An Equation Involving the Neighborhood (Two-Step) and Line Graphs

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ABSTRACT. The neighborhood or two-step graph, $N(G)$, of a graph G is the intersection graph of the open neighborhoods of the vertices of G, and $L(G)$ is the line graph of G. The class of graphs for which $N[L(G)] \cong L[N(G)]$ consists of those graphs for which every component is either K_1, $K_{1,3}$, or C_n where $n \geq 3$ and $n \neq 4$.

1 Introduction

All graphs considered have no loops or multiple edges. A *graph equation* is an equality involving one or more graphs. A number of such equations have been studied [2-6, 9-12, 16-19, 24-28] and Cvetkovic and Simic [13] present an extensive bibliography of early work. The *neighborhood* or *two-step graph*, $N(G)$, of a graph G is the intersection graph of the open neighborhoods of G, that is, $N(G)$ can be considered to have the same vertex set as G with two vertices adjacent if and only if, in G, they are joined by a path of length two. Neighborhood graphs play an important role in the study of competition graphs [20-23]. Properties of neighborhood graphs are developed in [1, 7, 8, 14].

In this paper we let $NL(G)$ and $LN(G)$ represent the graphs obtained by the composite operations $N[L(G)]$ and $L[N(G)]$, respectively, and characterize those graphs G for which equality holds. The characterization theorem follows.

Theorem. *Every component of a graph G is one of K_1, $K_{1,3}$, or C_n where $n \geq 3$ and $n \neq 4$ if and only if $NL(G) \cong LN(G)$.*

That $NL(G) \cong LN(G)$ whenever the components of G are as required by the theorem is established in Section 2 after some simple observations about neighborhood graphs and line graphs are presented. The proof in the opposite direction is given for connected graphs in Section 3 and disconnected graphs in Section 4.

2 Elementary Properties

Certain basic properties of line graphs and neighborhood graphs will be useful in the subsequent development. The first observation states well-known facts about line graphs.

Observation 1.

(a) *If G is a connected graph, $L(G)$ also is connected.*

(b) *Let $n \geq 4$. Then $L(G)$ is the complete graph K_n if and only if G is $K_{1,n}$.*

(c) *A graph is a line graph if and only if it contains none of nine special graphs as an induced subgraph [15], four of which are $K_{1,3}$, K_5 minus an edge (denoted by $K_5 - e$), and the two graphs shown in Figure 1.*

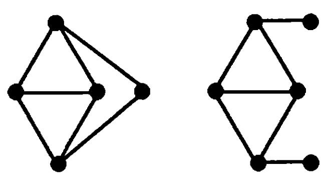


Figure 1

We now examine basic properties of neighborhood graphs. The first is well known, but we include a proof of necessity since we have been unable to locate one in the literature. The second is a simple, but useful, property.

Observation 2. *If G is connected, $N(G)$ is disconnected if and only if G is bipartite.*

Proof: Sufficiency is shown by Exoo and Harary [14]. Suppose $N(G)$ is disconnected but G is not bipartite. Then G contains an odd cycle C, implying $N(G)$ contains an odd cycle D on the same vertices as those in C. Let u be a vertex in a component of $N(G)$ different from the component which contains D. Since G is connected, there is in G a path $\langle u, v_1, v_2, \ldots, v_k, w, x\rangle$ where w is the first vertex on this path which lies in C, and x is a neighbor of w on C. Thus, in G, there is an even length path between u and either w or x, that is, an even length path from u to at least one vertex of C, implying u is joined to D by a path in $N(G)$, a contradiction. □

Observation 3. *Let G be a connected bipartite graph with at least two vertices in each bipartition. Then $N(G)$ has exactly two components, both of which are nontrivial.*

Proof: Since G is connected, any two vertices of G which lie in a single bipartition are joined by an even length path. Thus all the vertices of one of the bipartitions are in a single component of $N(G)$, and there must be two components by Observation 2. Furthermore, each component has at least one edge since each bipartition has at least two vertices. □

The next two observations present helpful counting and structural results. The number of vertices and edges of a graph G are indicated by $|V(G)|$ and $|E(G)|$, respectively.

Observation 4. *If $NL(G) \cong LN(G)$, then $|E(G)| = |E[N(G)]|$.*

Proof: Using the definitions of $N(G)$ and $L(G)$, it is clear that $|V[NL(G)]| = |V[L(G)]| = |E(G)|$ and $|V[LN(G)]| = |E[N(G)]|$. □

Observation 5. *If G has at least four edges, is not bipartite, and $NL(G) \cong LN(G)$, then $NL(G)$ is not complete.*

Proof: Suppose $NL(G) \cong LN(G) \cong K_n$ with $n \geq 4$. By Observation 1(b), $N(G)$ must be $K_{1,n}$. But G is not bipartite so both G and $N(G)$ contain an odd cycle. □

The proof of Lemma 1 follows directly from the Observations and the definitions of the composite operations on the specific graphs involved and will not be given. The lemma also establishes one direction of the theorem: If every component of G is one of K_1, $K_{1,3}$, or C_n where $n \geq 3$ and $n \neq 4$, then $NL(G) \cong LN(G)$.

Lemma 1.

(i) $NL(C_n) \cong LN(C_n) \cong C_n$ *if n is odd,*

(ii) $NL(C_n) \cong LN(C_n) \cong C_{n/2}$ *if $n \geq 6$ is even,*

(iii) $NL(K_{1,3}) \cong LN(K_{1,3}) \cong C_3$, *and*

(iv) $NL(K_1) \cong LN(K_1) \cong \emptyset$, *the graph with no vertices.*

The proof that these are the only graphs for which $NL(G) \cong LN(G)$ is the subject of the next two sections, Section 3 for connected graphs and Section 4 for disconnected graphs. Both sections proceed by developing a sequence of results which identify a class of graphs of the specified type, connected or disconnected, for which $NL(G) \not\cong LN(G)$ and whose union is all graphs of that type, except for those identified in the theorem.

3 Connected Graphs

We first show that for all connected graphs G with maximum degree, denoted $\Delta(G)$, at most two, other than those allowed by the theorem, $NL(G) \ncong LN(G)$. As with Lemma 1, the proof is straightforward and not given.

Lemma 2. *If a graph G is either a C_4 or a path P_n with $n \geq 2$, then $NL(G) \ncong LN(G)$.*

All remaining connected graphs G for which $NL(G)$ and $LN(G)$ may be isomorphic must have $\Delta(G) \geq 3$. The bipartite graphs $K_{1,n}$ are treated next in Lemma 3. All other bipartite graphs have at least two vertices in each bipartition, in which case Observation 3 becomes useful, and are covered in Lemma 4.

Lemma 3. *If $n \neq 3$, then $NL(K_{1,n}) \ncong LN(K_{1,n})$.*

Proof: The cases when $n \leq 2$ are included in the results of the previous two lemmas, so we may assume $n \geq 4$ and that $NL(K_{1,n}) \cong N(K_n) \cong K_n$ while $LN(K_{1,n}) \cong L(K_1 \cup K_n) \cong L(K_n)$. When $n \geq 4$, K_n has a pair of nonadjacent edges. Thus, $L(K_n)$ and, hence, $LN(K_{1,n})$ is not complete while $NL(K_{1,n})$ is complete. □

Lemma 4. *Let G be a connected bipartite graph with $\Delta(G) \geq 3$ and at least two vertices in each bipartition. Then $NL(G) \ncong LN(G)$.*

Proof: By Observation 2, $N(G)$ is disconnected and by Observation 3 each component of $N(G)$ has at least one edge. Thus, $LN(G)$ is disconnected. Since G is connected, so is $L(G)$ by Observation 1(a). Furthermore, $L(G)$ contains a triangle since G has a vertex of degree at least three, meaning $NL(G)$ is connected, again by Observation 2. □

Lemmas 2, 3, and 4 combine to show that the only connected bipartite graphs for which $NL(G) \cong LN(G)$ are exactly those specified in the theorem. It remains to show that the only nonbipartite graphs for which $NL(G) \cong LN(G)$ are the odd cycles. This class of connected graphs is partitioned into two subclasses: those that have induced C_4's, and those that do not.

We first consider the non-bipartite graphs which do not contain induced C_4's. Two results needed to establish a basis for the induction argument used in the proof of Lemma 7 are now given.

Lemma 5. *Let G_1 be the graph obtained by including a new vertex u with edges to at least one and at most $n-1$ of the vertices of a K_n, $n \geq 3$. Then $|E[N(G_1)]| > |E(G_1)|$.*

Proof: Note that $N(G_1)$ includes all edges of the K_n. Suppose u is adjacent to vertex v in G_1. Then, in $N(G_1)$, u is adjacent to all of the vertices of the K_n except possibly v, and it is adjacent to v if, in G_1, it is adjacent to at least two vertices of the K_n. The result follows. □

Lemma 6. *Let G_1 be the graph obtained from a C_n, n odd and $n \geq 5$, by including a new vertex u with edges to at least one of the vertices of the C_n. Then $|E[N(G_1)]| > |E(G_1)|$.*

Proof: Let the vertices of the cycle be labeled in order by $v_0, v_1, \ldots, v_{n-1}$. All subscript arithmetic is to be taken modulo n. Observe that $N(G_1)$ contains the n-cycle $\langle v_0, v_2, v_4, \ldots, v_{n-1}, v_1, v_3, \ldots, v_{n-2}, v_0\rangle$. If u is adjacent to all vertices of the cycle, $N(G_1)$ includes all the edges of G_1 as well, and we are done. Otherwise, let $v_i, v_{i+1}, \ldots, v_{i+k}$ be a maximal sequence of consecutive vertices to which u is adjacent. If $k \geq 1$, u is adjacent to all of these vertices in $N(G_1)$ and, in addition, is adjacent to v_{i+k+1}. If $k = 0$, u is adjacent to v_{i+1}. Thus, if any such sequence has $k \geq 1$, $N(G_1)$ will have more edges than G_1. On the other hand, if all k values are zero, then, for at least one v_i to which u is adjacent, neither v_{i-1} nor v_{i-2} is adjacent to u. But, in $N(G_1)$, v_{i-1} is adjacent to u and represents an edge not counted elsewhere, again giving the result. □

We now are in a position to show that $NL(G) \not\cong LN(G)$ when G has no induced C_4's.

Lemma 7. *Let G be connected, not bipartite, not an odd cycle, and have no induced C_4's. Then $NL(G) \not\cong LN(G)$.*

Proof: We may assume G is not complete, otherwise the result follows directly from Observation 5. The proof is by induction starting with an induced subgraph G_1 of G; adding vertices one at a time to create a sequence of induced subgraphs $G_2, G_3, \ldots, G_k \cong G$; and showing that $|E[N(G_i)]| > |E(G_i)|$, for $1 \leq i \leq k$. Since G is not bipartite, it contains either a maximal complete subgraph K_n, $n \geq 3$, in which case G_1 is taken as the graph of Lemma 5, or an induced odd cycle of length at least five and then G_1 is taken to be the graph of Lemma 6. In either case, $|E[N(G_1)]| > |E(G_1)|$. Suppose G_i, $i < k$, has been obtained and $|E[N(G_i)]| > |E(G_i)|$. Since G is connected, there is a vertex $u \in V(G) - V(G_i)$ which has $r \geq 1$ neighbors in G_i. Let G_{i+1} be the subgraph of G induced by $V(G_i) \cup \{u\}$. Note that $N(G_i)$ is a subgraph of $N(G_{i+1})$. Consider any edge ux in G_{i+1}. If there is also an edge uy with x and y adjacent in G_i, then both ux and uy are edges in $N(G_{i+1})$. If no such edge uy exists, then uz is an edge of $N(G_{i+1})$ for any z in G_i which is adjacent to x. Moreover, $\langle u, v, z\rangle$ is not a path in G_{i+1} for any $v \neq x$ since then G_{i+1}, and hence G, would have an induced C_4. Thus $N(G_{i+1})$ has at least r edges incident to u. It follows from this and the inductive hypothesis that $|E[N(G_{i+1})]| \geq |E[N(G_i)]| + r > |E(G_i)| + r = |E(G_{i+1})|$. Since eventually this process creates the graph G, the result follows from Observation 4. □

The rest of the proof of the theorem for connected graphs deals with the case when G is not bipartite and contains an induced 4-cycle $\langle a, b, c, d, a \rangle$. Let $X = \{a, b, c, d\}$ and E_X be the set of edges in G with exactly one end vertex in X. The following two structural lemmas, and the corollary to the first of them, facilitate the remainder of the proof. The vertices of $L(G)$ and $NL(G)$ often will be indicated by the two symbol string defining the end vertices of the corresponding edge in G. Thus, the 4-cycle becomes the cycle in $L(G)$ with vertices ab, bc, cd, and da. The degree of vertex v is denoted d_v.

Lemma 8. *Let e and f be any two edges in G. Then vertices e and f are not adjacent in $NL(G)$ if and only if, in G, the end vertices of e and f induce either $2K_2$ or P_3 where the common vertex of the P_3 has degree two in G.*

Proof: Assume e and f are not adjacent vertices in $NL(G)$ and, in G, the end vertices of the edges e and f do not induce $2K_2$ and, if they induce P_3, the common vertex has degree at least three. Suppose e and f are independent edges in G. Then there must be an edge h joining them, for otherwise their end vertices would induce $2K_2$. In this case $\langle e, h, f \rangle$ is a path in $L(G)$, and e and f are adjacent in $NL(G)$, a contradiction. Therefore, we must have $e = xy$ and $f = yz$, where either x and z are adjacent in G, or y has degree at least three. In either case, we again have e and f adjacent in $NL(G)$. Now assume the end vertices of e and f induce $2K_2$ in G. Then e and f have no common neighbor in $L(G)$ and, hence, are not adjacent in $NL(G)$. Finally, suppose e and f induce P_3 in G, that is, $e = xy$ and $f = yz$, where x and z are not adjacent and y has degree two in G. Then e and f have no common neighbor in $L(G)$ and, hence, are not adjacent in $NL(G)$. □

Corollary. *In $NL(G)$, vertex*

(1) *ab is adjacent to cd,*

(2) *bc is adjacent to da,*

(3) *ab is adjacent to bc if and only if $d_b > 2$,*

(4) *bc is adjacent to cd if and only if $d_c > 2$,*

(5) *cd is adjacent to da if and only if $d_d > 2$, and*

(6) *da is adjacent to ab if and only if $d_a > 2$.*

Lemma 9. *Let $e \in E_X$. Then, in $NL(G)$, vertex e is adjacent to all of vertices ab, bc, cd, and da.*

Proof: Without loss of generality, assume $e = xa$. Then, in $L(G)$, e forms a triangle with ab and da and has a length two path to each of bc and cd. □

The proof of the theorem, in the connected case, will be completed by the next two lemmas. The first assumes E_X has edges e and f which correspond to nonadjacent vertices in $NL(G)$. The second deals with the case when no such pair exists.

Lemma 10. *Let G be a connected nonbipartite graph having edges e and f in E_X corresponding to nonadjacent vertices in $NL(G)$. Then $NL(G) \ncong LN(G)$.*

Proof: From Lemma 8, the end vertices of e and f induce, in G, either $2K_2$ or P_3 where, in the latter case, the common vertex has degree two in G. By Lemma 9, e and f are adjacent to all of ab, bc, cd, and da in $NL(G)$.

Suppose first that the end vertices of e and f induce $2K_2$. Then, e and f must meet the 4-cycle in diagonally opposite corners, or otherwise they would be adjacent in $NL(G)$. Without loss of generality, assume $e = aa'$ and $f = cc'$. If there is a third edge in E_X, G must include one of the subgraphs shown in Figure 2.

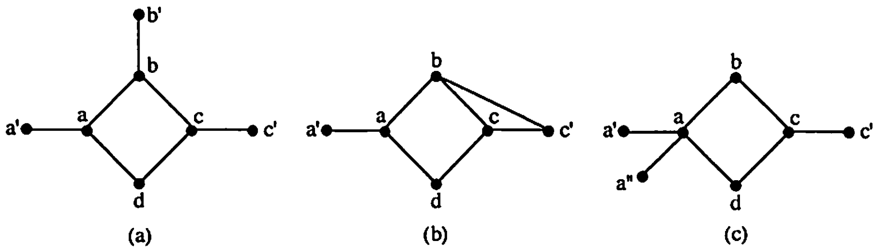


Figure 2

In the subgraphs of Figure 2(a) and 2(b), vertices aa', ab, bb' (bc' for 2(b)), bc, and cc' induce a $K_5 - e$ in $NL(G)$ where the missing edge is between aa' and cc'. Thus, NL(G) is not a line graph and therefore $NL(G) \ncong LN(G)$. If neither 2(a) nor 2(b) applies, then $d_b = d_d = 2$ and 2(c) yields, in $NL(G)$, a $K_5 - e$ on vertices aa', aa'', ab, da, and bc where the edge between ab and bc is guaranteed to be missing by the corollary.

If aa' and cc' are the only edges of E_X, we know that at least one of a' or c' has another incident edge since otherwise the graph would be bipartite. Without loss of generality, assume there is additional edge ga'. First, suppose g is the only neighbor of a' other than a and let H be the subgraph of G induced by the edges ab, bc, cd, da, aa', and cc'. Then, in $NL(G)$, ga' is adjacent only to ab and da of the edges of H, and, furthermore, H is transformed to the line graph of K_4. If $NL(G) \cong LN(G)$, $N(G)$ must contain a structure which includes a K_4 and which, when the line graph is taken, generates the subgraph H along with a vertex equivalent to ga' and its two edges to H. This can be done only if some vertex x is adjacent to at least one vertex of the K_4 in $N(G)$. But then x is adjacent to at least three of the K_4 vertices in $LN(G)$ which means it is impossible to create in $LN(G)$ the equivalent of vertex ga' which is joined to only two of the vertices.

Thus, if $NL(G) \cong LN(G)$, we may assume vertex a' has degree at least three in G. Since G is not bipartite, one of the subgraphs of Figure 3 must appear, where, in 3(b), $h \neq c'$.

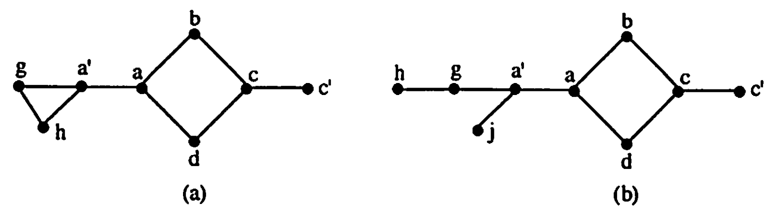


Figure 3

In either case, the subgraph of $NL(G)$ induced by the vertices gh, aa', ab, and bc is $K_{1,3}$ with the central vertex being aa'. In this subgraph, gh is not adjacent to ab or bc since, in G, there is no edge between either g or h and any of a, b, and c. Vertices ab and bc are not adjacent by the corollary and the fact that $d_b = 2$.

The only remaining situation is if the end vertices of e and f induce a P_3 in G and the common vertex has degree two. Then, G must contain one of the subgraphs shown in Figure 4 where the edge with end vertex h must appear since the graph is not bipartite and, hence, must contain additional structure.

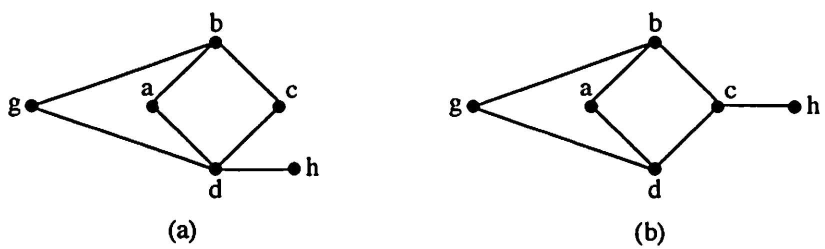


Figure 4

In 4(a), the vertices dh, ab, bc, gb, and gd in $NL(G)$ induce $K_5 - e$ with gb and gd not adjacent by assumption. In 4(b), the vertices bc, cd, ch, gb, and gd perform the same function, again with gb and gd not adjacent. All cases have been examined and show that $NL(G) \ncong LN(G)$. □

In all remaining cases, any two edges of G which are in E_X become adjacent vertices in $NL(G)$, implying all edges of E_X induce a complete subgraph of $NL(G)$ and each such edge is adjacent to every one of the vertices *ab*, *bc*, *cd*, and *da*. The next lemma completes the proof of the theorem for connected graphs.

Lemma 11. *Let G be a connected nonbipartite graph having every pair of edges e and f in E_X correspond to adjacent vertices in $NL(G)$. Then $NL(G) \not\cong LN(G)$.*

Proof: There are several cases depending on the size of E_X and the way edges of E_X are incident to the vertices of the 4-cycle.

Case 1: $|E_X| = 1$. Without loss of generality, assume the single edge in E_X is aa'. Then $d_b = d_c = d_d = 2$ and $d_a = 3$. Since G is not bipartite, there must be more to the graph and thus G must have one of the subgraphs depicted in Figure 5.

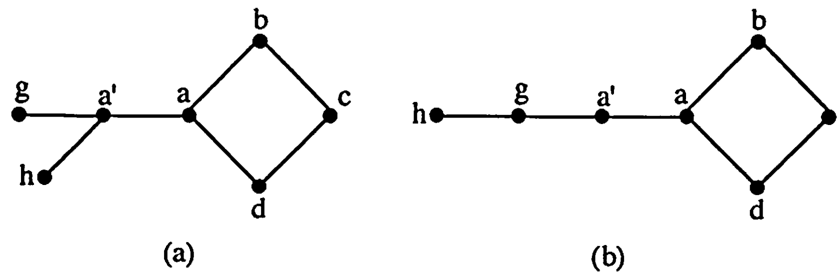


Figure 5

For 5(a), $NL(G)$ contains a $K_{1,3}$ induced by vertices ga', aa', *bc*, and *cd* while, for 5(b), $NL(G)$ contains a $K_{1,3}$ induced by *hg*, aa', *bc*, and *cd*.

Case 2: $|E_X| = 2$. Either one of the subgraphs shown in Figure 6 or the subgraph of Figure 2(c) without edge cc' must appear in G. For the subgraphs in Figure 6, two vertices of the cycle must have degree two, and, in (a), it is possible that $a' = b'$.

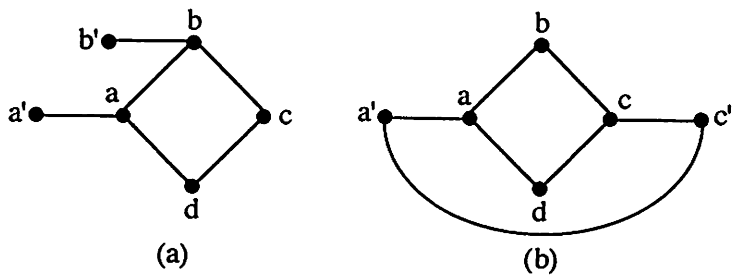


Figure 6

In the case of Figure 2(c), without edge cc', the same argument as in the proof of Lemma 10 shows the presence of a $K_5 - \dot{e}$ in $NL(G)$. The subgraph of 6(a) yields in $NL(G)$ an induced $K_5 - e$ on vertices aa', ab, bc, bb', and cd (even if $a' = b'$); and 6(b) also does on vertices aa', ab, da, bc, and cc'.

Case 3: $|E_X| \geq 3$ with at least one vertex, assumed without loss of generality to be a, of the 4-cycle having degree two in G. By the corollary, ab and da are not adjacent in $NL(G)$. Thus the vertices of $NL(G)$ associated with any three of the edges of E_X along with ab and da induce a $K_5 - e$.

Case 4: Every vertex of X has degree at least three in G. Define the *basic structure* to be the subgraph of G induced by the edges of the 4-cycle along with those in E_X. Note that the corresponding vertices in $NL(G)$ induce a complete subgraph. Furthermore, no single edge can be adjacent in G to all the edges of the basic structure, nor can any pair of adjacent edges between them be adjacent to all those edges if the 4-cycle is to have no chords. Now let Z be the set of vertices of $NL(G)$ which form a maximal complete subgraph and which contains the vertices corresponding to the edges of the basic structure. In $L(G)$, partition Z into Z_1, the vertices having at most one neighbor in Z; Z_2, the vertices having exactly two neighbors in Z; and Z_3, the vertices having at least three neighbors in Z. Notice that Z_3 includes all the vertices corresponding to the edges of the basic structure. We will show that either $NL(G)$ contains a $K_5 - e$ or it is complete with more than three vertices, a contradiction in both cases. The following discussion concerns the situation in $L(G)$.

Suppose $z \in Z_3$. We may assume z has no neighbors outside of Z, for such a neighbor would be adjacent in $NL(G)$ to at least three vertices of Z, but not to all vertices of Z, and a $K_5 - e$ would result.

Next suppose $z \in Z_1$. Then z must have a path of length 2 to every vertex of Z_3. The common neighbor y must be in Z since vertices in Z_3 are adjacent only to vertices in Z by the previous paragraph. Thus y must dominate all of Z_3 which means an edge of G is adjacent to all edges of the basic structure, which we have seen is not the case. It follows that Z_1 is empty.

Finally consider $z \in Z_2$ where z is adjacent to z_1 and z_2 in Z. It must be that z_1 and z_2 also are adjacent in order for there to be a length two path between z and each of them. Also, z must have a length two path to all vertices of Z_3. The central vertices of all such paths must be z_1 and z_2, implying these two vertices dominate all the vertices of the basic structure, that is, two adjacent edges are adjacent in G to all the edges of the basic structure, which is not possible. Therefore Z_2 also is empty.

We have shown $L(G)$ consists only of the vertices of the basic structure implying the impossible situation, by Observation 5, that $NL(G)$ is complete on more than three vertices. This completes the proof of the lemma. □

For connected graphs G satisfying any of Lemmas 2, 3, 4, 7, 10, or 11, $NL(G) \ncong LN(G)$. Further, the collection of such graphs is exactly the collection of all connected graphs except for those identified in the theorem. Therefore, the theorem holds for connected graphs. In the next section, we establish the theorem for disconnected graphs.

4 Disconnected Graphs

As with connected graphs in Section 3, we establish a sequence of lemmas which show that a graph G with any component other than one of K_1, $K_{1,3}$, or C_n where $n \geq 3$ and $n \neq 4$ results in $NL(G) \ncong LN(G)$. We may assume that G has no component H isomorphic to any of those of the theorem statement since $NL(H) \cong LN(H)$ for each such component. Furthermore, induced C_4's can only appear in bipartite components. To see this, the arguments of Lemmas 8 through 11 can be applied to any non bipartite component H having an induced C_4. In all but one case, a forbidden subgraph appears in $NL(H)$. However, in one instance of the proof of Lemma 11, the condition $NL(H) \cong K_n$ with $n \geq 4$ represents a contradiction for connected graphs. In the disconnected case, it may be possible for $LN(G)$ to possess a different component which is isomorphic to K_n. If so, $N(G)$ must have a component isomorphic to $K_{1,n}$, which implies G must have a bipartite component H' with a vertex x so that $\Delta(H') \geq d_x = n \geq 4$ and includes Figure 7(a) as an induced subgraph. No pair of $x_1, x_2, \ldots, x_n$, the vertices at distance two from x, can have common neighbors. In $N(H')$, x corresponds to the central vertex and $x_1, x_2, \ldots, x_n$ are the leaves of the $K_{1,n}$. Then, $L(H')$ must contain the induced subgraph of Figure 7(b) which, in turn, produces the forbidden subgraph $K_5 - e$ in $NL(H')$.

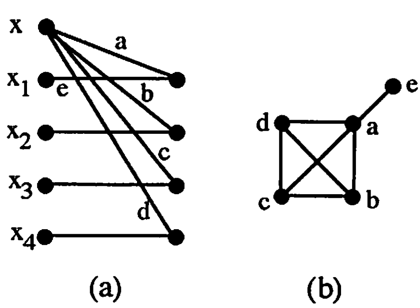


Figure 7

Thus, the only possibility for induced C_4's is in bipartite components of G.

In view of the preceding comments, the following table lists the possible types of components and properties of the corresponding NL and LN graphs with which we must still be concerned.

Class	Component H	$NL(H)$	$LN(H)$	Components in $NL(H)$	Components in $LN(H)$
A	P_2	P_1	$\emptyset$	1	0
B	P_3	$2P_1$	P_1	2	1
C	$P_n, n \geq 4$	$P_{\lceil\frac{n-1}{2}\rceil} \cup P_{\lfloor\frac{n-1}{2}\rfloor}$	$P_{\lceil\frac{n-2}{2}\rceil} \cup P_{\lfloor\frac{n-2}{2}\rfloor}$	2	2
D	C_4	$2P_2$	$2P_1$	2	2
E	$K_{1,n}, n \geq 4$	K_n	$L(K_n)$	1	1
F	Bipartite, not $K_{1,n}$, $\Delta \geq 3$	Not bipartite	At least one component not bipartite	1	2
G	Not bipartite, no induced C_4, $\Delta \geq 3$	Not bipartite	Not bipartite	1	1

Notice that components of Classes A and B each produce one more component in NL than in LN, and this extra component is a P_1. The only components which can make up this difference are those in Class F. Analogously, the components of Class F each produce one more component in LN than in NL. The only components which can make up this difference are those in Classes A and B. Therefore, the number of Class A and B components must be the same as the number of Class F components. Our first result assumes no component isomorphic to P_2 or P_3, that is, no components in classes A or B, and therefore none in Class F.

Lemma 12. *If all components of G are in classes C, D, E, and G, then $NL(G) \not\cong LN(G)$.*

Proof: Notice that, for every member H of classes E or G, both $NL(H)$ and $LN(H)$ contain a triangle while no component in classes C or D produces a triangle. It follows that for $NL(G) \cong LN(G)$ we must have $NL(E \cup G) \cong LN(E \cup G)$ and, thus, $NL(C \cup D) \cong LN(C \cup D)$.

In the proof of Lemma 7 we saw $|V[LN(H)]| > |V[NL(H)]|$ for any component H in Class G. It is easy to see that the inequality also applies to members of E. Thus, $NL(E \cup G) \not\cong LN(E \cup G)$ and implies that we can have no components from classes E or G.

Therefore, all components are C_4's or P_n's with $n \geq 4$, and $NL(C \cup D) \cong LN(C \cup D)$ which are both unions of paths. Let m be the largest index such that there is a component P_m, and suppose $m \geq 6$. Then the largest component in $NL(C \cup D)$ is $P_{\lceil\frac{m-1}{2}\rceil}$, where $\lceil\frac{m-1}{2}\rceil \geq 3$, and must be matched in $LN(C \cup D)$. This is impossible when m is even since then the largest component in $LN(C \cup D)$ is $P_{\frac{m-2}{2}}$. When m is odd, there are two $P_{\frac{m-1}{2}}$ components of $NL(C \cup D)$ for each original P_m in G, but only one of these can be matched in $LN(C \cup D)$. It follows that the only components we can have are isomorphic to P_4, P_5, or C_4. An easy check shows no combination of these is allowable. This final contradiction proves the lemma. □

We now consider the P_2 and P_3 components of Classes A and B, respectively. The previous result shows that if there is any component other than those described in the theorem, some of them must be from Class A or B, with an equal number from Class F. Our next result shows the possibilities in Class F are limited to exactly six graphs. Lemma 14 will finally establish the theorem by showing that none of these six is allowable.

First, recall that a P_2 or P_3 component produces one more component, a P_1, in $NL(G)$ than in $LN(G)$ and must be paired with a Class F component which generates exactly one P_1 in $LN(G)$, that is, exactly one P_2 in $N(G)$. Furthermore, if $NL(G) \cong LN(G)$ and H is a component of G in Class F, then H must produce a P_2 in $N(G)$.

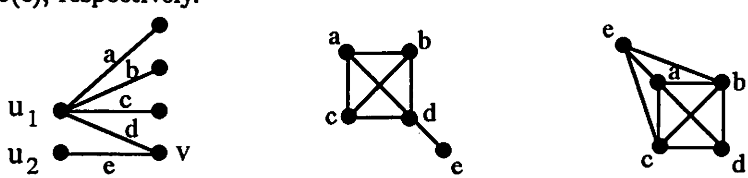
Lemma 13. *Let G be a graph such that $NL(G) \cong LN(G)$ and H be a component in Class F with partite sets $U = \{u_1, u_2, \ldots, u_m\}$ and V, where $m \geq 2$ and $|V| \geq 2$. Without loss of generality, assume $N(H)$ has a component isomorphic to P_2 on vertices u_1 and u_2. Then*

(a) $m = 2$,

(b) *the degree of at least one vertex of V is two, and*

(c) $\Delta(H) = 3$.

Proof: Let $v \in V$ be adjacent to u_1 and/or u_2. Then V has no other neighbors in U, for otherwise u_1 (or u_2) would have a neighbor in $N(H)$ other than u_2 (or u_1). Since H is connected, $m = 2$. For u_1 and u_2 to be adjacent in $N(H)$, they must have a common neighbor, that is, some vertex in V must have degree two. Thus, (a) and (b) are established. Since H is in Class F, u_1 or u_2 must have degree at least three. If, say, vertex u_1 has degree four or more, there is, in H, the subgraph shown in Figure 8(a), with corresponding subgraphs in $L(H)$ and $NL(H)$ shown in Figures 8(b) and 8(c), respectively.



(a) (b) (c)

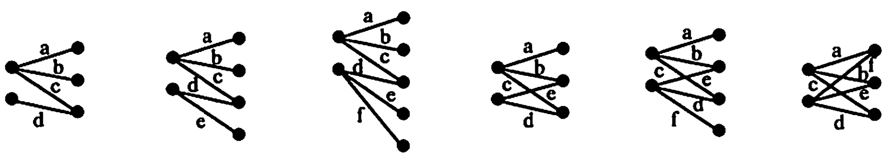
Figure 8

There is no path of length two between d and e in $L(H)$ since that would require either an edge between u_1 and u_2, which is impossible because H is bipartite, or another edge incident to v, which is also impossible. The vertices in Figure 8(c) then induce a forbidden $K_5 - e$ in $NL(H)$ with de being the missing edge. This contradiction proves (c). □

Observe that, for H in Class F, if the component in $N(H)$ induced by the vertices of V has an edge clique cover number of k, then at least k vertices must exist in U. This follows because the end vertices in V of all edges of H which are incident to the same vertex of U form a clique in this component, and all edges of the component are created in this way. Since k cliques are required to cover all the edges of the component, U must contain at least k vertices. Recall that every component in Class F has $|U| = 2$.

Lemma 14. *Let G be a graph such that $NL(G) \cong LN(G)$. Then no component in Class F can produce a P_2 in $N(G)$.*

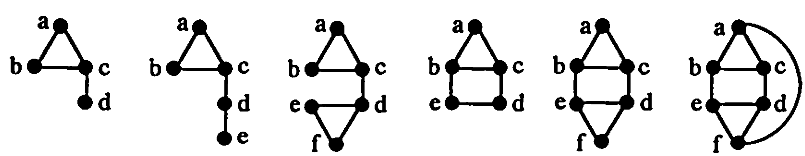
Proof: From Lemma 13, the only possible Class F graphs are shown in Figure 9.



(a) (b) (c) (d) (e) (f)

Figure 9

The corresponding line graphs are given in Figure 10.



(a) (b) (c) (d) (e) (f)

Figure 10

The corresponding NL graphs are shown in Figure 11.

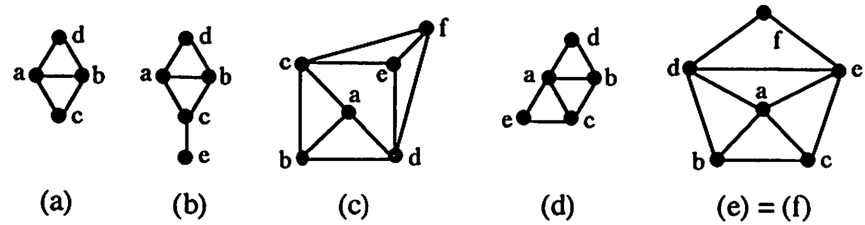


Figure 11

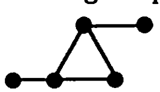
If a graph of Figure 11 is a component of $NL(G)$, there must be an isomorphic component in $LN(G)$. We show for each that this is impossible. We take them slightly out of order.

(b) For the graph of Figure 11(b) to be a component in $LN(G)$, a component isomorphic to the one in Figure 10(b) must appear in $N(G)$. This component can arise in $N(G)$ as either the connected neighborhood graph of a nonbipartite component H' of G or as one of multiple components in the neighborhood graph of a bipartite component H' of G. If H' is nonbipartite, then H' is a five vertex graph in Class G. Examination of all five vertex graphs (see, e.g., the table in [15]) reveals none having this graph as its neighborhood graph. Thus H' is bipartite and therefore in Class F. By the comment preceding the lemma, U must contain at least three vertices. Since $NL(G) \cong LN(G)$, H' must produce a P_2 in $N(H')$ so that Lemma 13 applies and we have a contradiction.

(a) We must have Figure 10(a) as a component in $N(G)$. No four vertex graph has this as its neighborhood graph. However, it can be generated from the graph of Figure 9(b). This graph, though, is not allowed as shown in the proof to (b) above.

(c) We have the forbidden subgraph induced by vertices a, b, c, d, and e.

(d) $N(G)$ must contain the following component which can not arise from



any five vertex graph. The only members of Class F which can produce it have at least three vertices in U, again, contradicting Lemma 13.

(e) = (f) This graph is $L(K_4)$, so we need a K_4 component in $N(G)$. Such a component can not arise from any graph in Class F for which $|U| = 2$ and $\Delta = 3$. However, a K_4 is the neighborhood graph of either another K_4 or a $K_4 - e$. Suppose K_4 is a component of G. Now, $NL(K_4) = K_6$ is the line graph of $K_{1,6}$. This implies $N(G)$ has a component isomorphic to $K_{1,6}$. Such a graph can arise only from a member of Class F with at least six vertices in U, and contradicts Lemma 13. Since $NL(K_4 - e)$ is K_5, a similar argument eliminates this case.

All possibilities have been considered and the lemma is proved. □

The proof of the characterization theorem is now complete.

5 Acknowledgement

The authors wish to express their appreciation to the referee for the careful reading given this manuscript and the suggestions which have improved its clarity.

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