On Skolem Labelling of Windmills

E. Mendelsohn*
Department of Mathematics
University of Toronto, Toronto
Ontario, Canada M5A 1A7

N. Shalaby
Department of Mathematics and Computer Science
Mount Allison University
Sackville, New Brunswick
Canada, E0A 3C0

ABSTRACT. In a previous work "Skolem labelled graphs" [4] we defined the Skolem labelling of graphs, here we prove that the necessary conditions are sufficient for a Skolem or minimum hooked Skolem labelling of all k-windmills. A k-windmill is a tree with k leaves each lying on an edge-disjoint path of length, m, to the centre. These paths are called the vanes.

1 Introduction

If one wishes to test the reliability of a communications network for node reliability, link reliability and distance reliability one would want a testing schedule so that

1. Every node is tested.
2. Every link is tested.
3. Every distance (in the network hop sense) is tested.

An efficient schedule for this leads to the concept of a Skolem labelled graph and where a graph does not have a Skolem labelling, a "hooked" – Skolem labelled graph with as few hooks (untested nodes) as possible. As a

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hub and spoke system is a common communication network configuration it makes sense to develop such schedules for these. In this case we restrict ourselves to spokes of equal length. In the language of trees these are k-windmills with vanes of length m. Skolem 1957 [11] introduced the concept of what is now called a Skolem sequence. It can be represented as a sequence S_j of numbers from $\{1, \dots, n\}$ of length $2n$ with each number appearing exactly twice. We define

$$\begin{aligned} a_j &= u \quad \text{if } S_u = j \text{ for the first time} \\ b_j &= v \quad \text{if } S_v = j \text{ for the second time} \end{aligned}$$

A sequence is Skolem if $b_j - a_j = j$, $j = 1, 2, \dots, n$. For example the sequence $4, 1, 1, 3, 4, 2, 3, 2$ has

$$\begin{aligned} b &= 3, 8, 7, 5 \\ a &= 2, 6, 4, 1 \end{aligned}$$

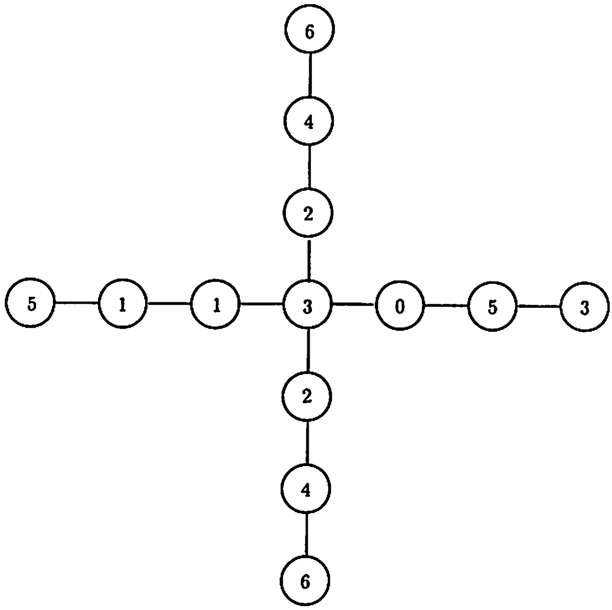
and thus is a Skolem Sequence. Informally the two 1's are 1 apart, the two 2's are 2 apart etc.

He also raised the question of the existence of similar sequences in the case $n \equiv 2, 3 \mod 4$ in which the sequence is of length $2n+1$ and a "hook" 0 is placed at S_{2n}. We normally write a "$*$" for "0". Thus the sequence $3\ 1\ 1\ 3\ 2\ *\ 2$ has $b = 3, 7, 4 \dots$; $a = 2, 5, 1$, and thus is hooked Skolem with a hook at 6.

O'Keefe [9] proved that for a hooked Skolem Sequence of order n to exist, n must be $\equiv 2, 3 \pmod 4$. The combined works of Skolem and O'Keefe were used to construct cyclic Steiner triple systems of order $v \equiv 1 \pmod 6$. Later, the same method was used by Rosa (1966) [10] for the case of $v \equiv 3 \pmod 6$. There are several generalizations and applications of Skolem sequences; for more details the reader may consult [8]. For more details about cyclic Steiner triple systems the reader may consult [2].

When we introduced the concept of Skolem labelling of graphs [4], we proved that any tree can be embedded in a Skolem labelled tree with $0(v)$ vertices and that this is sharp. Further, any graph can be embedded in an induced subgraph of a Skolem labelled graph with $0(v^3)$ vertices, and we exhibited the minimum embedding and labelling of all paths and cycles.

It is natural then to extend the investigation to finding the Skolem or minimum (hooked) Skolem labelling of trees. Initial discussions seem to lead to a conjecture as difficult as Ringel's graceful labelling of trees [5].



A minimum Skolem labelling of a 4-windmill with vanes of length 3

We restate the definitions of Skolem and hooked Skolem labelled graph.

Definition 1. *A d-Skolem labelled graph is a triple (G, L, d), where*

(a) $G = (V, E)$ *is an undirected graph*

(b) $L : V \rightarrow \{d, d+1, ..., d+n-1\}$

(c) $L(v) = L(w) = d+i$ *exactly once for* $i = 0, 1, \ldots, n-1$ *and* $d(v, w) = d+i$

(d) *If* $G' = (V, E')$ *and* $E' \subset E$ *then* (G', L, d) *violates (c).*

Definition 2. *A d-hooked Skolem labelled graph is a triple (G, L, d) satisfying Definition 1 with (b') substituted for (b):*

(b') $L : V \rightarrow \{0\} \cup \{d, d+1, \ldots, d+n-1\}$.

That is hooked sequences can have some vertices labelled 0 , but every edge is still (essential) i.e. the removal of that edge violates (c) . Note that if condition (d) in the above definitions is not satisfied it will be called a weak (hooked) Skolem labelled graph. A *minimum hooked* labelling of G is one with as few hooks as possible. A *k-windmill* is a tree with k leaves or endpoints (vertices of degree one) which are equidistant from a unique vertex of degree > 2 (the centre). The paths from the leaves to the centre are called *vanes.*

2 The necessary conditions

There are two necessary conditions for a tree to be Skolem labelled, called parity and degeneracy. In this section we establish the parity condition for a Skolem labelling for an arbitrary tree and a degeneracy condition for a (hooked) Skolem labelling for k-windmills.

The parity condition

We define the *Skolem parity* of a vertex u of a tree $T = (V, E)$, to be the sum (over all $v_i \in V$) $\sum_{v \in V} d(u, v) \mod 2$. If $|V|$ is even, this parity is independent of u and is called the Skolem parity of the tree.

Lemma 2.1. *Let $T = (V, E)$ be a tree with $2n$ vertices. Then the Skolem parity is independent of $u \in V$.*

Proof: Consider a vertex u and let $P_u = \sum_{v \in V} d(u, v) \mod 2$. Let w be adjacent to u, by edge $e = \{u, w\}$. The components of $T - e$ partition V into disjoint sets, A which contains u and B which contains w. If $x \in B$ then $d(x, u) = d(x, w) + 1$. If $x \in A$ then $d(x, u) = d(x, w) - 1$. Thus $P_u \equiv P_w + |B| - |A| \mod 2$. Since $|V|$ is even $|B| - |A|$ is even. Thus $P_u = P_w$.

Note that in case of a tree with an odd number of vertices the number $|B| - |A|$ is odd, so that adjacent vertices will be of opposite parity. (Hence for trees with an even number of vertices we can define the (Skolem) *parity of the tree* to be the parity of any of its vertices.

Lemma 2.2. *Let $T = (V, E)$ be a tree and V_1 and V_2 subsets of V satisfying $|V_1| = |V_2| + 2$ and $V_1 = V_2 \cup \{u, w\}$. For any $x \in V$ and $W \subseteq V$ define $D(x, W) = \sum_{v \in W} d(x, v) \mod 2$. Then for every $x \in V$*

$$D(x, V_1) - D(x, V_2) \equiv d(u, w) \mod 2.$$

Proof: For V_1 and V_2 as described we observe that

$$\begin{aligned} \sum_{v \in V_1} d(x, v) &= \sum_{v \in V_2} d(x, v) - d(x, u) + d(x, w) \\ &= \sum_{v \in V_2} d(x, v) + d(u, w) + 2d(x, z) \end{aligned}$$

where z is that vertex on the u, w path closest to x. The assertion follows immediately.

We observe that $D(x, v)$ equals the Skolem parity of T when $|V|$ is even.

Now we can prove the necessary parity condition for all trees.

Lemma 2.3. *Necessary conditions for the existence of a Skolem labelling of any tree with $2n$ vertices are as follows:*

1) *If $n \equiv 0, 3 \pmod 4$ the parity of T must be even.*

2) *if $n \equiv 1, 2 \pmod 4$ the parity of T must be odd.*

Proof: Assume vertices a_i, b_i are labelled i, $1 \le i \le n$, with $d(a_i, b_i) = i$. Apply Lemma 2.2 repeatedly, first with $V_1 = V$ and $V_2 = V - \{a_n, b_n\}$, with x chosen arbitrarily, to obtain $D(x, V) \equiv D(x, V - \{a_n, b_n\}) + n \bmod 2$. Repeat this process, removing successive pairs, to obtain $D(x, V) \equiv$ Skolem parity of $T = D(x, V - \{a_n, b_n, a_{n-1}, b_{n-1}\}) + n + (n-1) \equiv n(n+1)/2 \bmod 2$. The conditions follow immediately from these equalities.

The degeneracy conditions

It is obvious that a graph with $2n$ vertices must have a path of length n if it is to be Skolem labelled. Thus we see that all windmills with more than 4 vanes cannot have such a labelling.

For a (possibly hooked) Skolem labelling of a k-windmill, given that the largest label is $2m$, the maximum number of edges in the corresponding path, none used in any other path, is $2m$, this covering all edges of 2 vanes. Labels greater than m must cover parts of 2 vanes, with the label in each case acting as an upper bound on the number of edges used by that label and by no other label. The label m may cover all of a single vane. Thus for all labels m_i with $m \le m_i \le 2m$ the maximum number of edges covered is no more than

$$[2m + (2m-1) + \cdots + m] \;=\; \frac{3}{2}(m^2 + m). \tag{1}$$

In order to satisfy property (d), the labels that are $n_i \le m$ must cover at least one edge covered by another label, so that the total number of edges for these labels is at most

$$[(m-2) + \cdots + 1] \;=\; \frac{m^2 - 3m + 2}{2}. \tag{2}$$

Thus the maximum number of edges is $\le (1) + (2) \;=\; 2m^2 + 1$. Since the total number of edges in a k-windmill is km we have

$$km \;\le\; 2m^2 + 1 \quad or \quad k \;\le\; 2m,$$

k being an integer, except for the case $k = 3$ and $m = 1$. This is called the degeneracy condition.

3 Sufficiency

In this section we show that the above conditions necessary for obtaining the minimum (hooked) Skolem labelling for all n-windmills are sufficient except for a few small cases. For a k-windmill with k vanes we arbitrarily number the vanes (say clockwise) 1 to k, let m denote the length of the vane of the windmill, then to every vertex v we associate two coordinates (i, j) where i is the vane number and j is its distance from the centre, denote the vertex v_{ij}. In the following tables the centre of the windmill has the coordinates $(0,0)$ or $(i,0)$.

3-windmills

Lemma 3.1. *All 3-windmills with vane length* $\equiv 1, 7 \pmod 8$, *have a Skolem labelling.*

Proof:

Case (1) $m \equiv 1 \pmod 8$, $m > 9$

a_{ij}	b_{ij}	$\le r \le$	label
$(1, m-r+1)$	$(3, \frac{1}{2}(m+1)-r)$	$1 \le r \le \frac{1}{4}(m-1)$	$\frac{1}{2}(3m+3)-2r$
$(1, \frac{3}{4}(m-1)-r+2)$	$2, \frac{3}{4}((m-1)-r+2)$	$1 \le r \le \frac{1}{2}(m+3)$	$\frac{1}{2}(3m+5)-2r$
$(1, r)$	$(3, r+1)$	$1 \le r \le \frac{1}{4}(m-5)$	$2r+1$
$(2, r-1)$	$(2, m-r+1)$	$1 \le r \le \frac{1}{4}(m-1)$	$m-2r+2$
$(3, 1)$	$(3, \frac{1}{2}(m+3))$	–	$\frac{1}{2}(m+1)$
$(3, \frac{1}{2}(m+1))$	$(3, \frac{3}{4}(m-1)+1)$	–	$\frac{1}{4}(m-1)$
$(3, \frac{1}{2}(m+3)+r)$	$(3, m-r+1)$	$1 \le r \le \frac{1}{8}(m-9)$	$\frac{1}{2}(m-1)-2r$
$(3, \frac{1}{4}(3m+1)-r)$	$(3, \frac{1}{4}(3m+1)+r)$	$1 \le r \le \frac{1}{8}(m-9)$	$2r$
$(3, \frac{1}{8}(7m+9))$	$(3, \frac{1}{8}(7m+1))$	–	1

For $m = 1$

label $(1,1), (2,1)$ by 2; $(0,0), (3,1)$ by 1

For $m = 9$

label $(0,0), (3,9)$ by 9; $(3,1), (3,8)$ by 7;
$(1,1), (1,2)$ by 1; $(1,3), (3,3)$ by 6;
$(1,4), (3,4)$ by 8; $(1,5), (3,5)$ by 10;
$(1,6), (3,6)$ by 12; $(1,7), (3,7)$ by 14;
$(1,8), (2,3)$ by 11; $(1,9), (2,4)$ by 13;
$(3,2), (2,1)$ by 3; $(2,2), (2,7)$ by 5;
$(2,5), (2,9)$ by 4; $(2,6), (2,8)$ by 2.

Case (2) $m \equiv 7 \pmod 8$, $m > 7$

a_{ij}	b_{ij}	$\leq r \leq$	label
$(1, m-r+1)$	$(3, \frac{1}{2}(m+3)-r)$	$1 \leq r \leq \frac{1}{4}(m+1)$	$\frac{1}{2}(3m+5)-2r$
$(1, \frac{3}{4}(m+1)-r)$	$(2, \frac{3}{4}(m+1)-r)$	$1 \leq r \leq \frac{1}{2}(m+1)$	$\frac{1}{2}(3m+3)-2r$
$(1, r)$	$(3, r)$	$1 \leq r \leq \frac{1}{4}(m-3)$	$2r$
$(2, r-1)$	$(2, m-r+1)$	$1 \leq r \leq \frac{1}{4}(m+1)$	$m-2r+2$
$(3, \frac{1}{4}(m+1))$	$(3, \frac{1}{4}(3m-1))$	-	$\frac{1}{2}(m-1)$
$(3, \frac{1}{2}(m+3))$	$(3, \frac{1}{4}(3m+3))$	-	$\frac{1}{4}(m-3)$
$(3, \frac{1}{2}(m+3)+r)$	$(3, m-r+1)$	$1 \leq r \leq \frac{1}{8}(m-7)$	$\frac{1}{2}(m-1)-2r$
$(3, \frac{1}{4}(3m-1)-r)$	$(3, \frac{1}{4}(3m+3)+r)$	$1 \leq r \leq \frac{1}{8}(m-15)$	$2r+1$
$(3, \frac{7}{8}(m+1))$	$(3, \frac{1}{8}(7m-1)$	-	1

For $m = 7$

label $(1,7),(3,4)$ by 11 ; $(1,5),(2,5)$ by 10;
$(1,6),(3,3)$ by 9 ; $(1,4),(2,4)$ by 8;
$(0,0),(2,7)$ by 7; $(1,3),(2,3)$ by 6;
$(2,1),(2,6)$ by 5; $(3,2),(3,6)$ by 4;
$(2,2),(3,1)$ by 3; $(3,5),(3,7)$ by 2;
$(1,1),(1,2)$ by 1.

Lemma 3.2. *For all 3-windmills with vane length $m \equiv 0, 2, 4, 6 \pmod 8$, there is a minimum hooked Skolem labelling (i.e. with one hook), with the exception of $m = 2$.*

Proof:
Case (1) $m \equiv 2, 6 \pmod 8$, $m > 2$

a_{ij}	b_{ij}	$\leq r \leq$	label
$(1, 1)$	$(1, m)$	-	$(m-1)$
$(1, m-r)$	$(1, \frac{1}{2}m+r)$	$1 \leq r \leq \frac{1}{4}(m-6)$	$\frac{1}{2}m-2r$
$(1, r+1)$	$(3, \frac{1}{2}m+r-1)$	$1 \leq r \leq \frac{1}{4}(m-6)$	$\frac{1}{2}m+2r$
$(1, \frac{1}{4}(m+2))$	$(1, \frac{1}{4}(3m+2))$	-	$\frac{1}{2}m$
$(1, \frac{1}{2}(m+2)-r)$	$(3, m-r+1)$	$1 \leq r \leq \frac{1}{4}(m-2)$	$\frac{3}{2}m-2r+2$
$(3, \frac{1}{4}(3m+2))$	$(2, \frac{1}{4}(m+2))$	-	$m+1$
$(2, \frac{1}{4}(m+2)-r)$	$(2, \frac{1}{4}(m+2)+r)$	$1 \leq r \leq \frac{1}{4}(m+2)$	$2r$
$(3, r)$	$(2, \frac{1}{2}m+r+1)$	$1 \leq r \leq \frac{1}{2}m-1$	$\frac{1}{2}m+2r+1$
$(3, \frac{1}{4}(3m-6))$	$(3, \frac{1}{4}(3m-2))$	-	1

$m = 2$ does not satisfy (d).

Case (2) $m \equiv 0, 4 \pmod 8$, $m > 4$

a_{ij}	b_{ij}	$\leq r \leq$	label
$(1, r)$	$(3, \frac{1}{2}m + r + 1)$	$1 \leq r \leq \frac{1}{4}m - 1$	$\frac{1}{2}m + 2r + 1$
$(3, \frac{1}{2}m + 1)$	$(3, m - 1)$	–	$\frac{1}{2}m - 2$
$(2, 1)$	$(3, m)$	–	$m + 1$
$(1, m)$	$(2, \frac{1}{2}m - 1)$	–	$3m/2 - 1$
$(3, m - r - 1)$	$(1, \frac{1}{2}m - r)$	$1 \leq r \leq \frac{1}{4}m - 2$	$3m/2 - 2r - 1$
$(2, \frac{1}{4}m - r)$	$(2, \frac{1}{4}m + r)$	$1 \leq r \leq \frac{1}{4}m - 2$	$2r$
$(3, r)$	$(2, \frac{1}{2}m + r)$	$0 \leq r \leq \frac{1}{2}m$	$\frac{1}{2}m + 2r$
$(1, \frac{1}{4}m + r - 1)$	$(1, \frac{3}{4}m - r + 2)$	$1 \leq r \leq 2$	$\frac{1}{2}m - 2r + 3$
$(1, 3m/4 + r + 1)$	$(1, \frac{3}{4}m - r)$	$1 \leq r \leq \frac{1}{4}m - 2$	$2r + 1$
$(1, \frac{1}{2}m)$	$(1, \frac{1}{2}m + 1)$	–	1

For $m = 4$

label $(2,2), (1,4)$ by 6; $(2,4), (3,1)$ by 5;
$(0,0), (3,4)$ by 4; $(3,2), (1,1)$ by 3;
$(2,1), (2,3)$ by 2; $(1,2), (1,3)$ by 1.

Lemma 3.3. *All 3-windmills with vane length* $\equiv 3, 5 \pmod 8$, *have a minimum hooked Skolem labelling with 2 hooks, with the exception of* $m = 3$.

Proof: In this case $m = 3$ does not satisfy condition (d).

We subdivide into two cases

Case (1) $m \equiv 3 \pmod 8$, $m > 3$

a_{ij}	b_{ij}	$\leq r \leq$	label
$(2, \frac{1}{4}(m+1) - r)$	$(2, \frac{1}{4}(m+1) + r)$	$1 \leq r \leq \frac{1}{4}(m-3)$	$2r$
$(1, \frac{1}{4}(3m-1))$	$(2, \frac{1}{4}(m+1))$	–	m
$(2, m - r + 1)$	$(3, \frac{1}{2}(m+1) - r)$	$1 \leq r \leq \frac{1}{2}(m+1)$	$\frac{3}{2}(m+1) - 2r$
$(1, r)$	$(3, \frac{1}{2}(m-1) + r)$	$1 \leq r \leq \frac{1}{4}(m-3)$	$\frac{1}{2}(m-1) + 2r$
$(3, m - r + 1)$	$(1, \frac{1}{2}(m-1) - r)$	$1 \leq r \leq \frac{1}{4}(m-3)$	$\frac{3}{2}(m-1) - 2r + 2$
$(1, m - r + 1)$	$(1, \frac{1}{2}(m-1) + r)$	$1 \leq r \leq \frac{1}{4}(m-3)$	$\frac{1}{2}(m+3) - 2r$
$(3, \frac{1}{4}(3m-1))$	$(3, \frac{1}{4}(3m+3))$	–	1

Case (2) $m \equiv 5 \pmod 8$ and $m > 5$

a_{ij}	b_{ij}	$\leq r \leq$	label
$(2, \frac{1}{4}(m+3) - r)$	$(2, \frac{1}{4}(m+3) + r)$	$1 \leq r \leq \frac{1}{4}(m-1)$	$2r$
$(1, \frac{1}{4}(3m+5))$	$(2, \frac{1}{4}(m+3))$	–	$m + 2$
$(2, m - r + 1)$	$(3, \frac{1}{2}(m-1) - r)$	$1 \leq r \leq \frac{1}{2}(m-1)$	$\frac{3}{2}(m-1) - 2r + 2$
$(1, r)$	$(3, \frac{1}{2}(m-3) + r)$	$1 \leq r \leq \frac{1}{4}(m+3)$	$\frac{1}{2}(m+1) + 2r - 2$
$(3, m - r + 1)$	$(1, \frac{1}{2}(m+1) - r)$	$1 \leq r \leq \frac{1}{4}(m-5)$	$\frac{3}{2}(m+1) - 2r$
$(1, m - r + 1)$	$(1, \frac{1}{2}(m+1) + r)$	$1 \leq r \leq \frac{1}{4}(m-5)$	$\frac{1}{2}(m+1) - 2r$
$(3, \frac{1}{4}(3m+1))$	$(3, \frac{1}{4}(3m+5))$	–	1

For $m = 5$

label	$(1,5),(2,2)$ by 7;	$(2,5),(3,1)$ by 6;
	$(0,0),(3,5)$ by 5;	$(1,1),(2,3)$ by 4;
	$(2,1),(3,2)$ by 3;	$(3,3),(3,4)$ by 1.
	$(1,2),(1,4)$ by 2	

4-windmills

All 4-windmills have odd number of vertices. So the minimum hooked Skolem labelling in this case has at least one hook.

Lemma 3.4. *All 4-windmills with $m > 2$, have a minimum hooked Skolem labelling with exactly one hook.*

Proof: All the following cases have this construction in common: In vanes 2, 4 we distribute the even numbers as follows

a_{ij}	b_{ij}	$\leq r \leq$	label
$(4,r)$	$(2,r)$	$1 \leq r \leq m$	$2r$

Case (1) $m \equiv 0 \pmod 3$

a_{ij}	b_{ij}	$\leq r \leq$	label
$(3,m-r+1)$	$(1,m-r)$	$1 \leq r \leq 2m/3-1$	$2m-2r+1$
$(1,m)$	$(1,(m-3)/3)$	-	$(2m+3)/3$
$(1,r-2)$	$(3,1+r)$	$1 \leq r \leq m/3$	$2r-1$

Note that $(1,-x)$ is the same as $(3,x)$.

Case (2) $m \equiv 1 \pmod 3$

a_{ij}	b_{ij}	$\leq r \leq$	label
$(3,m-r+1)$	$(1,m-r)$	$1 \leq r \leq 2(m-1)/3$	$2m-2r+1$
$(1,m)$	$(1,(m-1)/3)$	-	$(2m+1)/3$
$(1,r-2)$	$(3,1+r)$	$1 \leq r \leq (m-1)/3$	$2r-1$

Case (3) $m \equiv 2 \pmod 6$, $m > 2$

a_{ij}	b_{ij}	$\leq r \leq$	label
$(3,m-r+1)$	$(1,m-r)$	$1 \leq r \leq m/2$	$2m-2r+1$
$(1,m)$	$(1,1)$	-	$m-1$
$(3,\frac{1}{2}m-1)$	$(3,\frac{1}{2}m)$	-	1
$(3,r)$	$(1,r+1)$	$1 \leq r \leq \frac{1}{2}m-2$	$2r+1$

Case (4) $m \equiv 5 \pmod 6$

a_{ij}	b_{ij}	$\leq r \leq$	label
$(3, m-r+1)$	$(1, m-r)$	$1 \leq r \leq \frac{1}{2}(m-1)$	$2m-2r+1$
$(1, m)$	$(0,0)$	–	m
$(3, \frac{1}{2}(m-1))$	$(3, \frac{1}{2}(m+1))$	–	1
$(3, r)$	$(1, r+1)$	$1 \leq r \leq \frac{1}{2}(m-3)$	$2r+1$

k-windmills $k > 4$

In this case there is no Skolem labelling, thus the only possibility is a minimum hooked Skolem labelling.

Lemma 3.5. *For any k-windmill the condition $k \leq 2m$ is sufficient for a minimum hooked Skolem labelling.*

Proof: Fix m

Case (1) $k = 2t$, $k \leq 2m$

Label the vanes $L_1, L_{2m}, L_2, L_{2m-1}, \ldots, L_t, L_{2m+1-t}$. For $k < 2m$:

a_{ij}	b_{ij}	$\leq r \leq$	label
$(2m-r, m)$	$(r+1, m-r)$	$0 \leq r \leq t-1$	$2m-r$
(r, m)	$(r, m-r)$	$2 \leq r \leq t$	r
$(2m, r)$	$(1, r)$	$t+1 \leq 2r \leq 2m-t$	$2r$
$(2m-1, r)$	$(2m-2, r+1)$	$t+1 \leq 2r+1 \leq 2m-t$	$2r+1$
$(0,0)$	$(2m-1, 1)$	–	1

For $k = 2m$, use only the first two rows, with label 1 in $(1,1)$, $(1,2)$.

Case(2) $k = 2t+1$, $k \leq 2m-1$, $t > 2$.

Label the vanes $L_1, L_{2m}, L_2, L_{2m-1}, \ldots, L_t, L_{2m+1-t}, L_{2m-t}$.

a_{ij}	b_{ij}	$\leq r \leq$	label
$(2m-r, m)$	$(r+1, m-r)$	$0 \leq r \leq t-1$	$2m-r$
(r, m)	$(r, m-r)$	$2 \leq r \leq t$	r
$(2m-t, m)$	$(2, m-t)$	–	$2m-t$
$(2m, r)$	$(1, r)$	$t+1 \leq 2r \leq 2m-t-1$	$2r$
$(2m-1, r)$	$(2m-2, r+1)$	$t+1 \leq 2r+1 \leq 2m-t-1$	$2r+1$
$(0,0)$	$(2m-1, 1)$	–	1

For $k = 2m-1$ use only one of the 4^{th} or 5^{th} lines of the table, depending on the parity of m.

Case (3) $k = 5$.

Label the vanes $L_1, L_{2m}, L_2, L_{2m-1}, L_{2m-2}$.

a_{ij}	b_{ij}	$\leq r \leq$	label
$(2m-r, m)$	$(r-1, m-r)$	$0 \leq r \leq 1$	$2m-r$
$(2m-2, m)$	$(2m, m-2)$	-	$2m-2$
$(2, m)$	$(2, m-2)$	-	2
$(2m, r-1)$	$(1, r+1)$	$3 \leq 2r \leq 2m-3$	$2r$
$(2m-1, r)$	$(2m-2, r+1)$	$1 \leq 2r+1 \leq 2m-3$	$2r+1$

This completes the proof.

Note that the case $k = 2$ is included in the results obtained in [4].

Thus we have,

Theorem I. *All n-windmills satisfying $k \leq 2m$ can be Skolem labelled or hooked Skolem labelled with the minimum number of hooks. With the exceptions:*

$$\begin{aligned} k &= 2, \quad m = 1,2,3,4 \\ k &= 3, \quad m = 1,2,3 \\ k &= 4, \quad m = 1,2 \end{aligned}$$

There are many scattered results on special classes of trees such as caterpillars, or trees with exactly one vertex of maximum degree 3, but this is the first class to be completely settled.

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