Regular Subgraphs of Hypercubes

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ABSTRACT. We consider several families of regular bipartite graphs, most of which are vertex-transitive, and investigate the problem of determining which ones are subgraphs of hypercubes. We define $H_{k,r}$ as the graph on k vertices $0, 1, 2, \ldots, k-1$ which form a k-cycle (when traversed in that order), with the additional edges $(i, i+r)$ for i even, where $i+r$ is computed modulo k. Since this graph contains both a k-cycle and an $(r+1)$-cycle, it is bipartite (if and) only if k is even and r is odd. (For the "if" part, the bipartition (X, Y) is given by $X =$ even vertices and $Y =$ odd vertices.) Thus we consider on the cases $r = 3, 5, 7$. We find that $H_{k,3}$ is a subgraph of a hypercube precisely when $k \equiv 0 \pmod 4$. $H_{k,5}$ can be embedded in a hypercube precisely when $k \equiv 0 \pmod{16}$. For $r = 7$ we show that $H_{k,7}$ is embeddable in a hypercube whenever $k \equiv 0 \pmod{16}$.

The question of which graphs can be embedded as a subgraph of a hypercube goes back to 1965. V.V. Firsov [5] asked which graphs could be *isometrically* embedded in a hypercube. An embedding is *isometric* if distances are preserved. That is, $\phi\colon H \to G$ is isometric if for all $x, y \in V(H)$, $\mathrm{dist}_H(x, y) = \mathrm{dist}_G(x, y)$. This was answered in 1973 by Djoković [3]. Garey and Graham [4] in 1975 raised the general embedding question (dropping the isometry requirement) and showed that a nice characterization was unlikely. This was confirmed in 1986 by Cybenko, Krumme and Venkataraman [2], who proved that deciding whether or not a graph has an embedding as a subgraph of a hypercube is NP-complete. In 1990, Wagner and Corneil [8] showed that the problem is NP-complete even for trees. On the other hand, many bipartite graphs are known to be subgraphs of hypercubes. A good reference for this is Leighton's book [7]. In particular, an $M_1 \times M_2 \times \ldots M_k$-array is a subgraph of an N-node hypercube if and only if $N \geq 2^{\lceil \log M_1 \rceil + \lceil \log M_2 \rceil + \cdots + \lceil \log k \rceil}$ [7, exercise 3.20], and an $N \times N$ mesh of trees is a subgraph of the $4N^2$-node hypercube [7, Exercise 3.60]. The purpose of this paper is to consider several families of regular bipartite graphs, most of which are vertex-transitive, and investigate the problem of determining which ones are subgraphs of hypercubes. We denote the n-dimensional hypercube by Q_n. For an edge $e = (x, y)$ of Q_n, the n-tuples x and y differ in exactly one coordinate i. We call this i the *dimension* of e and denote it by $d(e)$.

Definition 1. *$H_{k,r}$ is the graph on k vertices $0, 1, 2, \ldots, k-1$ where for all $0 \leq i\ j \leq k-1$, $(i, j) \in E \Leftrightarrow$ (1) $|i-j| \equiv 1 \pmod k$ or (2) i is even and $j - i \equiv r \pmod k$.*

Note: Since $H_{k,r}$ contains both a k-cycle and an $(r+1)$-cycle, the graph is not bipartite if either k is odd or r is even. So we shall only consider the graphs $H_{k,r}$ for k even and r odd. In this case $H_{k,r}$ is bipartite and 3-regular. Furthermore, it is vertex-transitive.

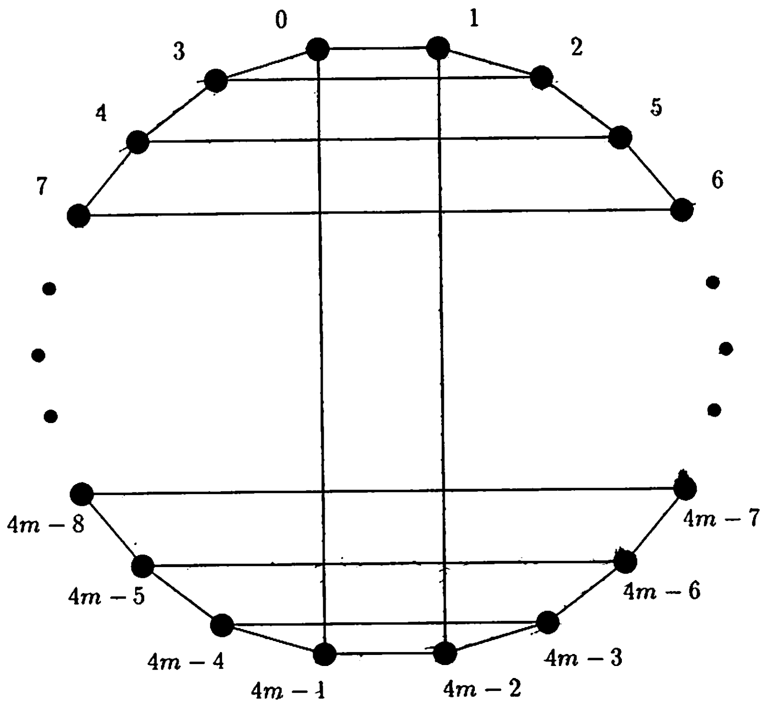

Figure 1. $H_{4m,3}$

Proposition 1. *If k is even then $H_{k,3}$ is a subgraph of some hypercube $\Leftrightarrow k \equiv 0 \pmod 4$. Furthermore, $H_{4m,3}$ is a subgraph of $Q_n \Leftrightarrow m \leq 2^{n-2}$.*

Proof: ($\Rightarrow$) Let $k = 4m+2$, and suppose $H_{k,3}$ is a subgraph of a hypercube. For i even, $(i, i+1)$ belongs to the 4-cycle $(i, i+1, i+2, i+3)$. Hence

$$d((i+2, i+3)) = d((i, i+1))$$

Let $d((0,1)) = q$. Then for all even integers i with $0 \le i \le k-1$,

$$d((i, i+1)) = q.$$

Let E^* be the set of edges e of the k-cycle $C = (0, 1, \ldots, k-1)$ with $d(e) = q$. So $|E^*| \ge \frac{k}{2} = 2m+1$. Since E^* must be an independent set of edges, $|E^*| = \frac{k}{2}$. This is a contradiction since $\frac{k}{2}$ is odd. Thus $H_{k,3}$ is not a subgraph of any hypercube.

($\Leftarrow$) $H_{4m,3}$ can be drawn as follows: The vertices $\{i \mid i \equiv 0 \pmod 4\}$ lie on one $2m$-cycle and the remaining vertices of $H_{4m,3}$ lie on another. The edges $\{(i, i+1) \mid 0 \le i \le 4m-2\}$ form a pairing of these two $2m$-cycles so that $H_{4m,3} \simeq C_{2m} \times K_2$. Now the latter is a subgraph of $Q_n \Leftrightarrow C_{2m}$ is a subgraph of Q_{n-1}, and since hypercubes are Hamiltonian and contain cycles of all even orders, C_{2m} a subgraph of $Q_{n-1} \Leftrightarrow 2m \le 2^{n-1}$, *i.e.* $\Leftrightarrow m \le 2^{n-2}$. □

Remark. Every occurrence of $H_{4m,3}$ in a hypercube is contained in some $(m+1)$-dimensional subcube. For by the argument given at the start of the proof of Proposition 1, for all even integers i, $d((i, i+1)) = d((0,1))$. Hence $2m$ of the $4m$ edges of the Hamiltonian cycle $(0, 1, \ldots, 4m-1)$ have the same d-value. Since every d-value must occur an even number of times on any cycle, at most m other values can occur. Hence $H_{4m,3}$ is contained in an $(m+1)$-dimensional subcube.

Proposition 2. *$H_{4k+2,2k+1}$ is not a subgraph of any hypercube.*

Proof: For all i, $(i, i+1, i+2k+2, i+2k+1)$ is a 4-cycle of H, and thus $d((i, i+2k+1)) = d((i+1, i+2k+2))$. So this value, call it q, occurs on every chord of the Hamiltonian cycle $(0, 1, \ldots, 4k+1)$ of H. Since every edge of this cycle is incident with two of these chords, the value q occurs on *no* edge of the Hamiltonian cycle. But then the value q occurs exactly once on the $(2k+2)$-cycle $(0, 1, \ldots, 2k+1)$, namely on the chord $(0, 2k+1)$. This contradicts the fact that any value must occur an even number of times on any cycle. □

Remark. $H_{8,5} \simeq H_{8,3} \simeq Q_3$.

Proposition 3. *If k is an even integer, $k > 8$ and $H_{k,5}$ is embeddable in Q_n for some n then $k \equiv 0 \pmod{16}$.*

Proof: We need three preliminary results.

Lemma 1. *If the d-values on 3 consecutive edges of a 6-cycle in a hypercube are a, b, a, then the d-values on the next 3 edges are c, b, c for some c distinct from a and b.*

Proof: The third d-value which occurs on this 6-cycle, c, must occur twice on the remaining 3 edges, and b must occur once. Since edges sharing a vertex must have different d-values, these values, in sequence, are c, b, c. □

Lemma 2. *If $H_{k,5}$ is a subgraph of a hypercube, and if for some even integer j, $d((j, j+1)) = a$, $d((j+1, j+2)) = b$, and $d((j+2, j+3)) = a$, then $d((j+4, j+5)) = b$, $d((j+5, j+6)) = a$, and $d((j+6, j+7)) = b$.*

Proof: Consider the subgraph H^* of $H_{k,5}$ shown in Figure 2. By Lemma 1, $d((j+4, j+5)) = b$, and $d((j+3, j+4)) = d((j, j+5)) = c$, $c \neq a$, $c \neq b$. Since $(j+2, j+7)$ and $(j+1, j+2)$ are incident, $d((j+2, j+7)) \neq b$. So in the 6-cycle $(j+2, j+3, j+4, j+5, j+6, j+7)$, the only possible edge whose d-value is b, other than $(j+4, j+5)$, is $(j+6, j+7)$. Hence the only edge of the 6-cycle with d-value a, other than $(j+2, j+3)$, is $(j+5, j+6)$. □

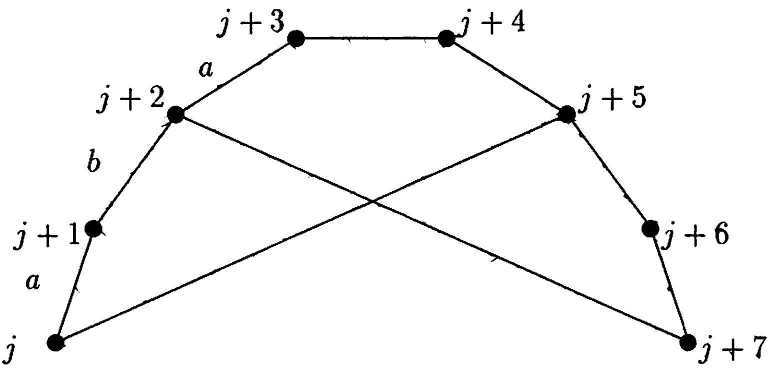

Figure 2. H^*

Lemma 3. *Suppose k is an even integer, $k > 8$, and $H_{k,5}$ is a subgraph of a hypercube. If for some even integer j, $d((j, j+1)) = d((j+2, j+3))$, then $k \equiv 0 \pmod{16}$.*

Proof: It follows from Lemma 2 that the sequence

$$\{d((j+2i, j+2i+1))\}_{i=0}^{i=\frac{k}{2}-1}$$

alternates $a, a, b, b, \ldots$ where $a = d((j, j+1))$ and $b = d((j+1, j+2))$. Now if this sequence ends with an a, *i.e.* $d((j-2, j-1)) = a$, then the 6-cycle $(j-2, j-1, j, j+1, j+2, j+3)$ has 3 edges whose d-value is a. Since this is impossible, the sequence ends with b. In fact, since k is even, the sequence ends with 2 b's, and so $k = 4q$ for some integer q. By Lemma 2, the sequence $\{d((j, j+1))\}_{j=0}^{j=k-1}$ has the form

$$a, b, a, *, b, a, b, *, a, b, a, *, b, a, b, *, \ldots$$

where the omitted values are all different from both a and b. Since $k = 4q$, the sequence consists of q 4-tuples. If q is odd, then the last 4-tuple is $a, b, a, *$. But then the 6-cycle $(k-2, k-1, 0, 1, 2, 3)$ has 3 edges whose d-value is a, which is impossible. So q is even. Thus $\frac{q}{2}$ of the 4-tuples have exactly 2 a's and the other $\frac{q}{2}$ 4-tuples have exactly 1 a. So exactly $\frac{3q}{2}$ of the edges of the k-cycle $(0, 1, 2, \ldots, k-1)$ have the d-value a. Hence $\frac{q}{2}$ is even. Say $\frac{q}{2} = 2p$. Then $k = 4q = 16p$. □

Proof of Proposition 3: Let $E_0 = \{(j, j+1) \mid j \text{ is even}\}$. Suppose that, contrary to the assumption in Lemma 3, *consecutive* edges of E_0 have different d-values, *i.e.* that for all even integers j, $d((j, j+1)) \neq d((j+2, j+3))$.

Claim 1: $d((j-2, j+3)) = d((j, j+1))$, for all even j. For in the 6-cycle $(j-2, j-1, j, j+1, j+2, j+3)$, 2 of the 3 edges not incident with $(j, j+1)$ belong to E_0 and are consecutive with $(j, j+1)$. Thus neither one can have d-value $= d((j, j+1))$. The chord $(j-2, j+3)$ is therefore the only possible edge $e \neq (j, j+1)$ such that $d(e) = d((j, j+1))$. This proves Claim 1.

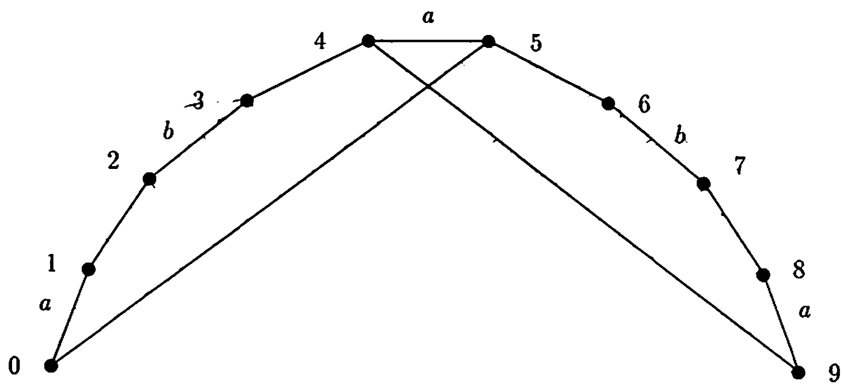

Figure 3. The induced subgraph

Claim 2: $|\{d(e) \mid e \in E_0\}| \geq 3$.

For suppose there were only 2 distinct d-values, say a and b, for edges $e \in E_0$. Consider the induced subgraph of $H_{k,5}$ on the vertices $\{0, 1, 2, \ldots, 9\}$. The d-values on the edges in E_0 alternate between a and b, as shown in Figure 3. Since $d((4,5)) = d((0,1)) = a$, it follows from Lemma 1, applied to the 6-cycle $(0, 1, 2, 3, 4, 5)$ that $d((3,4)) = d((1,2))$. Call this value c. In the 6-cycle $(2, 3, 4, 5, 6, 7)$, $d((2,3)) = d((6,7)) = b$, so by Lemma 1 $d((5,6)) = d((3,4)) = c$. Finally, in the 6-cycle $(4, 5, 6, 7, 8, 9)$, $d((8,9)) = d((4,5)) = a$, so a third application of Lemma 1 yields $d((7,8)) = d((5,6)) = c$. But then the sequence $\{d((j, j+1))\}_{j=0}^{j=7}$ is a, c, b, c, a, c, b, c. Since each d-value occurs an even number of times, vertices 0 and 8 coincide, which is a contradiction. This proves Claim 2. Now by our initial assumption, consecutive edges of

E_0 have different d-values, and by Claim 2, there at least 3 different d-values on edges of E_0. Hence there must be 3 consecutive edges of E_0 with 3 different d-values. Without loss of generality we may suppose $d((0,1)) = a$, $d((2,3)) = b$, and $d((4,5)) = c$, where a, b, and c are all distinct (see Figure 4). By Claim 1, $d((2,7)) = d((4,5)) = c$. Since the $d$values a, b, and c occur on the edges of the 6-cycle $(0,1,2,3,4,5)$, $d((1,2))$ must be one of these. Since a and b are the values on edges incident with $(1,2)$, $d((1,2))$ must be c. But then $(2,7)$ and $(1,2)$ are incident edges with the same d-value, which is impossible. Hence for some even integer j, $d((j,j+1)) = d((j+2,j+3))$, and so by Lemma 3, $k \equiv 0 \pmod{16}$. □

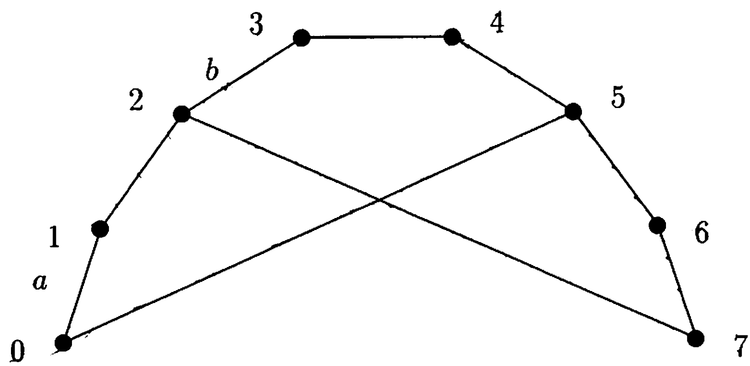

Figure 4.

The next proposition is the converse to Proposition 3.

Proposition 4. *For all k, $H_{16k,5}$ is a subgraph of Q_n, where $2^{n-1} \leq 16k \leq 2^n$.*

Proof: We shall construct a special $(16k)$-cycle on Q_n as follows. For $4 \leq j \leq n$ let $P(j)$ be the sequence $1213212j$. If $k = 1$ then $n = 4$ and we let $S = P(4)$, $P(4) = (12132124)^2$. It is not hard to check that S is the sequence of edge dimensions of a Hamiltonian cycle on Q_4. We now generalize the construction of S for $k > 1$ and $n > 4$. Let $a_1, a_2, \ldots, a_{2k}$ be the edge-dimension sequence of a $(2k)$-cycle on the $(n-3)$-dimensional subcube $\{000*^{n-3}\} = \{\vec{x} \in Z_2^n \mid x_i = 0 \text{ for } 1 \leq i \leq 3\}$ of Q_n. (Note that since $16k \leq 2^n$, we have $2k \leq 2^{n-3}$, and thus such a $2k$-cycle exists.) So for each i, $4 \leq a_i \leq n$. Let $S = P(a_1), P(a_2), \ldots, P(a_{2k})$. Then S is an edge-dimension sequence of length $8 \cdot 2k = 16k$. By starting at vertex $\vec{0}$ and traversing the sequence of edges whose corresponding sequence is S, we obtain a walk of length $16k$ on Q_n which we claim is a cycle. To see this, note that by Theorem 1 of [6], an edge-dimension sequence corresponds to a closed walk if and only if every integer occurs an even number of times, and corresponds to a path if and only if for every proper segment of the sequence, some integer occurs an odd number of times. Using this and the assumption that $a_1, a_2, \ldots, a_{2k}$ corresponds to a cycle, it is easy to see that S cooresponds to a closed walk, each $P(j)$ corresponds to a path, and furthermore, every proper segment of S corresponds to a path. This means that S corresponds to a cycle. Finally, if we let $S = b_0, b_1, \ldots, b_{16k-1}$ then we claim that for all i,

$$|\{b_{2i}\}\Delta\{b_{2i+1}\}\Delta\ldots\Delta\{b_{2i+4}\}| = 1.$$

To see this, note that b_{2i} occurs in some segment $1213212j$ of length 8. Let $\beta = \{b_{2i}\}\Delta\{b_{2i+1}\}\Delta\ldots\Delta\{b_{2i+4}\}$.

$$\begin{aligned}
&\text{If } 2i \equiv 0 \pmod 8, \quad \beta = \{1\}\Delta\{2\}\Delta\{1\}\Delta\{3\}\Delta\{2\} = \{3\}.\\
&\text{If } 2i \equiv 2 \pmod 8, \quad \beta = \{1\}\Delta\{3\}\Delta\{2\}\Delta\{1\}\Delta\{2\} = \{3\}.\\
&\text{If } 2i \equiv 4 \pmod 8, \quad \beta = \{2\}\Delta\{1\}\Delta\{2\}\Delta\{j\}\Delta\{1\} = \{j\}.\\
&\text{If } 2i \equiv 6 \pmod 8, \quad \beta = \{2\}\Delta\{j\}\Delta\{1\}\Delta\{2\}\Delta\{1\} = \{j\}.
\end{aligned}$$

Thus in all cases $|\beta| = 1$. If we denote the corresponding $(16k)$-cycle by $v_0, v_1, \ldots, v_{16k-1}$ then it follows that for all i, (v_{2i}, v_{2i+5}) is an edge of Q_n.

Thus $H_{16k,5}$ is a subgraph of Q_n. □

A modification of the construction of Proposition 4 shows an analogous result for $H_{16k,7}$.

Proposition 5. *For all k, $H_{16k,7}$ is a subgraph of Q_n, where $2^{n-1} < 16k \le 2^n$.*

Proof: The only change we make is in the definition of $P(j)$. We let $P(j) = 1232123j$. As the rest of the proof is virtually the same as that for Proposition 4, we omit it. □

Example 1. *Let $G(X,Y)$ be the bipartite graph associated with the projective plane of order 3 (or equivalently, with the (13,4,1) perfect difference set). Then G is a 4-regular, vertex-transitive graph which cannot be embedded in Q_n for any n.*

Proof: G contains $H_{26,5}$ as a subgraph and since $26 \not\equiv 0 \pmod{16}$, the result follows from Proposition 4. □

Example 2. *For n odd, Q_n has an $(n+1)/2$-regular subgraph H on $2 \cdot \binom{n}{\frac{n-1}{2}}$ vertices. None of the graphs is an $H_{k,r}$.*

Proof: $V(H) = \{$nodes of Hamming weight $(n-1)/2$ or $(n+1)/2\}$. Since $n - (\frac{n-1}{2}) = \frac{n+1}{2}$,

$$|V(H)| = 2 \cdot \binom{n}{\frac{n-1}{2}}$$

Now each v with *weight*$(v) = \frac{n-1}{2}$ is adjacent to exactly $n - (\frac{n-1}{2}) = \frac{n+1}{2}$ nodes of weight $\frac{n-1}{2} + 1 = \frac{n+1}{2}$. For each of the heavier neighbors of v can be obtained by flipping exactly one of the $\frac{n+1}{2}$ bits of v which is 0. Similarly, by flipping exactly one of the 1's of a node of weight $\frac{n+1}{2}$, we obtain an adjacent node of weight $\frac{n-1}{2}$, and this can also be done in exactly $\frac{n+1}{2}$ ways. Thus H is $\frac{n+1}{2}$-regular. For $n = 5$, the 3-regular subgraph H of Q_5 can be drawn as shown in Figure 5. Thus it is the generalized Petersen graph $GP(10,3)$ (see [1] for the general definition).

None of these graphs belongs to the family of $H_{k,r}$ graphs. For if $n \neq 5$ then $\frac{n+1}{2} \neq 3$ and so H is not 3-regular. On the other hand, if $n = 5$ then $H = GP(10,3)$ has no 4-cycle, whereas in $H_{20,3}$ the vertices $0,1,2,3$ span a 4-cycle. □

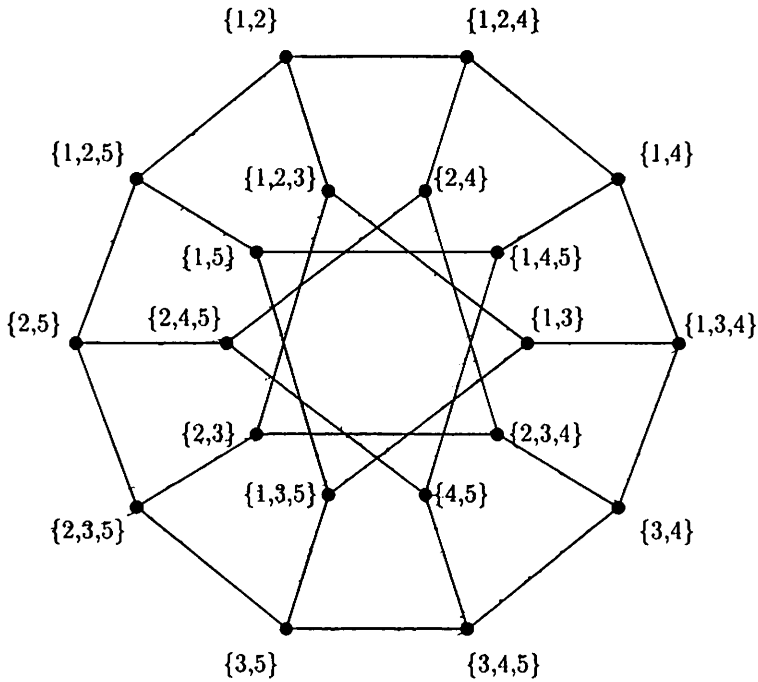

Figure 5

Proposition 6. *For $k \geq 3$, no k-regular bipartite graph on 10 vertices can be embedded in a hypercube.*

Proof: Since a k-regular bipartite graph is 1-factorable, it has a 3-regular subgraph. Hence it suffices to prove the result for the case $k = 3$. Assume the contrary, letting G be a 3-regular counter-example. Since the complete bipartite graph $K_{2,3}$ can not be embedded in a hypercube, $K_{2,3}$ is not a subgraph of G. Thus if (X, Y) is a bipartition of G and $y, y' \in Y$ with $y \neq y'$, then $N(y) \neq N(y')$, where $N(z)$ denotes the neighbor set of z. Hence

$\{N(y) \mid y \in Y\}$ consists of 5 distinct 3-sets of X. Similarly, $\{N(x) \mid x \in X\}$ consists of 5 distinct 3-sets of Y. Furthermore, for any 2-set $B \subset X$, let $S = \{(B,y) \mid y \in Y \text{ and } B \subset N(y)\}$. Let $S_B = \{y \in Y \mid (B,y) \in S\}$, and let $S_y = \{B \subset X \mid (B,y) \in S\}$. Then

$$\sum_{B \subset X, |B|=2} |S_B| = |S| = \sum_{y \in Y} |S_y|.$$

$$\sum_{y \in Y} |S_y| = \sum_{y \in Y} \binom{3}{2} = 5 \cdot \binom{3}{2} = 15.$$

Therefore

$$\sum_{B \subset X, |B|=2} |S_B| = 15.$$

Since $K_{2,3}$ is not a subgraph of G, $|S_B| \leq 2$ for all 2-sets $B \subset X$. On the other hand, if $B = \{x, x'\}$ then $B \subset N(y) \Leftrightarrow y \in N(x) \cap N(x')$. So $|S_B| = N(x) \cap N(x')$. But since $|Y| = 5$, the 3-sets $N(x)$ and $N(x')$ cannot be disjoint. Hence $|S_B| \geq 1$ for all B. So $|S_B| = |N(x) \cap N(x')|$. Hence $|S_B| \geq 1$ for all B.

Since $1 \leq |S_B| \leq 2$, $\sum |S_B| = 15$, and there are 10 summands, we must have $|S_B| = 2$ for 5 B's and $|S_B| = 1$ for the other 5 B's. These same remarks apply with the roles of X and Y reversed. In particular, for $y, y' \in Y$, $N(y) \cap N(y') \neq \emptyset$. Choose $x_1, x_2 \in X$ such that $|N(x_1) \cap N(x_2)| = 2$. Let $N(x_1) = \{y_1, y_2, y_3\}$ and $N(x_2) = \{y_2, y_3, y_4\}$. Since $y_1 \notin N(x_2)$, we have $x_2 \notin N(y_1)$, and since $y_4 \notin N(x_1)$, we have $x_1 \notin N(y_4)$. Now $x_2 \in N(y_4)$, so calling the other 2 members of $N(y_4)$ x_3 and x_4, we have $N(y_4) = \{x_2, x_3, x_4\}$. By a remark above, $N(y_1) \cap N(y_4) \neq \emptyset$. Since $x_2 \notin N(y_1)$, either x_3 or x_4 is in $N(y_1) \cap N(y_4)$. Without loss of generality we may assume that $x_3 \in N(y_1) \cap N(y_4)$. Suppose that $x_4 \in N(y_1) \cap N(y_4)$ also. Then $N(y_1) = \{x_1, x_3, x_4\}$, and so $x_5 \notin N(y_1)$. Since $x_5 \notin N(y_4) = \{x_2, x_3, x_4\}$, we must have $N(x_5) = \{y_2, y_3, y_5\}$. But then $N(y_2) = N(y_3) = \{x_1, x_2, x_5\}$, contradicting the distinctness of the $N(y)$'s. Thus $x_4 \notin N(y_1)$. Hence $N(y_1) = \{x_1, x_3, x_5\}$. Now by our choice of x_1 and x_2, neither of them is in $N(y_5)$. Hence $N(y_5) = \{x_3, x_4, x_5\}$. Thus $N(x_3) = \{y_1, y_4, y_5\}$. Hence $x_3 \notin N(y_2) \cup N(y_3)$. Since $N(y_2) \cap N(y_3) \supset \{x_1, x_2\}$ and $N(y_2) \neq N(y_3)$, one of them must be $\{x_1, x_2, x_4\}$, and the other $\{x_1, x_2, x_5\}$. With no loss of generality, we may assume that $N(y_2) = \{x_1, x_2, x_4\}$ and $N(y_3) = \{x_1, x_2, x_5\}$. So G is as shown in Figure 6. But this is precisely the graph $H_{10,3}$ (see Figure 6b) which by Proposition 1 is not a subgraph of any hypercube. □

Finally, we show just how special the number 10 was in the preceding result.

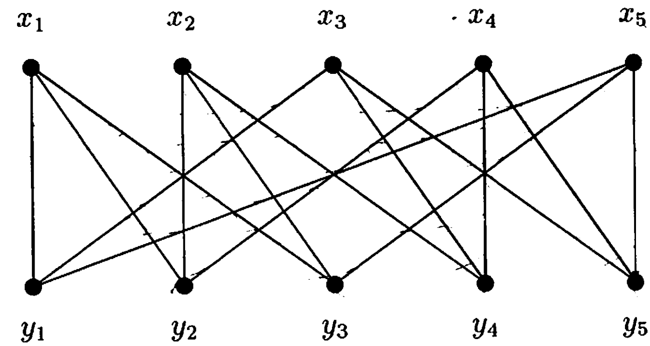

Figure 6a. G

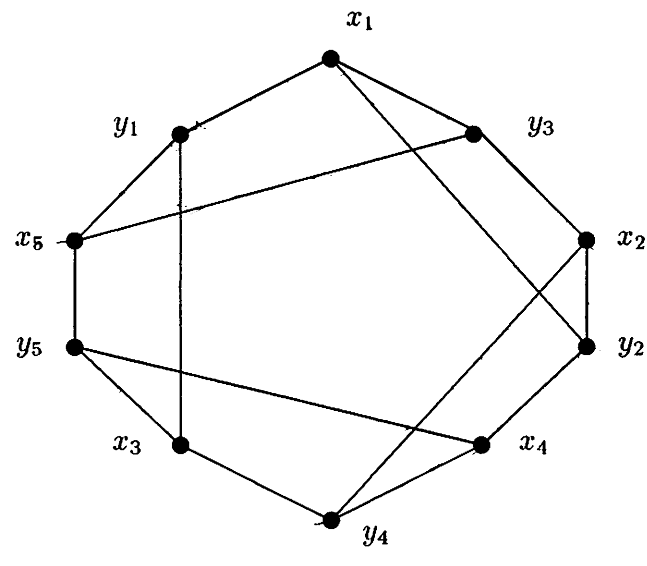

Figure 6b. A redrawing of G

Proposition 7. *Let k be an integer ≥ 14 and suppose that $k \equiv 2 \pmod 4$. Then there is a 3-regular graph on k vertices which is a subgraph of a hypercube.*

Proof: We argue by induction on q, where $k = 4q + 2$. For $q = 3$ (so $k = 14$), we give an example of a 3-regular subgraph of Q_4 on 14 vertices (see Figure 7a). Now to establish the result in general we shall show that if G is a 3-regular subgraph of Q_n and if x, y, z, w is a 4-cycle in Q_n such that all four vertices belong to G and edges (x, y) and (z, w) belong to G, then we can adjoin an additional four vertices of Q_{n+1} to form a larger 3-regular subgraph. For viewing Q_{n+1} as two copies of Q_n, joined by a perfect matching, choose vertices x', y', z', and w' in Q_{n+1} so that the edges (x, x'), (y, y'), (z, z'), (w, w') are part of this perfect matching. Let G' be the graph obtained from G by deleting edges (x, y) and (z, w), and adjoining the vertices and edges of the 4-cycle x', y', z', w', along with the four edges matching the two 4-cycles. Clearly G' is 3-regular. This construction is illustrated by Figure 7b. □

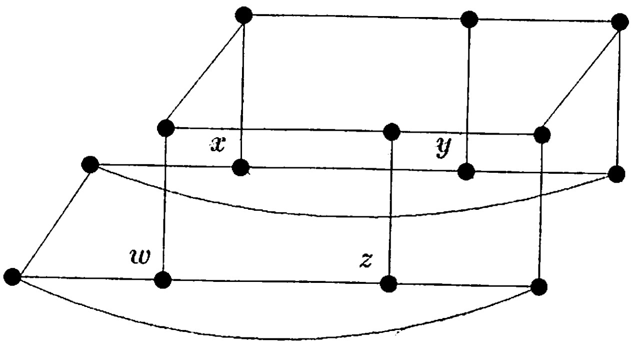

Figure 7a

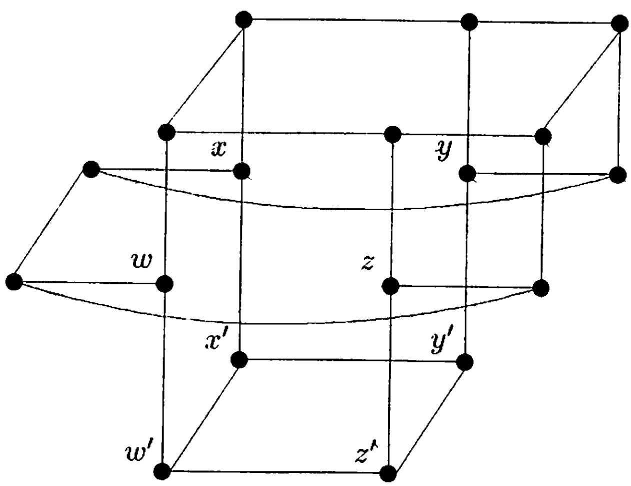

Figure 7b

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