Generalized Steiner Systems With Block Size Three and Group Size Four

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Abstract

Generalized Steiner Systems, $GS(2,3,n,g)$, are equivalent to maximum constant weight codes over an alphabet of size $g+1$ with distance 3 and weight 3 in which each codeword has length n. We construct Generalized Steiner Triple Systems, $GS(2,3,n,g)$, when $g=4$.

1 Introduction and Background Information

A *pairwise balanced design* of order v ($PBD(v)$) is an ordered pair (S,B), where S is a finite set of symbols with $|S|=v$, and B is a set of subsets of S called blocks, such that each pair of distinct elements of S occurs together in exactly one block in B. A *group divisible design* is an ordered triple (P,G,B) where P is a finite set, G is a collection of sets called groups which partition P, and B is a set of subsets called blocks of P, such that $(P, G\cup B)$ is a PBD. The number $|P|$ is the *order* of the group divisible design. If a group divisible design has n groups of the same size, say g, and blocks of the same size, say k, then we will briefly refer to this design as a $k - GDD$ of type g^n. The reader is referred to [3] and [6] for any other words not defined in this paper.

A *$(g+1)$-ary constant weight code* (n, w, d) is a code $C \subseteq (\mathbb{Z}_{g+1})^n$ of length n and distance d, such that every $v \in C$ has Hamming weight w. Clearly, for $w = 3$ and $d = 3$, we have that $|C| \leq \frac{g^2 n(n-1)}{6}$. We can form a maximum $(g+1)$-ary constant weight code $(n, 3, d)$, $d \geq 2$, from a group divisible design 3-GDD of type g^n, (P, G, B), in the following way. Let $P = (\mathbb{Z}_{n+1} \backslash \{0\}) \times (\mathbb{Z}_{g+1} \backslash \{0\})$ with n groups, $G_\ell \in G$,

$$G_\ell = \{\ell\} \times (\mathbb{Z}_{g+1} \backslash \{0\}), 1 \leq \ell \leq n \text{ and blocks } [(a,i)(b,j),(c,k)] \in B,$$

$a, b, c, \in (\mathbb{Z}_{n+1} \backslash \{0\})$ and $i, j,\ k \in (\mathbb{Z}_{g+1} \backslash \{0\})$. Then for each block $[(a,i),(b,j),(c,k)]$, we form a codeword of length n by putting an i, j and k in positions a, b and c respectively and zeros elsewhere. Clearly this gives us a constant weight code over $\mathbb{Z}_{g+1}$, however, in general, the minimum distance is either 2 or 3. In fact, if two blocks share a point in common and cut across the same three groups, $d = 2$ is forced. If the minimum distance is 3, then the code is a maximum constant weight $(g+1)$-ary code $(n, 3, 3)$, and the group divisible design will be called a Generalized Steiner Triple System. It appears that most constructions of 3-GDD of type g^n in the literature that we have read produce a constant weight code with $d = 2$ and so finding these that have $d = 3$ is a more difficult problem.

Formally, a *Generalized Steiner Triple System*, $GS(2, 3, n, g)$, is a 3-GDD of type g^n, such that the constant weight code formed from the design has distance 3. Etzion [4] first introduced the name Generalized Steiner Systems, $GS(t, k, n, g)$ for group divisible designs k-GDD of type g^n, such that the constant weight code formed from the design has distance $1 + 2(k - t)$. Etzion [4] established a number of general results for constructing $GS(t, k, n, g)$ and completely solved the problem of finding a $GS(2, 3, n, g)$ when $g = 2$ and $g = 3$ as well. Phelps and Yin [8] solved the case for $g = 9$ and have solved the general case $g \equiv 3 \pmod 6$ with a handful of possible exceptions for each g.

In this paper, we will prove that a $GS(2, 3, n, 4)$ exists if and only if $n \equiv 0$ or $1 \pmod 3$, $n \geq 6$. We now give the background information needed in establishing the main result for forming a $GS(2, 3, n, 4)$.

Although it is a slight abuse of terminology, we will say that a subset of blocks of a 3-GDD of type g^n has distance d whenever the corresponding codewords formed from these blocks have distance d. Also, for notational purposes, we will use $\mathbb{Z}_k$ for the additive group modulo k consisting of the integers $\{0, 1, 2, \ldots, k-1\}$ and either $\mathbb{Z}\mathbb{Z}_k$ or $\mathbb{Z}_{k+1} \backslash \{0\}$ for the additive group modulo k consisting of the integers $\{1, 2, \ldots, k\}$. We will use the latter when we want to emphasize the fact that 0 cannot be included.

From the necessary and sufficient condition for a 3-GDD of type g^n that Hanani [5] established and by the observable fact that $n \geq g + 2$, we have the following theorem for the existence of a $GS(2, 3, n, g)$.

Theorem 1.1 ([4]) *The necessary conditions for the existence of a $GS(2, 3, n, g)$ is that:*

1. *If $g \equiv 0 \pmod 6$, then $n \geq g + 2$.*
2. *If $g \equiv 3 \pmod 6$, then $n \equiv 1 \pmod 2$ and $n \geq g + 2$.*
3. *If $g \equiv 2$ or $4 \pmod 6$, then $n \equiv 0$ or $1 \pmod 3$ and $n \geq g + 2$.*
4. *If $g \equiv 1$ or $5 \pmod 6$, then $n \equiv 1$ or $3 \pmod 6$ and $n \geq g + 2$.*

As mentioned before, Etzion [4] completely solved the case for $g = 3$, establishing that the necessary conditions were also sufficient. He also found some general results for forming $GS(2,3,n,g)$, $g \geq 5$. For example, Etzion pointed out that the codewords of a Hamming Code of length n over $GF(q)$, q a prime power, is perfect, includes the zero codeword and hence the codewords of weight 3 form a $GS(2,3,n,q-1)$. These Hamming codes exist for each $n = \frac{q^m-1}{q-1}$. In the case we will consider, i.e. $q-1=4$ implies that $q=5$ and $n = \frac{5^m-1}{5-1}$.

Also, it is easy to see that PBD closure also works for these designs and is given below.

Theorem 1.2 *For fixed g, the set of all n for which there exists a $GS(2,3,n,g)$ is PBD closed.*

Proof: We only need to consider two triples that share a point in common to make sure that $d \geq 3$ between the two triples. Two triples can share a point in common in two ways.

Case 1: The first way is for the two triples to cut across groups contained in a block from the PBD. But, if this is the case, then the size of this block, say m, is the order of a $GS(2,3,m,g)$ and so by putting a $GS(2,3,m,g)$ on these groups, $d \geq 3$ is guaranteed for two triples of this type.

Case 2: The second way for two triples to share a point in common is for the triples to cut across two different blocks of the PBD, sharing one group in common. Since this is a PBD, then the group from which the point is shared is the only group in common to each triple and so we have that $d \geq 3$ for triples of these types as well.

Since these are the only two cases we need to consider, then we have proved the theorem. □

The rest of the paper is organized as follows. In Section two and three we focus on the main constructions of these $GS(2,3,n,4)$ to be used in Section four. Specifically, in Section two, we give a product-type construction and in section three, we give a construction using cyclic group divisible designs. In section four, we settle the case g = 4 using the constructions from sections two and three and conclude in section five with some various open problems.

2 Product Constructions

In this section, we present two product-type constructions, due to Etzion [4], that will be very useful to us in constructing these $GS(2,3,n,4)$. He shows that these work for general g and found the t-partitions that he needed for the cases he solved. We will be finding the t-partitions that are needed specifically for $g = 4$. Etzion's two product theorems are as follows:

Theorem 2.1 ([4]) *If Q is a 3-GDD of type g^m and $R = GS(2,3,n,g)$, then there exists a $GS(2,3,mn,g)$, if Q can be partitioned into t sets $S_0, S_1, \ldots, S_{t-1}$, such that $t \leq n$ and the minimum distance in S_r, $r \in \mathbb{Z}_t$ is 3.*

Theorem 2.2 ([4]) *If Q is a 3-GDD of type g^m and $R = GS(2,3,n+1,g)$, then there exists a $GS(2,3,mn+1,g)$, if Q can be partitioned into t sets $S_0, S_1, \ldots, S_{t-1}$, such that $t \leq n$ and the minimum distance in S_r, $r \in \mathbb{Z}_t$ is 3.*

Corollary 2.1 *If Q is a 3-GDD of type g^m and $R = GS(2,3,n+p,g)$, which contains a sub $GS(2,3,p,g)$ then there exists a $GS(2,3,mn+p,g)$, if Q can be partitioned into t sets $S_0, S_1, \ldots, S_{t-1}$, such that $t \leq n$ and the minimum distance in S_r, $r \in \mathbb{Z}_t$ is 3.*

Proof: Pull out the subsystem $GS(2,3,p,g)$ of order p. The construction then proceeds as in the case $p = 1$ (Theorem 2.2). □

In the case $g = 4$, we will need a t-partition for Theorem 2.1 and Theorem 2.2 when $m = 3$ and $m = 4$. This can be done by using the following two lemmas.

Lemma 2.1 *There exists a 3-GDD of type* 4^3 *and a partition of* Q *into* t *sets* $S_0, S_1, \ldots, S_{t-1}$*, such that* $t \leq 4$ *and the minimum distance in* S_r*,* $r \in \mathbb{Z}_t$ *is* 3.

Proof: We form a 3-GDD, of type 4^3, (P, G, B) of order 12 as follows. Let $P = \{1,2,3,4\} \times \mathbb{Z}\mathbb{Z}_3$, $G_i \in G$, $G_i = (\mathbb{Z}_5 \backslash \{0\}) \times \{i\}$, $1 \leq i \leq 3$. The following four sets form our t-partition.

$$\begin{aligned}
S_0 &= \{[(1,1),(1,2),(4,3)],[(2,1),(3,2),(3,3)],[(3,1),(4,2),(1,3)],\\
&\quad [(4,1),(2,2),(2,3)]\}\\
S_1 &= \{[(1,1),(2,2),(1,3)],[(2,1),(4,2),(2,3)],[(3,1),(3,2),(4,3)],\\
&\quad [(4,1),(1,2),(3,3)]\}\\
S_2 &= \{[(1,1),(3,2),(2,3)],[(2,1),(1,2),(1,3)],[(3,1),(2,2),(3,3)],\\
&\quad [(4,1),(4,2),(4,3)]\}\\
S_3 &= \{[(1,1),(4,2),(3,3)],[(2,1),(2,2),(4,3)],[(3,1),(1,2),(2,3)],\\
&\quad [(4,1),(3,2),(1,3)]\}
\end{aligned}$$

So, we get $t \leq 4$. □

Lemma 2.2 *There exists a 3-GDD of type* 4^4 *and a partition of* Q *into* t *sets* $S_0, S_1, \ldots, S_{t-1}$*, such that* $t \leq 10$ *and the minimum distance in* S_r*,* $r \in \mathbb{Z}_t$ *is* 3.

Proof: We form a 3-GDD of type 4^4, (P, G, B) of order 16 as follows. Let $P = \{1,2,3,4\} \times \mathbb{Z}\mathbb{Z}_4$, $G_i \in G$, $G_i = (\mathbb{Z}_5 \backslash \{0\}) \times \{i\}$, $1 \leq i \leq 4$. The following ten sets form our t-partition.

$$
\begin{aligned}
S_0 &= \{[(1,1),(1,2),(1,3)],[(2,1),(2,2),(2,3)],[(3,1),(3,2),(1,4)],\\
&\quad [(4,1),(3,3),(2,4)]\}\\
S_1 &= \{[(1,1),(2,2),(1,4)],[(2,1),(1,2),(1,3)],[(3,1),(1,3),(4,4)],\\
&\quad [(4,1),(3,2),(2,3)]\}\\
S_2 &= \{[(1,1),(2,3),(2,4)],[(2,1),(1,3),(1,4)],[(3,1),(1,2),(3,3)],\\
&\quad [(4,1),(2,2),(3,4)]\}\\
S_3 &= \{[(1,1),(3,2),(3,3)],[(2,1),(4,2),(4,3)],[(3,1),(2,3),(3,4)],\\
&\quad [(4,1),(1,2),(4,4)]\}\\
S_4 &= \{[(1,1),(4,2),(3,4)],[(2,1),(3,2),(4,4)],[(3,1),(2,2),(4,3)]\}\\
S_5 &= \{[(1,1),(4,3),(4,4)],[(2,1),(3,3),(3,4)],[(3,1),(4,2),(2,4)]\}\\
S_6 &= \{[(4,1),(4,2),(1,3)],[(1,2),(2,3),(1,4)],[(2,2),(3,3),(4,4)],\\
&\quad [(3,2),(4,3),(2,4)]\}\\
S_7 &= \{[(4,1),(4,3),(1,4)],[(2,2),(1,3),(2,4)],[(4,2),(2,3),(4,4)]\}\\
S_8 &= \{[(1,2),(4,3),(3,4)],[(4,2),(3,3),(1,4)]\}\\
S_9 &= \{[(3,2),(1,3),(3,4)]\}
\end{aligned}
$$

So we get $t \leq 10$. □

It is worth noting that we will be using Lemma 2.2 along with Theorems 2.1 and 2.2 when $n \geq 17$ and so the fact that we get that $t = 10 \leq 17$ is sufficient.

3 Cyclic Group Divisible Designs

In this section, we present a general construction that uses cyclic group divisible designs [6]. A group divisible design, (P, G, B) is said to be cyclic if it has an automorphism consisting of an ng-cycle which cyclically permutes the elements P, mapping groups to groups and blocks to blocks. We will denote a cyclic 3-GDD of type g^n as 3-$CGDD$ of type g^n. To form this 3-$CGDD$ of type g^n, we let $G_i \in G$, $0 \leq i \leq n-1$, be defined by $G_i \in \{i, i+n, i+2n, \ldots, i+(g-1)n\}$ with $|P| = ng$. The blocks (triples) $b \in B$ are formed by finding difference triples, $\{x, y, z\}$, that cover all remaining differences d ($d \not\equiv 0$ mod n), and then forming the appropriate base blocks $\{0, x, x+y\}$ and then the triples $\{i, x+i, x+y+i\}$ for each base block.

Now, we need to see if we can form a $GS(2,3,n,g)$ from a 3-$CGDD$ of type g^n. Unfortunately, Jiang's construction [6] of 3-$CGDD$ of type g^n do not form Generalized Steiner Systems, $GS(2,3,n,g)$. We need to find the additional properties necessary for a 3-$CGDD$ of type g^n to be a $GS(2,3,n,g)$; that is, for any two triples to be distance $d \geq 3$ apart. The next theorem establishes when this can be done.

Theorem 3.1 *Let (P, G, B) be a 3-$CGDD$ of type g^n with P, G and B defined as above. This 3-$CGDD$ of type g^n will be a $GS(2,3,n,g)$ if and only if the following conditions hold:*

(a) for any two triples, $\{x, y, z\}$ and $\{x', y', z'\}$ in different orbits, we have that the triples reduced (mod n) are not equal; i.e. $\{x, y, z\}$ (mod n) $\neq \{x', y', z'\}$ (mod n) and

(b) if $n \equiv 0$ (mod 3), say $n = 3t$, then no orbit contains a block $\{0, a, b\}$ with $\{a, b\} \equiv \{t, 2t\}$ (mod n) unless $\{a, b\} = \{gt, 2gt\}$.

Proof: For two triples to have distance $d = 2$, they must cut across the same groups and share a point in common. This means the triples reduced (mod n) must be equal and thus must come from the same orbit. We have two types of orbits that are possible. If the orbit is short, which covers the distance $\frac{gn}{3}$, then the blocks are all disjoint. If the orbit is full length, then because of the extra condition (b), we can be assured that the orbit length (mod n) is full also and so if any two blocks share a point in common, then they do not cut across the same three groups. In either of these cases, we would have $d \geq 3$ occurring. □

4 $GS(2, 3, n, 4)$

As stated previously, Etzion completely solved the case $g = 2$ and so $g = 4$ is the next even case to consider. Recall that if $g = 4$, then $n \equiv 0$ or 1 (mod 3), $n \geq 6$. To apply PBD-closure, we need to establish the existence of $GS(2, 3, n, 4)$ for all $n \in S = \{6, 7, 9, 10, 12, 13, 15, 16, 18, 19, 21, 22, 24, 25, 27, 28, 30, 33, 34, 36, 37, 39, 40, 45, 46, 51, 52, 69, 93, 94\}$ (see [3], p. 213). We begin by showing how to construct a $GS(2, 3, n, 4)$ when $n = 6$ and $n = 7$ since so many of the constructions we have given depend on the existence of these two systems.

Corollary 4.1 *There exists a* $GS(2, 3, n, 4)$, $n = \frac{5^m - 1}{4}$.

Proof: We just apply the comments from Section one when $q = 5$. This gives us a $GS(2, 3, n, 4)$ when $n = 6, 31, \ldots$. □

We now present a construction when $n = 7$, since this is a specific construction that we use only in this case.

Lemma 4.1 *There exists a* $GS(2, 3, 7, 4)$.

Proof: We first take a 4-GDD of type 2^7 with groups $\{i, i + 7 \mid 0 \leq i \leq 6\}$ and blocks $\{i, i + 1, i + 4, i + 6 \mid 0 \leq i \leq 13\}$ where all addition is done modulo 14. Now blow up each point by two. Take each of the old blocks of size four, now with groups of size two and put a $GS(2,3,4,2)$ on each.

Now, do we have $d \geq 3$? We only need to check two points that share a point in common. This can happen in one of two ways.

Case 1: The first way is for the two triples to cut across groups contained in a block from the 4-GDD of type 2^7. This argument is the same as the argument in the proof of Theorem 1.2, Case 1. So $d \geq 3$ for any two triples of this type.

Case 2: The second way for two triples to share a point in common is for the triples to cut across two different blocks of the PBD, sharing one group in common. The only way that $d \leq 2$ could be possible would be if in our 4-GDD of type 2^7, we had combinations of j and k or $k+7$ and m or $m+7$, $0 \leq j \neq k \neq m \leq 13$, where all addition is done modulo 14, in two different blocks. But by observing the actual blocks, this never occurs. Therefore $d \geq 3$ for any two triples of this type.

So, we have formed a $GS(2,3,7,4)$. □

In the next corollary, we give the values of n for which Section three gives us a $GS(2,3,n,4)$.

Corollary 4.2 *There exists a* $GS(2,3,n,4)$ *for* $n = 9, 10, 12, 13, 15, 22, 24, 33, 40$ *and* 51.

Proof: We apply Theorem 3.1 from section three to get these. The base blocks with the properties needed are listed below.

Base Blocks for a $GS(2,3,9,4)$

$\{0,1,3\}, \{0,4,11\}, \{0,5,19\}, \{0,6,26\}, \{0,8,21\}, \{0,12,24\}$

Base Blocks for a $GS(2,3,10,4)$

$\{0,1,8\}, \{0,2,19\}, \{0,3,14\}, \{0,4,16\}, \{0,5,18\}, \{0,6,15\}$

Base Blocks for a $GS(2,3,12,4)$

$\{0,1,9\},\{0,2,23\},\{0,3,22\},\{0,4,18\},\{0,5,15\},\{0,6,17\},\{0,7,20\},$
$\{0,16,32\}$

Base Blocks for a $GS(2,3,13,4)$

$\{0,1,24\},\{0,2,14\},\{0,3,22\},\{0,4,25\},\{0,5,20\},\{0,6,16\},\{0,7,18\},$
$\{0,8,17\}$

Base Blocks for a $GS(2,3,15,4)$

$\{0,1,12\},\{0,2,16\},\{0,3,13\},\{0,4,27\},\{0,5,24\},\{0,6,28\},\{0,7,25\},$
$\{0,8,29\},\{0,9,26\},\{0,20,40\}$

Base Blocks for a $GS(2,3,22,4)$

$\{0,1,3\},\{0,4,9\},\{0,6,13\},\{0,8,18\},\{0,11,23\},\{0,14,45\},\{0,15,47\},$
$\{0,16,46\},\{0,17,50\},\{0,19,53\},\{0,20,49\},\{0,21,48\},\{0,24,52\},$
$\{0,25,51\},$

Base Blocks for a $GS(2,3,24,4)$

$\{0,1,18\},\{0,2,29\},\{0,3,25\},\{0,4,23\},\{0,5,42\},\{0,6,47\},\{0,7,43\},$
$\{0,8,46\},\{0,9,30\},\{0,10,45\},\{0,11,44\},\{0,12,40\},$
$\{0,13,39\},\{0,14,34\},\{0,15,31\},\{0,32,64\}$

Base Blocks for a $GS(2,3,33,4)$

$\{0,44,88\},\{0,1,57\},\{0,2,47\},\{0,3,62\},\{0,4,27\},\{0,5,55\},\{0,6,37\},$
$\{0,7,32\},\{0,8,61\},\{0,9,63\},\{0,10,39\},\{0,11,60\},\{0,12,42\},$
$\{0,13,35\},\{0,14,38\},\{0,15,41\},\{0,16,52\},\{0,17,51\},\{0,18,58\},$
$\{0,19,65\},\{0,20,48\},\{0,21,64\}$

Base Blocks for a $GS(2,3,40,4)$

$\{0,1,62\},\{0,2,76\},\{0,3,73\},\{0,4,60\},\{0,5,37\},\{0,6,72\},\{0,7,71\},$
$\{0,8,39\},\{0,9,78\},\{0,10,44\},\{0,11,47\},\{0,12,79\},\{0,13,48\},$
$\{0,14,52\},\{0,15,57\},\{0,16,59\},\{0,17,63\},\{0,18,51\},\{0,19,77\},$
$\{0,20,49\},\{0,21,75\},\{0,22,50\},\{0,23,68\},\{0,24,65\},\{0,25,55\},$
$\{0,26,53\}$

Base Blocks for a $GS(2,3,51,4)$

$\{0,1,77\},\{0,2,38\},\{0,3,48\},\{0,4,86\},\{0,5,100\},\{0,6,43\},\{0,7,92\},$
$\{0,8,99\},\{0,9,97\},\{0,10,49\},\{0,11,101\},\{0,12,70\},\{0,13,96\},$
$\{0,14,89\},\{0,15,62\},\{0,16,71\},\{0,17,67\},\{0,18,52\},\{0,19,54\},$
$\{0,20,79\},\{0,21,78\},\{0,22,63\},\{0,23,87\},\{0,24,84\},\{0,25,81\},$
$\{0,26,72\},\{0,27,80\},\{0,28,93\},\{0,29,69\},\{0,30,74\},\{0,31,73\},$
$\{0,32,98\},\{0,33,94\}$ □

Corollary 4.3 *There exists a $GS(2,3,n,4)$ for $n = 16,18,19,21,25,27,28,30,34,36,39,45$.*

Proof: Apply the product constructions of Theorem 2.2, 2.1 for $m = 3$ and $n = 6,7,9,10,12,13,15$. □

Corollary 4.4 *There exist a $GS(2,3,n,4)$ for $n = 46,52,69,93$.*

Proof: Same as before. Use product constructions for $m = 3$ and appropriate values of $n < 45$. □

Lemma 4.2 *There exists a $GS(2,3,94,4)$.*

Proof: First, we note that there exists a $GS(2,3,36,4)$ with a sub $GS(2,3,7,4)$ by applying Theorem 2.2 using a $GS(2,3,7,4)$ and a $GS(2,3,6,4)$ (i.e. $36 = 7 \cdot (6-1)+1$). Second, we apply the general product construction of Corollary 2.1 with $p = 7$, $m = 3$, $n = 36$ (i.e. $94 = 3 \cdot (36-7)+7$). □

Theorem 4.1 *A $GS(2,3,n,4)$ exists if and only if $n \equiv 0$ or 1 (mod 3), $n \geq 6$.*

Proof: PBD closure. □

5 Conclusions and Open Problems

In this paper we have shown that a $GS(2,3,n,4)$ exists if and only if $n \equiv 0$ or 1 (mod 3) and $n \geq 6$. The discussion in this paper leads to many open problems, such as:

1. Find more constructions for generalized Steiner Triple Systems.
2. Find t-partitions for other g-values.
3. Find constructions for generalized Steiner Quadruple Systems, $GS(3,4,n,g)$. Etzion [4] has a few constructions in his paper for g = 2.

The reader is referred to Etzion [4] and to Phelps and Yin [8] for even more open problems dealing with Generalized Steiner Systems.

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