On the existence of $(v, n, 4, \lambda)$-IPMD for even λ

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ABSTRACT. Let v, k, λ and n be positive integers. $(x_1, x_2, ..., x_k)$ is defined to be $\{(x_1, x_2), (x_2, x_3), ..., (x_{k-1}, x_k), (x_k, x_1)\}$, and is called a cyclically ordered k-subset of $\{x_1, x_2, ..., x_k\}$. An incomplete perfect Mendelsohn design, denoted by $(v, n, 4, \lambda)$-IPMD, is a triple $(X, Y, \mathcal{B})$, where X is a v-set (of points), Y is an n-subset of X, and $\mathcal{B}$ is a collection of cyclically ordered k-subsets of X (called blocks) such that every ordered pair $(a, b) \in X \times X \setminus Y \times Y$ appears t-apart in exactly λ blocks of $\mathcal{B}$ and no ordered pair $(a, b) \in Y \times Y$ appears in any block of $\mathcal{B}$ for any t, where $1 \leq t \leq (k-1)$. In this paper the necessary condition for the existence of a $(v, n, 4, \lambda)$-IPMD for even λ, namely $v \geq (3n + 1)$, is shown to be sufficient.

1 Introduction

Let v, u, k and λ be positive integers. $(x_1, x_2, ..., x_k)$ is defined to be $\{(x_1, x_2), (x_2, x_3), ..., (x_{k-1}, x_k), (x_k, x_1)\}$, and is called a cyclically ordered k-subset of $\{x_1, x_2, ..., x_k\}$. An incomplete holey perfect Mendelsohn design, denoted by (v, u, k, λ)-IHPMD, is a quadruple $(X, Y, \mathcal{G}, \mathcal{A})$ which satisfies the following properties:

1. X is a v-set of points, and Y is a u-subset of X;
2. $\mathcal{G}$ is a partition of X into groups;
3. $\mathcal{A}$ is a collection of cyclically ordered k-subsets of X (called blocks) each

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of which intersects each group in at most one point;
4. No block contains two points of Y; and
5. Every ordered pair of points (x, y) from distinct groups such that at least one of x, y is in $X \setminus Y$, appear t-apart in exactly λ blocks of $\mathcal{A}$, for $t = 1, 2, ..., k-1$.

If $\mathcal{G} = \{G_i : 1 \le i \le h\}, |G_i| = g_i$ and $|G_i \cap Y| = u_i$, we say that $\{(g_1, u_1), (g_2, u_2), ..., (g_h, u_h)\}$ is the type of the IHPMD .

A (v, k, λ)-HPMD can be viewed as an IHPMD with $Y = \emptyset$ and the vector $(g_1, g_2, ..., g_h)$ is called the type of the HPMD

A (v, n, k, λ)-IPMD can be viewed as an HPMD with the type $(n, 1, 1, ..., 1)$, and a (v, k, λ)-PMD can be viewed as an IPMD with $n = 1$.

For more details on the above terminology, the reader is referred to [3,4,7].

A $\diamondsuit$-IPMD is a quadruple $(X, Y_1, Y_2, \mathcal{A})$, where Y_1 is a w_1-subset of X, Y_2 is a w_2-subset of X, and $\mathcal{A}$ is a collection of cyclically ordered k-subsets of X, such that every ordered pair (x, y) appears t-apart in exactly λ blocks of $\mathcal{A}$, unless $(x, y) \subset Y_1$, or $(x, y) \subset Y_2$, in which case the pair appears in no block, where $1 \le t \le (k-1)$. We also say that the $\diamondsuit$-IPMD is a $(v, w_1, w_2, w, k, \lambda)$-$\diamondsuit$-IPMD where $w = |Y_1 \cap Y_2|$.

IPMDs are not only useful tools in the construction of PMDs and other structures, but finding when they exists is an interesting question itself (see [3,9,11]). The necessary conditions for the existence of a (v, k, λ)-IPMD were developed in [2], namely, $\lambda(v-n)(v-(k-1)n-1) \equiv 0 \pmod{k}$ and $v \ge (k-1)n + 1$. In the case of $k = 4$ and even λ, it is, $v \ge 3n + 1$. These basic necessary conditions were shown to be sufficient for the case $k = 3$ and $\lambda = 1$, with one exception of $v = 6$ and $n = 1$. In this paper, we shall investigate the case $k = 4$ and even λ, and it will be shown that a $(v, n, 4, \lambda)$-IPMD exists for even λ if and only if $v \ge 3n + 1$.

The existence of $(v, 4, \lambda)$-PMDs forms the basis for most of our constructions. The problem of existence was initially studied by N.S.Mendelsohn, and now we have a complete result in the form of the following theorem (see [1,3,4,9]).

Theorem 1.1 *A $(v, 4, \lambda)$-PMD exists if and only if $\lambda v(v-1) \equiv 0 \pmod 4$ with the exception of $v = 4$ and odd λ, and $v = 8$ and $\lambda = 1$.*

The following results are from [4,10].

Theorem 1.2 *The necessary conditions $v \equiv 2, 3 \pmod 4$ and $v \ge 7$ for the existence of a $(v, n, 4, 1)$-IPMD for $n = 2$ are sufficient except for $v = 7$ and possibly excepting $v = 15, 19, 23, 27$.*

Theorem 1.3 *There exists a $(v, n, 4, 1)$-IPMD for $n = 3$ if and only if $v \equiv 2, 3 \pmod 4$ and $v \ge 10$.*

We assume that the reader is familiar with the basic concepts in design theory, such as paiwise balanced design (PBD), group divisible design (GDD), incomplete group divisible design (IGDD) and transversal design (TD). For convenience, the reader can be referred to [6, 8].

The following results can be found in [5].

Lemma 1.4 *If m is odd and $m \neq 3, 5, 15, 33, 35, 39, 45, 51$, then there is a resolvable TD(6,m).*

Lemma 1.5 *If $m \geq 4, m \neq 6, 10$, then there is a TD(5,m).*

2 Recursive construction methods

The following theoreem provides a way to obtain an IPMD from an IHPMD and some $\diamondsuit$-IPMD, which is a variation of construction 3.3 in [8].

Theorem 2.1 *(Filling in groups). Suppose that the following designs exist:*
1. a (v, u, k, λ)-IHPMD of type $\{(g_1, u_1), (g_2, u_2), ..., (g_h, u_h)\}$;
2. a $(g_i + b, u_i + a, b, a, k, \lambda)$-$\diamondsuit$-IPMD, for $1 \leq i \leq h$; and
3. a $(g_n + b, u_n + a, k, \lambda)$-IPMD, for some n with $1 \leq n \leq h$.
Then there eexists a $(v + b, u + a, k, \lambda)$-IPMD

To employ this theorem we need some IHPMDs to start with, which can be obtained by weighting.

Theorem 2.2 *Suppose $(X, \mathcal{G}, \mathcal{A})$ is a GDD with $\lambda = 1$, and let $s, t : X \longrightarrow Z^+ \cup 0$ be functions such that $t(x) \leq s(x)$ for every $x \in X$. For every block $A \in \mathcal{A}$, suppose that we have a $(\sum_{x\in A} s(x), \sum_{x\in A} t(x), k, \lambda) - IHPMD$ of type $\{(s(x), t(x)) : x \in A\}$. Then there exists a $(\sum_{x\in A} s(x), \sum_{x\in A} t(x), k, \lambda)$-IHPMD of type $\{(\sum_{x\in G} s(x), \sum_{x\in G} t(x)) : G \in \mathcal{G}\}$.*

Theorem 2.3 *There exist $(v, n, 4, 2)$-IHPMDs of the following types: $(3,1)^4, (3,1)^4(4,1)^1, (3,1)^4(3,0)^1$ and $(3,1)^5$.*

Proof: There exist $\{4,5\}$-IGDDs of types and $\lambda = 1 : (3,1)^4, (3,1)^4(4,1)^1$ and $(3,1)^4(3,0)^1$ from the proof of Lemma 3.6 in [6]. Replacing each block in the above IGDDs with a $(5,4,2)$-PMD and a $(4,4,2)$-PMD, we obtain the first three IHPMDs . For the last type, we start with a $(5,4,2)$-PMD and replace each block with a $\{4\}$-IGDD of type $(3,1)^4$ and $\lambda = 1$ to obtain an IHPMD of type $(3,1)^5$ and $\lambda = 2$. □

Theorem 2.4 *Let $m \geq 4$ and $m \neq 6, 10$. Suppose there exist a $(3m + b, m + a, b, a, 4, 2)$-$\diamondsuit$-IPMD and a $(r + b, n + a, 4, 2)$-IPMD. Then there exists a $(v, u, 4, 2)$-IPMD where $v = 12m + r + b, u = 4m + n + a, 0 \leq n \leq m, r = 0, 3, 6, ..., 3m$ when $n = 0, r = 3, 4, 6, 7, ..., 3m, 3m + 1$ when $n = 1$ and $3n \leq r \leq 3m + n$ when $n \geq 2$.*

Proof: From Lemma 1.5 there exist a TD(5,m) for $m \geq 4$ and $m \neq 6, 10$. Let X be the points set of the TD . Partition one group of the TD into $Y_1 \cup Y_2$ such that $|Y_1| = n$, $|Y_2| = m - n$, Define $s, t : X \longrightarrow Z^+ \cup \{0\}$, such that

$$(s(x), t(x)) = \begin{cases} (3,1) \text{ or } (4,1) \text{ if } x \in Y_1; \\ (3,0) \text{ or } (0,0) \text{ if } x \in Y_2; \\ (3,1) \text{ otherwise.} \end{cases}$$

we apply Theorem 2.2 to obtain an IHPMD of type $(3m, m)^4(r, n)^1$, here the required IHPMDs come from Lemma 2.3. Then we obtain the desired result from Theorem 2.1. □

The following theorem is essentially from Theorem 2.4 in [9].

Theorem 2.5 *Suppose that* $(X, \mathcal{B})$ *is a* $(v, k, 1)$*-PBD where* $\mathcal{B}$ *can be partitioned into* s *parallel classes* $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_s$*. Suppose there exists a* $(|B| + n_i, n_i, k, \lambda)$*-IPMD for every* $B \in \mathcal{B}_i$ *where* $1 \leq i \leq s$*. then there exists a* $(v + n, n, k, \lambda)$*-IPMD where* $n = n_1 + n_2 + ... + n_s$*.*

As an application of Theorem 2.5, we have the following useful lemmas.

Lemma 2.6 *Suppose that there exists a resolvable TD(t, m). Suppose there exists the following IPMDs :*
1. a $(t + n, n, k, \lambda)$*-IPMD for* $c \leq n \leq d$*;*
2. a $(m + w, w, k, \lambda)$*-IPMD for* $e \leq w \leq f$*.*
Then there exists a (v, u, k, λ)*-IPMD where* $v - u = mt$ *and* $mc + e \leq u \leq md + f$*.*

Proof: A resolvable TD(t, m) admits m parallel classes of blocks of size t and one parallel class (of groups) of size m. We can then apply theorem 2.5 to obtain the desired result. □

Lemma 2.7 *Suppose that there exists a resolvable TD(t, m). Suppose there exists the following IPMDs :*
1. a $(t - i + n, n, k, \lambda)$*-IPMD for* $i = 0, 1$ *and* $c \leq n \leq d$*;*
2. a $(m + w, w, k, \lambda)$*-IPMD for* $e \leq w \leq f$*;*
3. a $(m - s + w, w, k, \lambda)$*-IPMD for* $e \leq w \leq f$*.*
Then there exists a (v, u, k, λ)*-IPMD where* $v - u = mt - s$ *and* $mc + e \leq u \leq md + f$*.*

Proof: A resolvable PBD obtained by deleting s points in one group of a resolvable TD(t, m), admits m parallel classes of blocks of size t and $t - 1$ and one parallel class (of groups) of size m and $m - s$. We can then apply theorem 2.5 to obtain the desired result. □

Lemma 2.8 *Suppose that there exists a resolvable TD(t, m). Suppose there exists the following IPMDs :*
1. a $(t-i+n, n, k, \lambda)$-IPMD for $i = 0, 1, 2$ and $c \le n \le d$;
2. a $(m+w, w, k, \lambda)$-IPMD for $e \le w \le f$;
3. a $(m-s_i+w, w, k, \lambda)$-IPMD for $i = 1, 2$ and $e \le w \le f$.
Then there exists a (v, u, k, λ)-IPMD where $v-u = mt - s_1 - s_2$ and $mc+e \le u \le md+f$.

Proof: The Proof is similar to that of Lemma 2.7. □

3 Direct construction methods

We will directly construct some small examples of IPMDs by employing the effective and easy way which was developed in [10].

For odd $v-u$, we always present a set of base blocks $\mathcal{B}$ which can be developed under the cyclic group of $G = Z_{v-u}$. If $B = (0, a, a+b, a+b+c) \in \mathcal{B}$, we define $B(1) = (a, b, c, -(a+b+c))$ and $B(2) = (a+b, b+c, -(a+b), -(b+c))$, that is, the t-apart ordered difference set of B for $t = 1, 2$, similarly if $B = (\infty, 0, a, a+b)$, we define $B(1) = (a, b)$ and $B(2) = (a+b, -(a+b))$. It is easy to see if $\mathcal{B}(t) = \{B(t) : B \in \mathcal{B}\}$ for $t = 1, 2$ partitions $(Z_{v-u} \setminus \{0\}) \cup (Z_{v-u} \setminus \{0\})$, then $(X, dev\mathcal{B})$ is a $(v, u, 4, 2)$-IPMD where $X = Z_{v-u} \cup \{\infty_i : 1 \le i \le u\}$ and $dev\mathcal{B} = \{B+g, B \in \mathcal{B}, g \in Z_{v-u}\}$.

From Lemma 1.6 in [10], if $M = (a, b, c, -(a+b+c))$ such that $a, b, c, -(a+b+c), a+b, b+c \in Z_{v-u} \setminus \{0\}$, then there is one and only one base block B such that $B(1) = M$ and $B(2) = (a+b, b+c, -(a+b), -(b+c))$, and if $M = (a, b)$ such that $a, b, a+b \in Z_{v-u} \setminus \{0\}$, then there is one and only one base block B containing ∞ such that $B(1) = M$ and $B(2) = (a+b, -(a+b))$. For this reason and convenience we always present $\mathcal{B}(1)$ and instead of $\mathcal{B}$.

Lemma 3.1 *There exists a $(v, u, 4, 2)$-IPMD for $v-u = 2m+1$ and $u = m$.*

Proof: Let $\mathcal{B}(1) = \{(2k, -k) : k \in Z_{2m+1} \setminus \{0\}\}$. □

Lemma 3.2 *There exists a $(v, u, 4, 2)$-IPMD for $v \ge 4, u = 1$.*

Proof: Since there exists a $(v, 4, 2)$-IPMD for $v \ge 4$ from Theorem 1.1. □

A base block B is called *a special base block* if $B(1) = (a, b, a, -(2a+b))$.

Lemma 3.3 *If $\mathcal{B}$ is base blocks of a $(v, u, 4, 2)$-IPMD and there is a special base block of $\mathcal{B}$, then there exists a $(v+1, u+1, 4, 2)$-IPMD.*

Proof: Let B be a special base block and $B(1) = (a, b, a, -(2a+b))$. Let $A_1 = (\infty_{u+1}, 0, a, a+b)$, $A_2 = (\infty_{u+1}, 0, a, -(a+b))$. It is easily seen that $(\mathcal{B} \setminus B) \cup \{A_1, A_2\}$ is base blocks of a $(v+1, u+1, 4, 2)$-IPMD. □

Lemma 3.4 *There exists a $(v, u, 4, 2)$-IPMD for $v - u = 11, 19, 23, 31, 43, 47, 59, 67$ and $1 \le u \le (v-u-1)/2$.*

Proof: For $v - u = 11, 19, 23, 47, 59, 67$. Let $G = GF(v-u)$ and $\mathcal{B}(1) = \{(x^{2i}, x^{2i+1}, x^{2i}, -x^{2i+2}) : 1 \le u \le (v-u-1)/2\}$ where $x = 2$, a primitive root.
For $v - u = 31$. Let $G = GF(31)$ and $\mathcal{B}(1) = \{(x^{2i}, x^{2i+1}, x^{2i}, x^{2i+5}) : 1 \le i \le 15\}$ where $x = 3$, a primitive root.
For $v - u = 43$. Let $G = GF(43)$ and
$\mathcal{B}(1) = \{(x^{2i}, -x^{2i+14}, x^{2i}, -x^{2i+2}) : 1 \le i \le 21\}$ where $x = 3$, a primitive root. □

Lemma 3.5 *There exists a $(v, u, 4, 2)$-IPMD for $v - u = 7, 9, 13, 15$ and $1 \le u \le (v-u-1)/2$, and $v - u = 21, 27, 33$ and $1 \le u \le (v-u)/3 - 1$.*

Proof: See Lemma 5.1 in the section 5 of this paper. □

Lemma 3.6 *If $\mathcal{B}$ is base blocks of a $(v, u, 4, 2)$-IPMD and there is a pair of base blocks B_1 and B_2 of $\mathcal{B}$ such that $B_1(1) = (a, a, b, -(2a+b))$ and $B_2(1) = (-a, -a, -b, 2a+b)$. Then there is a $(v+n, u+n, 4, 2)$-IPMD for $n = 0, 1, 2$.*

Proof: Let A_1, A_2, A_3 be three new base blocks such that $A_1(1) = (a, b, a, -(2a+b))^*$, $A_2(1) = (-a, -a)$, $A_3(1) = (-b, 2a+b)$. It is readily checked that $(\mathcal{B} \setminus \{B_1, B_2\}) \cup \{A_1, A_2, A_3\}$ is a $(v+1, u+1, 4, 2)$-IPMD. Since A_1 is a special base block, we have a $(v+2, u+2, 4, 2)$-IPMD from Lemma 3.3. □

Lemma 3.7 *There exists a $(v, u, 4, 2)$-IPMD for $v - u = 17, 29, 41, 37, 61$ and $1 \le u \le (v-u-1)/2$.*

Proof: We present $\mathcal{B}(1)$ for each case to obtain the desired result by applying Lemma 3.6.
For $v - u = 17$, $\mathcal{B}(1) = \{\pm(x^{2i}, x^{2i}, x^{2i+1}, x^{2i+7}), : 1 \le i \le 4\}$ where $x = 5$.
For $v - u = 29$, $\mathcal{B}(1) = \{\pm(x^{2i}, x^{2i}, x^{2i+3}, x^{2i+9}), : 1 \le i \le 7\}$ where $x = 2$.
For $v - u = 37$, $\mathcal{B}(1) = \{\pm(x^{2i}, x^{2i}, -x^{2i+3}, -x^{2i+9}), : 1 \le i \le 9\}$ where $x = 2$.
For $v - u = 41$, $\mathcal{B}(1) = \{\pm(x^{2i}, x^{2i}, x^{2i+13}, -x^{2i+15}), : 1 \le i \le 10\}$ where $x = 7$.
For $v - u = 61$, $\mathcal{B}(1) = \{\pm(x^{2i}, x^{2i}, x^{2i+3}, -x^{2i+23}), : 1 \le i \le 15\}$ where $x = 2$. □

Lemma 3.8 *There exists a $(v,u,4,2)$-IPMD for $v-u=6,10,12,14,18$ and $2 \le u \le (v-u-2)/2, v-u=22,26,30$ and $3 \le u \le (v-u-2)/2$, and $v-u=34,38,58,62$ and $10 \le u \le (v-u-2)/2$ (see Theorem 2.13 in [10]).*

Lemma 3.9 *There exists a $(v,u,4,2)$-IPMD for $v-u=16,20,24,28$ and $u=2$ and $v-u=16,20,24,28,32$ and $4 \le u \le (v-u-2)/2$ (see Theorem 2.13 in [10]).*

Lemma 3.10 *There exists a $(v,u,4,2)$-IPMD for $v-u \in \{6,10,14,18,22, 26,30,34,38,58,62\} \cup \{12,16,20\} \cup \{24,28,32,64\}$, and $2 \le u \le (v-u-2)/2$.*

Proof: For $v-u=22,26,30,32, u=2$, a $(v,u,4,2)$-IPMD exists, since there is a $(v,7,4,2)$-IPMD for each v and there is a $(7,2,4,2)$-IPMD. For $v-u=16,20,24,28,32, u=3$, a $(v,u,4,2)$-IPMD exists from Theorem 1.3. For $v-u=34,38,58,62, 2 \le u \le 9$, to obtain a $(v,u,4,2)$-IPMD, we apply Lemma 2.7 with $t=5, c=e=0, d=1, f=2$ and $(m,s)=(7,1),(8,2),(12,2),(13,3)$. For $v-u=64, 2 \le u \le 31$. Let X be the points set of a TD(5,8). Partition one group of this TD into $Y_0 \cup Y_1 \cup Y_2 \cup Y_3$. Define $s: X \longrightarrow Z^+ \cup \{0\}$ such that

$$s(x)=\begin{cases} i \text{ if } x \in Y_i \\ 2 \text{ otherwise.} \end{cases}$$

We apply Theorem 2.2 to obtain an HPMD of type $16^4 r^1, 0 \le r \le 24$, where the required HPMDs come from [10]. It is easy to see that we have a $(v,u,4,2)$-IPMD for $v-u=64, 0 \le u \le r+a$ if there are a $(16+a,a,4,2)$-IPMD and a $(r+a,a,4,2)$-IPMD. Thus we have a $(v,u,4,2)$-IPMD for $v-u=64, 0 \le u \le 31$. By combining the above results with Lemma 3.8 and 3.9 we have obtain the desired results. □

4 $(v,u,4,2)$-IPMDs

In this section, it is shown there exists a $(v,u,4,2)$-IPMD for $v \ge 3u+1$ where
$u \ge 1$. It is fairly obvious that $\{(v,u): v \ge 3u+1, u \ge 1\}$
$=\{(v,u): 1 \le u \le [(v-u)/2], v-u \ge 3\}$ where $[a/2]=a/2$ when a is even and $[a/2]=(a-1)/2$ when a is odd.

Lemma 4.1 *There exists a $(v,u,4,2)$-IPMD for $v-u=3m, m \ge 4, m \ne 6, m \le u \le [(3m-1)/2]$.*

Proof: We apply Lemma 2.6 with $t=3, c=d=1, e=0$ and $f=[(m-1)/2]$ to obtain the desired result. □

Lemma 4.2 *There exists a* $(v, u, 4, 2)$*-IPMD for* $v - u = 4m, m \geq 4, m \neq 6, 10, 0 \leq u \leq m + [(m-1)/2]$.

Proof: We apply Lemma 2.6 with $t = 4, c = 0, d = 1, e = 0$ and $f = [(m-1)/2]$ to obtain the desired result. □

Lemma 4.3 *There exists a* $(v, u, 4, 2)$*-IPMD for* $v-u = 25, 35, 40, 45, 49, 55, 56, 60, 63, 65, 75, 77$ *and* 80, *and* $1 \leq u \leq [(v-u-1)/2]$.

Proof: We apply Lemma 2.6 with $t = 5, m = 5, 7, 8, 9, 11, 12, 13, 15$ and 16, $c = 0, d = 2, e = 0$ and $f = [(m-1)/2]$ to obtain the desired result for $v - u = 25, 35, 40, 45, 55, 60, 65, 75$ and 80. Similarly we can obtain the desired result for $v - u = 49, 56, 63, 77$ by applying Lemma 2.6 with $t = 7, m = 7, 8, 9, 11$. □

Let $K = \{v-u : 3 \leq v-u \leq 80\}$ and $L = \{44, 46, 50, 52, 53, 68, 70, 71, 73, 74, 76, 79\}$.

Theorem 4.4 *There exists a* $(v, u, 4, 2)$*-IPMD for* $v - u \in K \setminus L$ *and* $1 \leq u \leq [(v-u-1)/2]$.

Proof: It is easily seen that there exists a $(v, u, 4, 2)$-IPMD for $v - u = 3, 4, 5, 8$ and $1 \leq u \leq [(v-u-1)/2]$ from Lemma 3.1 and 3.2 and Theorem 1.2 and 1.3, and there exists a $(v, u, 4, 2)$-IPMD for $v - u = 36, 48$ and 72 from Lemma 4.1 and 4.2. By combining the results in the section 3 and this section, it is readily verified there esists a $(v, u, 4, 2) - IPMD$ for $v - u \in K \setminus (L \cup \{3, 4, 5, 8, 36, 48, 72\})$ and $1 \leq u \leq [(v-u-1)/2]$. This completes the proof. □

Lemma 4.5 *There exists a* $(3m+b, m+a, b, a, 4, 2)$*-*$\diamondsuit$ *-IPMD for* $b - a = 1, a = 0$.

Proof: Since there esists a $(3m+1, m, 4, 2)$-IPMD from Lemma 3.1. □

Lemma 4.6 *Let* $m \geq 4, m \neq 6, 10,$. *If there is an* $(m+a, a, 4, 2)$*-IPMD for* $0 \leq a \leq [(m-1)/2]$, *then there exists a* $(3m+b, m+a, b, a, 4, 2)$*-*$\diamondsuit$*-IPMD for* $b - a = m$ *and* $0 \leq a \leq [(m-1)/2]$.

Proof: Since there is a $(4, 4, 2)$-PMD and there is a $TD(4, m)$, we have a $(4m, 4, 2)$-HPMD of type m^4 (see Theorem 2.2 in [9]), say, G_1 and G_2 are two groups of the HPMD, and $\mathcal{B}$ is its blocks. Let $H = \{\infty_1, \infty_2, ..., \infty_a\}$ and $\mathcal{A}_i$ be blocks of a $(m+a, a, 4, 2)$-IPMD based on $G_i \cup H$ for $i = 1, 2$. It is readily verified that $\mathcal{B} \cup \mathcal{A}_1 \cup \mathcal{A}_2$ is the blocks of the desired $\diamondsuit$-IPMD. □

Lemma 4.7 *There exists a* $(v, u, 4, 2)$*-IPMD for* $v - u = 39, 42, 51, 54, 57, 66, 69, 78$ *and* $1 \le u \le [(v - u - 1)/2]$.

Proof: For each case of $v - u = 3m$, we only need to show that there exists a $(v, u, 4, 2)$-IPMD for $1 \le u \le m - 1$ from Lemma 4.1. To obtains the desired result, let $c = 0, d = 2, e = f = 0$. For $v - u = 42, 56, 66, 78$, we apply Lemma 2.6 with $t = 6, m = 7, 9, 11, 13$. For $v - u = 39, 51, 69$, we apply Lemma 2.7 with $t = 6, (m, s) = (7, 3), (9, 3), (13, 9)$. And for $v - u = 57$, we apply Lemma 2.8 with $t = 7, m = 9, s_1 = s_2 = 3$. □

Lemma 4.8 *There exists a* $(v, u, 4, 2)$*-IPMD for* $v - u = 44, 46, 50, 52, 53, 68, 70, 71, 73, 74, 76, 79$ *and* $1 \le u \le [(v - u - 1)/2]$.

Proof: For each pair $(v - u, u)$ shown in the table 1, it is readily checked there exists a $(v, u, 4, 2)$-IPMD from Lemma 5.2 in the section 5.

v-u	u	v-u	u
44	$18 \le u \le 21$	70	$30 \le u \le 34$
46	$18 \le u \le 22$	71	$30 \le u \le 35$
50	$21 \le u \le 24$	73	$30 \le u \le 36$
52	$22 \le u \le 25$	74	$30 \le u \le 36$
53	$22 \le u \le 26$	76	$30 \le u \le 37$
68	$30 \le u \le 33$	79	$30 \le u \le 38$

Table 1

To obtain the other desired results, for $v - u = 44, 46, 50, 53$, we apply Lemma 2.7 with $t = 6, c = 0, d = 2, e = 1$ and $(m, s, f) = (8, 4, 1), (8, 2, 1), (9, 4, 2), (9, 1, 3)$. For $v - u = 68, 70, 71$, we apply Lemma 2.7 with $t = 8, c = 0, d = 3, e = 0, f = 2$ and $(m, s) = (9, 4), (9, 2), (9, 1)$. For $v - u = 73, 74, 76, 79$, we apply Lemma 2.8 with $t = 9, c = 0, d = 3, e = 0, f = 2, m = 9$ and $(s_1, s_2) = (4, 4), (4, 3), (3, 2), (1, 1)$. □

Lemma 4.9 *Let* $v - u = 12h - g \ge 80, h \ge 7, 0 \le g \le 11$. *There exists a* $(v, u, 4, 2)$*-IPMD for* $1 \le u \le 4h + 1$.

Proof: We apply Lemma 2.8 with $t = 7, m = 2h + 1, c = 0, d = 2, e = 1, f = 3, 0 \le s_1, s_2 \le h - 1$ (since $s_1 + s_2 = 2h + g - 7$ and $0 \le g \le 4$) to obtain a $(v, u, 4, 2)$-IPMD for $h = 7, 0 \le g \le 4$ (since $12h - g \ge 80$). In the following, we always let $c = 0, d = 2, t = 6, e = 1, f = [(m - s - 1)/2]$ and $s = 6m - 2h + g$ and present m for each case of h and g. It is readily checked that there exists a $(v, u, 4, 2)$-IPMD for $v - u = 12h - g$ and $1 \le u \le 4h + 1$. For $h = 8, 16$ and $0 \le g \le 11$, we put $m = 2h$, for $h = 9$ and $0 \le g \le 5$, we put $m = 2h+1$, for $h = 9$ and $6 \le g \le 11$, we put $m = 2h-1$, and for $h = 17, 19, 22, 25$ and $0 \le g \le 11$, we put $m = 2h+3$. Finally for the remain cases of h, and $0 \le g \le 11$, we put $m = 2h+1$. We have completed the proof. □

Lemma 4.10 *There exists a $(v, u, 4, 2)$-IPMD for $120 \le v-u \le 109$ and $42 \le u \le 45$.*

Proof: We apply Lemma 2.8 with $t = 11, m = 11, 0 \le s_1, s_2 \le 6, c = 0, d = 4, e = f = 1$ to obtain the desired results. □

Lemma 4.11 *Let $m \ge 4, m \ne 6, 10$ and $2 \le n \le m$, there exists a $(v, u, 4, 2)$-IPMD for $9m + 2n \le v - u \le 12m$ and $4m + n \le u \le 4m + n + [(m-1)/2]$.*

Proof: Apply Theorem 2.4 with $b-a = m, 0 \le a \le [(m-1)/2], n \ge 2, 3n \le r \le 3m+n$ to obtain a $(v, u, 4, 2)$-IPMD for $v-u = 9m+r-n, u = 4m+n+a$, that is, for $9m+2n \le v-u \le 12m$ and $4m+n \le u \le 4m+n+[(m-1)/2]$, here the required $\diamondsuit$-IPMD comes from Lemma 4.6. □

Lemma 4.12 *Let $v-u = 12h-g \ge 80, h \ge 7, 0 \le g \le 11$. There exists a $(v, u, 4, 2)$-IPMD for $4h+2 \le u \le [(v-u-1)/2]$.*

Proof: For $h \ge 11$, by applying Lemma 4.11 with $m = h, n = 2, 3, ..., m$, we obtain a $(v, u, 4, 2)$-IPMD for $4h+2 \le u \le 4h+h+[(h-1)/2]$ when $v-u > 11h$ or $4h+2 \le u \le [(v-u-1)/2]$ when $v-u \le 11h$. If $v-u \ge 11h$, then $9(h+1)+4 \le v-u \le 12(h+1)$, we apply Lemma 4.11 with $m = h+1$ to obtain a $(v, u, 4, 2)$-IPMD for $4h+6 \le u \le 5h+5+[h/2]$ when $v-u > 11(h+1)$ or $4h+6 \le u \le [(v-u-1)/2]$ when $v-u \le 11(h+1)$. Obviously, we can obtain the desired result by applying Lemma 4.11 for finite times. For $7 \le h \le 10$ and $v-u = 12h-g \ge 80$ there exists a $(v, u, 4, 2)$ -IPMD for $4h+2 \le u \le [(v-u-1)/2]$ from Lemma 5.2 in the section 5 and Lemma 4.10. We have completed the proof. □

The main result of this paper can be sumarized by combining Theorem 4.4 and Lemma 4.7, 4.8, 4.9 and 4.12.

Theorem 4.13 *There exists a $(v, u, 4, 2)$-IPMD if and only if $v \ge 3u+1$.*

We have essentially established the following:

Theorem 4.14 *There exists a $(v, u, 4, \lambda)$-IPMD for even λ if and only if $v \ge 3u+1$.*

5 Appendix

Lemma 5.1 *There exists a* $(v,u,4,2)$*-IPMD for* $v-u=7,9,13,15$ *and* $1 \le u \le (v-u-1)/2$*, and* $v-u=21,27,33$ *and* $1 \le u \le (v-u)/3-1$.

Proof: For each $v-u$, we let $G = Z_{v-u}$ and present $\mathcal{B}(1)$. It is easy to obtain the base blocks $\mathcal{B}$ and $\mathcal{B}(2)$ from $\mathcal{B}(1)$, and check that $\mathcal{B}(t)(t=1,2)$ partitions $(Z_{v-u} \setminus \{0\}) \cup (Z_{v-u} \setminus \{0\})$. From the number of special base blocks which indicate with $*$, we can obtain the desired result.

For $v-u=7,9,13,15$ and $u=1,(v-u-1)/2$, there exists a $(v,u,4,2)$-IPMD from Lemma 3.1 and 3.2.

For $v-u=7, u=2$. Let
$\mathcal{B}(1)=\{(2,2,-1,-3),(3,-1),(-2,1),(1,3),(-2,-3)\}$

For $v-u=9, u=2,3$. Let
$\mathcal{B}(1)=\{(1,2,1,-4)^*,(3,-1,-4,2),(4,-3),(4,-3),(3,-1),(-2,-2)\}$

For $v-u=13, u=2,3,4,5$. Let
$\mathcal{B}(1)=\{(5,1,-4,-2),(-1,6,-1,-4)^*,(-5,4,-5,6)^*,(-6,2,-6,-3)^*,$
$(5,-3),(4,-2),(1,2),(3,3)\}$

For $v-u=15, u=2,3,4,5,6$. Let
$\mathcal{B}(1)=\{(3,2,3,7)^*,(-3,2,-3,4)^*,(6,-4,6,7)^*,(-6,-1,-6,-2)^*,$
$(5,1,5,4)^*,(-2,-1),(-5,-7),(-7,-4),(-5,1)\}$

For $v-u=21,27,33, u=1$, there exists a $(v,u,4,2)$-IPMD from Lemma 3.2.

For $v-u=21, 2 \le u \le 6$. Let
$\mathcal{B}(1)=\{(1,-6,1,4)^*,(2,8,2,9)^*,(7,-4,7,-10)^*,(3,5,3,10)^*,$
$(-1,-3,10,-6),(-2,4,-10,8),(-5,-7,-8,-1),(6,-7,9,-8),$
$(-3,-9),(5,-4),(6,-2),(-5,-9)\}$

For $v-u=27, 2 \le u \le 8$. Let
$\mathcal{B}(1)=\{(3,-8,3,2)^*,(6,-2,6,-10)^*,(-6,5,-6,7)^*,(9,-2,9,11)^*,$
$(-9,1,-9,-10)^*,(12,-1,12,4)^*,(-12,2,-12,-5)^*,(-3,-3,1,5)^*,$
$(4,10,-7,-7),(-4,-8,-4,-11)^*,(10,8,-5,-13),(13,8),(11,7),$
$(13,-11),(-1,-13)\}$

For $v-u=33, 2 \le u \le 16$. Let
$\mathcal{B}(1)=\{(3,2,3,-8)^*,(-3,1,-3,5)^*,(-6,5,-6,7)^*,(-9,-4,-9,-11)^*,$
$(12,-1,12,10)^*,(-12,2,-12,-11)^*,(-15,-1,-15,-2)^*,(6,8,6,13)^*,$

$(9,-2,9,-16)^*, (15,-7,15,10)^*, (16,-13,16,14)^*,$
$(-14,8,-14,-13)^*, (-5,-7,-5,-16)^*, (11,4,11,7)^*, (14,-10),$
$(-8,4), (1,-10), (-4,13)\}$ □

By applying Theorem 2.4 in conjunction with Lemma 4.5 and 4.6, we have

Lemma 5.2 *There exists a $(v,u,4,2)$-IPMD for $(v-u,u)$ shown in the table 2, table 3 and table 4.*

Let $[i,j]$ indicates $\{i, i+1, i+2, ..., j\}$.
In the table 2, $2 \le n \le m, b-a=m, 0 \le a \le [(m-1)/2], 9m+2n \le v-u \le 12m, 4m+n \le u \le 4m+n+[(m-1)/2]$.
In the table 3, $2 \le n \le m, b=1, a=0, 8m+1+2n \le v-u \le 11m+1, u = 4m+n)$
In the table 4, $n=1, v-u=8m+b-a+r-n, u=4m+n+a)$

m	n	(v-u, u)	m	n	(v-u, u)
4	2	[40, 48] × [18, 19]	4	3	[42, 48] × [19, 20]
4	4	[44, 48] × [20, 21]	5	2	[49, 60] × [22, 24]
5	3	[51, 60] × [23, 25]	5	4	[53, 60] × [24, 26]
5	5	[55, 60] × [25, 27]	7	2	[67, 84] × [30, 33]
7	3	[69, 84] × [31, 34]	7	6	[75, 84] × [34, 37]
7	7	[77, 84] × [35, 38]	8	2	[76, 96] × [34, 37]
8	3	[78, 96] × [35, 38]	8	7	[86, 96] × [39, 42]
8	8	[88, 96] × [40, 43]	9	2	[85, 108] × [38, 42]
9	3	[87, 108] × [39, 43]	9	9	[99, 108] × [45, 49]
11	2	[103, 132] × [46, 51]	11	11	[121, 132] × [55, 60]

Table 2

m	n	(v-u, u)	m	n	(v-u, u)
4	2	[37, 45] × 18	4	3	[39, 45] × 19
4	4	[41, 45] × 20	5	2	[45, 56] × 22
5	3	[47, 56] × 23	5	4	[49, 56] × 24
5	5	[51, 56] × 25	11	2	[93, 122] × 46
11	3	[95, 122] × 47	11	10	[109, 122] × 54
11	11	[111, 122] × 55			

Table 3

m	b	a	r	(v-u, u)
5	5	0	6	(50, 21)

Table 4

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