The possible number of cycles in cycle systems

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ABSTRACT: For $v \geq 3$, v odd, it is shown that there exists a decomposition of K_v into b cycles whose edges partition the edge set of K_v if and only if

$$\left\lfloor \frac{v-1}{2} \right\rfloor \leq b \leq \left\lfloor \frac{v(v-1)}{6} \right\rfloor .$$

For even v, $v \geq 4$, a similar result is obtained for K_v minus a 1-factor.

In recent times much work has been done on decompositions of graphs, especially complete graphs, into cycles. For example see the survey [3] and the references therein. Alspach [1] conjectured in 1981 that the obvious necessary conditions for the existence of a decomposition of the complete graph into cycles of lengths $m_1, m_2, \ldots, m_t$ are sufficient and to date the conjecture is still an open problem.

According to Jean Doyen (Auburn Combinatorics Conference, 1996), Erdös conjectured that if $v \geq 5$, and if $2v-4 \leq b \leq \binom{v}{2}$, with b not equal to $\binom{v}{2} - 1$ or $\binom{v}{2} - 3$, then there exists a linear space on v points with b lines. (According to [4] and [5], this question was asked by Doyen himself!)

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Here we consider the analogous problem for cycle decompositions of complete graphs. That is, we determine the possible number of cycles in cycle decompositions of K_n for n odd (or of $K_n \setminus F$, F a 1-factor, for n even).

In the case of cycle decompositions of K_n or $K_n \setminus F$, the least value of b arises with a hamilton decomposition of K_v or $K_v \setminus F$, and the greatest value of b arises when all cycles are of length 3, or just one cycle has length 4 and the rest have length 3.

We start with some definitions. We denote by $(v_1, v_2, \ldots, v_m)$ the m-cycle with vertices $v_1, v_2, \ldots, v_m$ and edges $v_1v_2, v_2v_3, \ldots, v_mv_1$. If H_1 and H_2 are edge-disjoint graphs, we denote by $H_1 + H_2$ the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. For convenience, define G_n to be the complete graph K_n if n is odd, and $K_n \setminus F$ (F a 1-factor) if n is even. We shall say that $B(n)$ is the set of all t such that the edge set of G_n can be partitioned into a set of t cycles. Further, let $e_n = |E(G_n)|$, so $e_n = \dfrac{n(n-1)}{2}$ if n is odd and $e_n = \dfrac{n(n-2)}{2}$ if n is even.

Now with this notation, our aim is to prove the following result.

MAIN THEOREM *Let t be an integer. Then $t \in B(n)$ if and only if t is in the interval* $\left[\dfrac{e_n}{n}, \dfrac{e_n}{3}\right]$.

We start with a specific hamilton decomposition of G_n. If $n = 2r+1$, let $V(G_n) = \{\infty, 1, 2, \ldots, 2r\}$, and if $n = 2r+2$, let $V(G_n) = \{\infty, \infty', 1, 2, \ldots, 2r\}$. Then let $\mathcal{H}_n$ be the hamilton cycle decomposition of G_n given by

$$\mathcal{H}_n = \{H_i \mid i \in \mathbb{Z}_r\}$$

where for each $i \in \mathbb{Z}_r$,

(i) $H_i = (\infty, 1+i, 2r+i, 2+i, 2r-1+i, \ldots, r+i, r+1+i)$ if $n = 2r+1$;

(ii) $H_i = (\infty, 1+i, 2r+i, 2+i, 2r-1+i, \ldots, \frac{r+1}{2}+i, \infty', \frac{3r+1}{2}+i, \frac{r+3}{2}+i, \ldots, r+i, r+1+i)$ if $n = 2r+2$ and r is odd;

(iii) $H_i = (\infty, 1+i, 2r+i, 2+i, 2r-1+i, \ldots, \frac{3r+2}{2}+i, \infty', \frac{r+2}{2}+i, \frac{3r}{2}+i, \ldots, r+i, r+1+i)$ if $n = 2r+2$ and r is even.

LEMMA 1 *For all* $i \in \mathbb{Z}_r$,

(i) *if* $n \geq 5$, $H_i + H_{i+1} = C_3 + C_{n-2} + C_{n-1}$; *and*

(ii) *if* $n \geq 7$, $H_i + H_{i+1} = C_3 + C_3 + C_{n-4} + C_{n-2}$.

Proof: (i) In H_i we replace the path $\infty, 1+i, 2r+i, 2+i$ with the path $\infty, 2+i$ to obtain an $(n-2)$-cycle, g_1 say. In H_{i+1} we replace the path $\infty, 2+i, 1+i$ with the path $\infty, 1+i$ to obtain an $(n-1)$-cycle, g_2 say. The unused edges form the 3-cycle $(2r+i, 1+i, 2+i)$.

(ii) In g_1 we replace the path $\infty, r+1+i, r+i, r+2+i$ with the path $\infty, r+2+i$ to obtain an $(n-4)$-cycle. In g_2 we replace the path $\infty, r+2+i, r+1+i$ with the path $\infty, r+1+i$ to obtain an $(n-2)$-cycle. The unused edges form two 3-cycles, $(2r+i, 1+i, 2+i)$ and $(r+i, r+1+i, r+2+i)$. □

COROLLARY 2

$$\text{For } n \geq 3, \quad \frac{e_n}{n}, \frac{e_n}{n}+1, \frac{e_n}{n}+2, \ldots, \frac{2e_n}{n}-1 \in B(n).$$

Proof: The existence of the hamilton cycle decomposition $\mathcal{H}_n$ tells us that $e_n/n \in B(n)$. We obtain e_n/n, $e_n/n+1$, $e_n/n+2$, $\ldots$, $2e_n/n-1 \in B(n)$ by applying Lemma 1 to H_0 and H_1, and then subsequently to H_2 and H_3, H_4 and H_5 and so on. When $|\mathcal{H}_n|$ is even we also have $2e_n/n \in B(n)$. □

LEMMA 3 *If* $t \in B(n)$ *then* $t + \frac{e_n}{n} + 1 \in B(n+2)$.

Proof: Let S be a set of cycles which partitions the edge set of G_n and suppose that $V(G_n) = \{1, 2, \ldots, n\}$ and $V(G_{n+2}) = V(G_n) \cup \{n+1, n+2\}$.

If n is even, then

$$S \cup \{(1, n+1, 2, n+2), (3, n+1, 4, n+2), \ldots, (n-1, n+1, n, n+2)\}$$

is a set of $t + n/2$ (that is, $t + e_n/n + 1$) cycles which partitions the edge set of G_{n+2}.

If n is odd, then

$$\begin{aligned} S \quad \cup \quad & \{(1, n+1, 2, n+2), (3, n+1, 4, n+2), \ldots, \\ & (n-2, n+1, n-1, n+2), (n, n+1, n+2)\} \end{aligned}$$

is a set of $t + (n+1)/2$ (that is, $t + e_n/n + 1$) cycles which partitions the edge set of G_{n+2}. □

The following proposition is a corollary of a result in [2]

PROPOSITION 4 *If $3x + 4y = e_n$ with x, $y \geq 0$, then there is a decomposition of G_n into x 3-cycles and y 4-cycles.*

COROLLARY 5 *For all integers t in the range $\left\lceil \frac{e_n}{4} \right\rceil \leq t \leq \left\lfloor \frac{e_n}{3} \right\rfloor$, $t \in B(n)$.*

Proof: It is straightforward to check that there exist non-negative integers x and y satisfying $3x + 4y = e_n$ for any $n \geq 3$. Let t_{min} and t_{max} be respectively the minimum and maximum number of cycles in a set of 3-cycles and 4-cycles which partition $E(G_n)$.

If $3x+4y = e_n$ then $3(x+4)+4(y-3) = e_n$, and if $y < 3$ and $3x+4y = e_n$ then $x+y = t_{\text{max}}$. Hence for all t in the range $t_{\text{min}} \leq t \leq t_{\text{max}}$, $t \in B(n)$.

We shall now show that $t_{\text{min}} \leq \left\lceil \frac{e_n}{4} \right\rceil$ and $t_{\text{max}} \geq \left\lfloor \frac{e_n}{3} \right\rfloor$.

Suppose, on the contrary, that $t_{\text{min}} > \left\lceil \frac{e_n}{4} \right\rceil$, $x+y = t_{\text{min}}$ and $3x+4y = e_n$. Then

$$t_{\text{min}} = \frac{e_n - 3x}{4} + x = \frac{e_n}{4} + \frac{x}{4}$$

and so $\frac{e_n}{4} + \frac{x}{4} > \left\lceil \frac{e_n}{4} \right\rceil$. Hence $(e_n+x)/4 \geq e_n/4+1$ and so $x \geq 4$. This is a contradiction, since $3(x-4)+4(y+3) = e_n$ and $x-4+y+3 = t_{\text{min}}-1$. Hence $t_{\text{min}} \leq \left\lceil \frac{e_n}{4} \right\rceil$. Similarly, if $t_{\text{max}} < \lfloor e_n/3 \rfloor$, $x+y = t_{\text{max}}$ and $3x + 4y = e_n$ then $y \geq 3$ and again we have a contradiction. Hence $t_{\text{max}} \geq \lfloor e_n/3 \rfloor$. □

We can now prove our main result.

MAIN THEOREM *Let t be an integer. Then $t \in B(n)$ if and only if t is in the interval $\left[\frac{e_n}{n}, \frac{e_n}{3}\right]$.*

Proof: Clearly, if $t < e_n/n$, then we are forced to have at least one cycle containing more than n edges (which is impossible), and if $t > e_n/3$, then we are forced to have at least one cycle containing fewer than three edges (which again is impossible).

Case 1: n is odd The proof is by induction on n. Clearly $1 \in B(3)$ and so the result is true for $n = 3$. Now assume it is true for $n = k - 2 \geq 3$ and consider the case $n = k$.

By Corollary 2,

$$\frac{k-1}{2}, \frac{k+1}{2}, \ldots, k-2 \in B(k). \tag{1}$$

By Lemma 3 and the induction hypothesis for $B(k-2)$, we have

$$\left\{\frac{k-3}{2}, \frac{k-1}{2}, \ldots, \left\lfloor \frac{(k-2)(k-3)}{6} \right\rfloor\right\} + \frac{k-1}{2} \subseteq B(k),$$

that is,

$$k-2,\ k-1,\ \ldots,\ \left\lfloor \frac{k^2-2k+3}{6} \right\rfloor \in B(k). \tag{2}$$

By Corollary 5,

$$\left\lceil \frac{k(k-1)}{8} \right\rceil,\ \left\lceil \frac{k(k-1)}{8} \right\rceil + 1,\ \ldots,\ \left\lfloor \frac{k(k-1)}{6} \right\rfloor \in B(k). \tag{3}$$

Now,

$$\begin{aligned} \left\lfloor \frac{k^2-2k+3}{6} \right\rfloor - \left(\left\lceil \frac{k(k-1)}{8} \right\rceil - 1 \right) &\geq \frac{k^2-2k+3}{6} - \frac{k^2-k}{8} \\ &= \frac{k^2-5k+12}{24} \\ &= \frac{(k-2)(k-3)+6}{24} \\ &\geq 0 \end{aligned}$$

and so the result follows from (1), (2) and (3).

Case 2: n is even Again, the proof is by induction on n. Clearly $1 \in B(4)$ and so the result is true for $n = 4$. Now assume it is true for $n = k-2 \geq 4$ and consider the case $n = k$. By Corollary 2,

$$\frac{k-2}{2},\ \frac{k}{2},\ \ldots,\ k-3 \in B(k). \tag{4}$$

By Lemma 3 and the induction hypothesis we have

$$\left\{ \frac{k-4}{2},\ \frac{k-2}{2},\ \ldots,\ \left\lfloor \frac{(k-2)(k-4)}{6} \right\rfloor \right\} + \frac{k-2}{2} \subseteq B(k).$$

That is,

$$k-3,\ k-2,\ \ldots,\ \left\lfloor \frac{k^2-3k+2}{6} \right\rfloor \in B(k). \tag{5}$$

By Corollary 5,

$$\left\lceil \frac{k(k-2)}{8} \right\rceil,\ \left\lceil \frac{k(k-2)}{8} \right\rceil + 1,\ \ldots,\ \left\lfloor \frac{k(k-2)}{6} \right\rfloor \in B(k). \tag{6}$$

Now,

$$\left\lfloor \frac{k^2-3k+2}{6} \right\rfloor - \left(\left\lceil \frac{k(k-2)}{8} \right\rceil - 1 \right) \geq \frac{k^2-3k+2}{6} - \frac{k^2-2k}{8}$$
$$= \frac{k^2-6k+8}{24}$$
$$= \frac{(k-3)^2-1}{24}$$
$$\geq 0 \quad \text{since } k \geq 4$$

and so the result follows from (4), (5) and (6). □

References

[1] B. Alspach, *Research Problems*, Discrete Mathematics **36** (1981), 333.

[2] K. Heinrich, P. Horák and A. Rosa, *On Alspach's conjecture*, Discrete Mathematics **77** (1989), 97–121.

[3] C.C. Lindner and C.A. Rodger, *Decomposition into cycles II: Cycle systems* in *Contemporary design theory: a collection of surveys* (J.H. Dinitz and D.R. Stinson, eds.), John Wiley and Sons, New York (1992), 325-369.

[4] P. Erdös, R.C. Mullin, V.T. Sós and D.R. Stinson, *Finite linear spaces and projective planes*, Discrete Mathematics **47** (1983), 49–62.

[5] P. Erdös, J.C. Fowler, V.T. Sós and R.M. Wilson, *On 2-designs*, J. Combinatorial Theory series A **38** (1985), 131–142.