set $\mathcal{F}_4$. Eight odd codewords cover at most $8 \cdot 5 = 40$ vectors of $\mathcal{F}_4$. The third even codeword $\mathbf{c}_3$ cannot have weight 2, since in this case $\mathbf{c}_2$ and $\mathbf{c}_3$ cover at most 29 vectors of $\mathcal{F}_4$. It cannot have weight 4 since we forbid distance 4 between the even codewords. It cannot have weight 8 since, in this case, $\mathbf{c}_3$ does not cover any vector of $\mathcal{F}_4$. If $\mathbf{c}_3$ has weight 6, it cannot have two ones in the first two coordinates, otherwise $d(\mathbf{c}_2, \mathbf{c}_3) = 4$. If $\mathbf{c}_3$ starts with 00, then $d(\mathbf{c}_2, \mathbf{c}_3) = 8$, and adding $\mathbf{c}_2$ to all codewords, we are not able to cover $\mathcal{F}_4$.

We are left with a possibility of $\mathbf{c}_3 = 01111110$, $\mathbf{c}_3 \in \varphi(1001)$. Let φ_0 be defined so that it contains $\mathbf{c}_3$. Then there are 3 vectors in $\varphi_0(1001)$ which are covered by $\mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_3$. The other five vectors of $\varphi(1001)$ should be covered by five vectors from C_o. To cover the class $\varphi(0110)$ we need at least 4 words from C_o. But there are no odd words which cover at once vectors belonging to complementary classes. So, we need at least 9 odd codewords, a contradiction to $|C_o| = 8$.

□

Lemma 2 *There are no two even codewords being at distance 8.*

Proof. Let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4$ be even codewords (it is possible that there is one more even codeword). Assume $d(\mathbf{c}_1, \mathbf{c}_2) = 8$.

Case 1: Let $d(\mathbf{c}_1, \mathbf{c}_3) = 4$. W.l.o.g. we may assume $\mathbf{c}_1 = 11110000$, $\mathbf{c}_2 = 00001111$, $\mathbf{c}_3 = 00000000$. The rest 8 codewords must cover $\mathcal{F}_4^{(2)}$. For this we need at least two even codewords $(2 \cdot 9 + 6 \cdot 3 = 36)$. Since there are at most five even codewords, then $|C_e|$ is necessarily 5, and $|C_o| = 6$. Moreover, each of the two additional even codewords have to cover exactly 9 vectors from $\mathcal{F}_4^{(2)}$, and their weights are 2,4 or 6. Since all odd codewords are necessary for covering $\mathcal{F}_4^{(2)}$, they are of weight at most 5, and vector 11111111 should be covered by an even codeword. Thus, at least one of the even codewords, say $\mathbf{c}_4$ is of weight 6, and w.l.o.g. we may assume $\mathbf{c}_4 = 11101110$. The fifth even codeword, $\mathbf{c}_5$, must cover 9 vectors in $\mathcal{F}_4^{(2)}$, and cannot have weight 6, otherwise the subsets of $\mathcal{F}_4^{(2)}$ covered by $\mathbf{c}_4$ and $\mathbf{c}_5$ intersect.

Case 1.1 Weight of $\mathbf{c}_5$ is 4.

a) Let $\mathbf{c}_5 \in \mathcal{F}_4^{(3)}$. Then $\mathbf{c}_5$ necessarily has the last coordinate equal to 1, otherwise $\mathbf{c}_4$ and $\mathbf{c}_5$ cover less than 18 vectors in $\mathcal{F}_4^{(2)}$. Then, w.l.o.g. $\mathbf{c}_5 = 11100001$ or $\mathbf{c}_5 = 11010001$. In both cases, if $\mathcal{V}$ stands for the set of even vectors that are not covered, we check that $A_6(\mathcal{V}) = 0$.

b) Let $\mathbf{c}_5 \in \mathcal{F}_4^{(2)}$. To cover 9 not covered vectors from $\mathcal{F}_4^{(2)}$, $\mathbf{c}_5$ must have 1 in the fourth and eighth coordinates. W.l.o.g. assume $\mathbf{c}_5 = 10011001$. However, we are not able to cover $\varphi_0(0000)$ (we need 3 odd codewords) along with the subclass of $\varphi(1111)$ containing c_5 (we need 5 odd codewords, different from the previous ones).

c) If $c_5 \in \mathcal{F}_4^{(1)}$, it is equivalent to a).

Case 1.2 Weight of c_5 is 2.

a) Let $c_5 \in \mathcal{F}_2^{(1)}$. The word c_5 must have 1 in the fourth or eighth coordinates, otherwise there is intersection in $\mathcal{F}_4^{(2)}$. By symmetry we may assume that the eighth coordinate of c_5 is 1. Now, w.l.o.g., there are two possibilities for c_5, namely, 10000001 and 00010001. In both cases, $A_6(\mathcal{V}) = 0$.

b) It is impossible that $c_5 \in \mathcal{F}_2^{(2)}$ or $c_5 \in \mathcal{F}_2^{(0)}$ since it does not cover 9 new vectors in $\mathcal{F}_4^{(2)}$.

Case 2: W.l.o.g. assume $c_1 = 00000000, c_2 = 11111111, c_3 = 11111100$. Clearly, $|C_e| = 5$, otherwise we are not able to cover $\mathcal{F}_4$ $(2 \cdot 15 + 7 \cdot 5 = 65 < 70)$. W.l.o.g. there are two possibilities for choosing c_4 and c_5.

Case 2.1: Both, c_4 and c_5, have weight 6.

a) Let $d(c_3, c_4) = 2$. Then, w.l.o.g., $c_4 = 11111010$. Since c_3 and c_4 cover jointly exactly 25 vectors of $\mathcal{F}_4$, and c_5 covers at most 13 additional vectors of $\mathcal{F}_4$, we are not able to cover $\mathcal{F}_4$ $(25 + 13 + 6 \cdot 5 = 68 < 70)$.

b) All three codewords, c_3, c_4 and c_5, are at distance 4. W.l.o.g. $c_4 = 11110011, c_5 = 11001111$. Now, the words from C_e do not cover any vector in $\varphi(1111)$, and we need at least 8 odd words to cover $\varphi(1111)$.

Case 2.2: Both c_4 and c_5 have weight 2.

Then c_4 must cover at least 10 vectors from $\mathcal{F}_4$ not covered by c_1, c_2 and c_3. So, c_4 cannot have two zeros in the last two coordinates, otherwise it covers at most 9 new vectors of $\mathcal{F}_4$. Hence, w.l.o.g., assume that $c_4 = 10000010$. Then there are the following possibilities to choose c_5: 10000001, 01000010, 00000011, 01000001. The first three possibilities are impossible since $c_1, \ldots, c_5$ cover all together 40 vectors of $\mathcal{F}_4$, and the rest 30 vectors must be covered by 6 odd codewords. Therefore, each of the odd vectors has to cover 5 vectors from $\mathcal{F}_4$. This is possible only if the odd codewords are chosen from the vectors of weight 3, starting with 00 and ending with 01, or from the vectors of weight 5, starting with 0 and ending with 11. Any three vectors of such form and the same weight (3 or 5) cover common vectors in $\mathcal{F}_4$, and among 6 codewords there are always at least three of the same weight, a contradiction. If $c_5 = 01000001$, there exist 26 vectors from $\mathcal{F}_4$ that are not covered. So, we need at least two odd codewords covering 5 vectors from $\mathcal{F}_4$. These vectors are of weight 5, start with 00 and end with 11. However, in this case any two such vectors (there are four) cover common vectors, a contradiction.

Case 2.3: Weight of c_4 is 6 and weight of c_5 is 2. Adding the all one vector to all the codewords we reduce the case to Case 2.2.

□

Lemma 3 *There exist three codewords,* $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 \in C_e$*, such that*

$$d(\mathbf{c}_1, \mathbf{c}_2) \leq d(\mathbf{c}_1, \mathbf{c}_3) \leq 4.$$

Proof. Among four even codewords it is always possible to choose a pair, say $\mathbf{c}_1$ and $\mathbf{c}_2$, being at distance at most 4. If $d(\mathbf{c}_1, \mathbf{c}_3) \leq 4$ then we are done, otherwise, $d(\mathbf{c}_1, \mathbf{c}_3) = d(\mathbf{c}_1, \mathbf{c}_4) = 6$, and $d(\mathbf{c}_3, \mathbf{c}_4) \leq 4$. Let $|C_e| = 5$, then or $d(\mathbf{c}_1, \mathbf{c}_5) \leq 4$, or $d(\mathbf{c}_3, \mathbf{c}_5) \leq 4$, and, thus, either $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_5$, or $\mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5$ is the sought triple.

If $|C_e| = 4$, we need more complex arguments. Assume $d(\mathbf{c}_1, \mathbf{c}_2) \geq d(\mathbf{c}_3, \mathbf{c}_4)$, and all other pairwise distances equal 6.

Case 1: $d(\mathbf{c}_1, \mathbf{c}_2) = 4$. W.l.o.g. assume $\mathbf{c}_1 = 00000000, \mathbf{c}_2 = 11110000$, then $\mathbf{c}_3$ and $\mathbf{c}_4$ are of weight 6 and end with four ones. In this situation we are not able to cover $\mathcal{F}_4^{(2)}$, since $\mathbf{c}_3$ and $\mathbf{c}_4$ cover at most 12 vectors from $\mathcal{F}_4^{(2)}$, and seven odd codewords can cover at most 21 vectors from $\mathcal{F}_4^{(2)}$ $(12 + 21 = 33 < 36)$.

Case 2: $d(\mathbf{c}_1, \mathbf{c}_2) = d(\mathbf{c}_3, \mathbf{c}_4) = 2$. W.l.o.g. assume $\mathbf{c}_1 = 00000000, \mathbf{c}_2 = 11000000, \mathbf{c}_3 = 10111110$. Then $\mathbf{c}_4$ is a vector of weight 6, and there are, w.l.o.g., two possibilities to choose it, namely,

a) $\mathbf{c}_4 = 01111110$. Then there are 40 vectors from $\mathcal{E}$ that should be covered by 7 odd codewords, and it is easy to check that an odd codeword covers at most 5 vectors from $\mathcal{E}$, a contradiction.

b) $\mathbf{c}_4 = 10111101$. Denote by $\mathcal{V}$ the set of 30 not covered vectors of $\mathcal{F}_4$. $A_7(\mathcal{V}) = 0$, i.e. $\mathcal{V}$ cannot be covered by any collection of 7 odd words.

□

Lemma 4 *There exists an even codeword* $\mathbf{c}_1$ *such that* $d(\mathbf{c}_1, \mathbf{c}_2)$*,* $d(\mathbf{c}_1, \mathbf{c}_3)$*,* $d(\mathbf{c}_1, \mathbf{c}_4)$*, are at most 4.*

Proof.

Case 1: Assume that in every triple of even codewords there exist a pair being at distance 6.

Case 1.1: $d(\mathbf{c}_1, \mathbf{c}_2) = 4, d(\mathbf{c}_1, \mathbf{c}_3) = 2, d(\mathbf{c}_2, \mathbf{c}_3) = 6$. In this case, w.l.o.g. assume $\mathbf{c}_1 = 00000000, \mathbf{c}_2 = 11110000, \mathbf{c}_3 = 00001100$. If weight of $\mathbf{c}_4$ is less than 6, then the statement of the lemma is valid with $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ and $\mathbf{c}_4$. Then there are the following possibilities for choosing $\mathbf{c}_4$, being of weight 6 and at distance at most 6 from the three chosen words.

a) $\mathbf{c}_4 = 11111100$. To cover $\varphi(1111)$ we need at least five even codewords $(7 \cdot 2 = 14 < 16)$. It is necessary that $d(\mathbf{c}_2, \mathbf{c}_5) \leq 4$ or $d(\mathbf{c}_3, \mathbf{c}_5) \leq 4$, since otherwise there exists triangle in the Hamming space with all sides equal to 6, which is impossible. Now, either $\mathbf{c}_2$ or $\mathbf{c}_3$, are at distance at most 4 from $\mathbf{c}_1, \mathbf{c}_4$ and $\mathbf{c}_5$, the situation we seek.

b) $\mathbf{c}_4 = 11101110$. To cover 24 vectors of $\mathcal{F}_4^{(2)}$ we need additional even codeword $\mathbf{c}_5$ $(7 \cdot 3 = 21 < 24)$. Now the same arguments as in a) prove the claim in this case.

c) $\mathbf{c}_4 = 11001111$. To cover $\varphi(1111)$ we need additional even codeword $\mathbf{c}_5$ $(7 \cdot 2 = 14 < 16)$. Since $\mathbf{c}_5$ covers at most 5 vectors in $\varphi(1111)$, at least 6 odd codewords are necessary to cover $\varphi(1111)$. These odd codewords do not cover any vector from $\varphi(0000)$, and, hence, $\mathbf{c}_5$ necessarily covers 00110011 which is not yet covered, and at least 4 vectors from $\varphi(1111)$. W.l.o.g. $\mathbf{c}_5 = 10111011$, and $A_5(\mathcal{V}) = 0$, where $\mathcal{V}$ is the set of even vectors not covered by $\mathbf{c}_1, \ldots, \mathbf{c}_5$. However, $A_6(\mathcal{V}) = 1$. Nevertheless, since $A_6(\mathcal{V})$ gives all the possible collections of 6 odd vectors that cover $\mathcal{V}$, it is easy to check that they, together with $\mathbf{c}_1, \ldots, \mathbf{c}_5$, do not cover the whole space.

d) $\mathbf{c}_4 = 11111010$. Consider the set $\mathcal{V}$ of 12 words consisting of the class $\varphi(1111)$ and the two vectors: 11000011 and 00110011. Since $A_7(\mathcal{V}) = 0$, we need a fifth even codeword $\mathbf{c}_5$. Necessarily, weight of $\mathbf{c}_5$ is 6, and $d(\mathbf{c}_4, \mathbf{c}_5) \leq 4$. Since $d(\mathbf{c}_4, \mathbf{c}_2) = 2$, we get $d(\mathbf{c}_2, \mathbf{c}_5) = 6$. So, $\mathbf{c}_5 \in \mathcal{F}_6^{(2)}$, and it is equivalent to subcase c.

e) $\mathbf{c}_4 = 11101011$. If there is an additional even codeword $\mathbf{c}_5$ then the same arguments as in d) prove that it is equivalent to c). Otherwise, $A_7(\mathcal{V}) = 0$, where $\mathcal{V}$ is the set of even vectors not covered by $\mathbf{c}_1, \ldots, \mathbf{c}_4$.

Case 1.2: By Lemma 3, we have $d(\mathbf{c}_1, \mathbf{c}_2) \leq d(\mathbf{c}_1, \mathbf{c}_3) \leq 4$. By assumption, $d(\mathbf{c}_2, \mathbf{c}_3) = 6$. By the triangle inequality $d(\mathbf{c}_1, \mathbf{c}_2) = d(\mathbf{c}_1, \mathbf{c}_3) = 2$ is impossible, and since we are not in Case 1.1, $d(\mathbf{c}_1, \mathbf{c}_2) = d(\mathbf{c}_1, \mathbf{c}_3) = 4$. W.l.o.g. assume $\mathbf{c}_1 = 00000000, \mathbf{c}_2 = 11110000, \mathbf{c}_3 = 10001110$. We consider several possibilities for choosing $\mathbf{c}_4$.

a) $\mathbf{c}_4 \in \mathcal{F}_6^{(4)}$, then $d(\mathbf{c}_1, \mathbf{c}_4) = 6, d(\mathbf{c}_2, \mathbf{c}_4) = 2$, and $\mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_4$ are as in Case 1.1.

b) $\mathbf{c}_4 = 01111110$. It is easy to check that there is no other even word $\mathbf{c}_5$ that can be added in such a way that it does not fall under previous cases. Hence, there are 7 odd codewords, and for the set of even vectors not covered by $\mathbf{c}_1, \ldots, \mathbf{c}_4$ we verify that $A_7(\mathcal{V}) = 0$.

c) $\mathbf{c}_4 \in \mathcal{F}_6^{(2)}$. The first coordinate of $\mathbf{c}_4$ must be 0, otherwise there exists a permutation giving a). W.l.o.g. we may assume $\mathbf{c}_4 = 01101111$. Again there is no place for additional even codeword, and $A_7(\mathcal{V}) = 0$.

d) $\mathbf{c}_4 \in \mathcal{F}_6^{(3)}$. The only (up to permutations) situation which has not been considered yet is that the first and the last coordinates of $\mathbf{c}_4$ equal to 1. W.l.o.g., $\mathbf{c}_4 = 11100111$. The same argument as in the cases b) and c) is valid here as well.

Case 2: $4 \geq d(\mathbf{c}_1, \mathbf{c}_2) \geq d(\mathbf{c}_1, \mathbf{c}_3) \geq d(\mathbf{c}_2, \mathbf{c}_3)$. W.l.o.g. we assume $\mathbf{c}_1 = 00000000$. There are four possibilities for the distances between the three codewords.

a) $d(\mathbf{c}_1, \mathbf{c}_2) = d(\mathbf{c}_1, \mathbf{c}_3) = d(\mathbf{c}_2, \mathbf{c}_3) = 4$. W.l.o.g. $\mathbf{c}_2 = 11110000, \mathbf{c}_3 =$

11001100. Necessarily, the distance from c_4 to c_1, c_2 and c_3 is 6. Indeed, otherwise the claim of the lemma is true. There is only one possibility to choose c_4, namely, $c_4 = 00111111$, and there is no more even codewords. Now, considering covering of $\varphi(1111)$ we conclude that it is impossible by 7 odd codewords.

b) $d(c_1, c_2) = d(c_1, c_3) = 4, d(c_2, c_3) = 2$. W.l.o.g., $c_2 = 11110000, c_3 = 11101000$. Then c_4 has five ones in the last five coordinates, and w.l.o.g. we assume $c_4 = 00111111$. If there are only four even codewords, we check that $A_7(\mathcal{V}) = 0$. Otherwise, there is an additional even codeword c_5. W.l.o.g. $c_5 = 01011111$. Surprisingly, $A_6(\mathcal{V}) = 1$, i.e. the noncovered odd vectors can be covered by 6 (even 5) odd codewords. However, since $A_6(\mathcal{V})$ gives all the possible collections of 6 odd vectors to cover $\mathcal{V}$, it is easy to check that they do not cover all the space.

c) $d(c_1, c_2) = 4, d(c_1, c_3) = d(c_2, c_3) = 2$. W.l.o.g. $c_2 = 11110000, c_3 = 11000000, c_4 = 01011111$. Since $A_7(\mathcal{V}) = 0$ there is necessarily an even codeword c_5. There are two nonequivalent possibilities for c_5, namely, 10011111 and 10101111. In the first case $A_6(\mathcal{V} = 0$. In the second case $A_5(\mathcal{V}) = 0$, but $A_6(\mathcal{V}) = 1$. However, checking all the possiblities giving $A_6(\mathcal{V}) = 1$ we never get complete covering of the whole space.

d) $d(c_1, c_2) = d(c_1, c_3) = d(c_2, c_3) = 2$. W.l.o.g. $c_2 = 11000000, c_3 = 10100000$. There are two nonequivalent choices of c_4. If $c_4 = 10011111$ and there four even codewords, then we have $A_7(\mathcal{V}) = 0$. Otherwise, if $c_4 = 01111110$, then $A_7(\mathcal{V}) = 0$, and if there exists c_5 it is 01111101 or 10011111. Continuing as in case c) we check that it is impossible.

□

Lemma 5 *Let* $c_1 = 00000000$, *and* $d(c_1, c_2)$, $d(c_1, c_3)$, $d(c_1, c_4)$ *are at most 4. Then* c_2, c_3 *and* c_4 *cannot belong simultaneously to* $\varphi(0000)$.

Proof. Seeking a contradiction assume that c_2, c_3 and c_4 are in $\varphi(0000)$. Then $|C_e| = 5$. Indeed, otherwise we are not able to cover $\varphi(1111)$. The same argument shows that c_5, the fifth even codeword, cannot belong to $\varphi(0000)$. On the other hand, c_5 does not belong to $\varphi(1111)$ since in this case we cannot cover $\varphi(1111)$. Now, $c_1, \ldots, c_5$ must cover all $\varphi(0000)$. Indeed, otherwise assume that there is one vector in $\varphi(0000)$ that is not covered by an even codeword. Since there are at least 12 vectors of $\varphi(1111)$ that are not covered by the even codewords ($c_1, \ldots, c_4$ cover nothing in $\varphi(1111)$, and c_5 covers at most 4 vectors in $\varphi(1111)$), we need at least 7 odd codewords to cover $\varphi(1111)$ and the not covered vector from $\varphi(0000)$.

We consider the four cases.

Case 1: c_2 and c_3 have weight 2. W.l.o.g. we may assume that $c_2 = 11000000$ and $c_3 = 00110000$. In this case the weight c_4 is 4, otherwise we cannot cover all $\varphi(0000)$. The following possibilities exist for the choice of c_4:

a) $c_4 = 11110000$. Then, w.l.o.g., $c_5 = 10101111$, and $A_6(\mathcal{V}) = 0$.

b) $c_4 = 11001100$. Then $\varphi(0000)$ cannot be covered.

c) $c_4 = 00001111$. Then c_5 must cover the three vectors 11111111, 11111100, 11110011, so c_5 necessarily belongs to $\varphi(0011)$. But then there exists a permutation of coordinates which puts all the even codewords in $\varphi(0000)$, a contradiction.

Case 2: c_2 has weight 2, c_3 has weight 4, and $d(c_2, c_3) = 2$. If we add c_2 to all the codewords we get Case 1.

Case 3: c_2 has weight 2, c_3 has weight 4, and $d(c_2, c_3) = 6$. W.l.o.g. $c_2 = 11000000, c_3 = 00001111$. Then c_4 has weight 4 (if it is 2 then we are in Case 1), and w.l.o.g $c_4 = 00111100$. Since 00110011 and 11111111 have to be covered, this yields, w.l.o.g. $c_5 = 10111011$, and $A_6(\mathcal{V}) = 0$.

Case 4: c_2 and c_3 have weight 4, and $d(c_2, c_3) = 4$. Then, to avoid the previous cases, the weight of c_4 should be 4, and, hence, the five even codewords cannot cover $\varphi(0000)$.

□

So, as a corollary of Lemma 4, we conclude that we always may assume $c_1 = 00000000$, the weights of c_2 and c_3 are at most 4, and $d(c_2, c_3) \leq 4$. As it is easy to check there are four possibilities to choose c_2 and c_3, namely,

a) $c_2 = 11000000, c_3 = 10100000$;

b) $c_2 = 11000000, c_3 = 00110000$;

c) $c_2 = 11000000, c_3 = 10111000$;

d) $c_2 = 11110000, c_3 = 11001100$.

For each of these cases we can find all possible c_4 such that it has distance at most 4 to at least one of the chosen even codewords, and distance at most 6 to all others. In the following table we give all these nonequivalent possibilities for c_4:

a) 01100000, 10010000, 01010000, 00011000,
10011100, 01111000, 01011100, 00001111;

b) 11101000, 10101100, 10001110, 11101110;

c) 10110100, 01111000, 10100110, 01110100, 10000111, 01100110,
00111100, 00110110, 00100111, 10111110, 10110111, 01111110;

d) 10101010, 00111010, 10100011, 00101011.

All in all we have 28 possible choices of four even codewords $c_1, \ldots, c_4$, and as it is easy to check that $A_7(\mathcal{V}) = 0$ in all these cases. It requires checking 28×8^7 collections of 7 odd codewords. So, there should be one more even codeword.

The problem now becomes tractable. For each of 28 choices of the four even codewords, the computer program goes through all possibilities for the fifth even codeword. Given the five even codewords, let $\mathcal{V}$ stand for the set of even vectors not covered by them. In all the cases $A_4(\mathcal{V}) = 0$, i.e. we always need at least 5 odd codewords to cover $\mathcal{V}$. If $A_5(\mathcal{V}) = 0$ and $A_6(\mathcal{V}) = 1$, as a result of the program we obtain all the sets of 6 odd codewords that cover

$\mathcal{V}$. For each such set we check if the resulting collection of 5 even and 6 odd codewords cover the complete space. It requires checking $28 \times 128 \times 8^6$ collections of 6 odd codewords. If $A_5(\mathcal{V}) = 1$ we just have to check the combination of 5 even and 5 odd codewords if the part of $\mathcal{F}$ not covered by them is within a sphere of radius 2. It requires checking $28 \times 128 \times 8^5$ collections of 5 odd codewords.

The program did not find any set of 11 codewords covering $\mathcal{F}$, thus confirming the conjecture.

□

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