# Codes of $C(n, n, 1)$ Designs

Chester J. Salwach
Department of Mathematics
Lafayette College
Easton, PA 18042

ABSTRACT. We associate codes with $C(n, n, 1)$ designs. The perfect $C(n, n, 1)$ designs obtained from perfect one-factorizations of $K_n$ yield codes of dimension $n-2$ over $\mathbb{F}_2$ and $n-1$ over $\mathbb{F}_p$ for $p \neq 2$. We also demonstrate a method of obtaining another $C(n, n, 1)$ design from a pair of isomorphic perfect $C(n, n, 1)$ designs and determine the dimensions of the resulting codes.

## 1 Introduction

A $C(n, n, 1)$ cycle design consists of a collection of $n$-cycles selected from the complete graph on $n$ vertices, $K_n$, so that each vertex occurs between each possible pair of vertices in precisely one $n$-cycle. This problem has had a long history stemming back to Judson [3] and Dudeney (see Problem 273 in [2]). They formulated it as a problem of seating $n$ persons at a round table on $(n-1)(n-2)/2$ days so that no person sat twice between the same pair of companions. Solutions are known for sporadic odd values of $n$ and recently Kobayashi, Kiyasu-Zen'iti, and Nakamura [4] constructed a solution for every even $n$.

Although perfect one-factorizations of $K_n$ are conjectured to exist for every even $n$, they are known to exist whenever $n$ is either one plus or twice an odd prime, for all even $n$ up to 50, and for a few other even values of $n$. See Anderson [1] and Wallis [6] for surveys on one-factorizations.

Perfect $C(n, n, 1)$ designs are constructed from perfect one-factorizations. In this paper, we investigate a code that we associate with these and other $C(n, n, 1)$ designs. The perfect $C(n, n, 1)$ designs yield codes of dimension $n-2$ over $\mathbb{F}_2$ and $n-1$ over $\mathbb{F}_p$ for $p \neq 2$. We also demonstrate a method of obtaining a different $C(n, n, 1)$ design from a pair of isomorphic perfect $C(n, n, 1)$ designs and determine the dimensions of the resulting codes. Basis vectors for these codes are provided as well.

## 2 Definitions and Remarks

Let $k \geq 3$. A $k$-cycle $(v_1, v_2, v_3, \ldots, v_{k-1}, v_k)$ in $K_n$ consists of the edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_1\}$ where the $k$ vertices are distinct. Note that there are $2k$ ways of writing the same $k$-cycle. A *2-path* (path of length 2) $a - b - c$ consists of the two edges $\{a, b\}$ and $\{b, c\}$ where $a$ and $c$ are distinct vertices. We take $a - b - c = c - b - a$. The number of 2-paths in $K_n$ is $n(n-1)(n-2)/2$. A $C(n, k, \lambda)$ *design* on a set of $n$ vertices consists of a collection, $\mathcal{D}$, of $k$-cycles in $K_n$ so that each 2-path in $K_n$ occurs in precisely $\lambda$ elements of $\mathcal{D}$. Thus, for a $C(n, n, 1)$ design, the number of $n$-cycles is $|\mathcal{D}| = (n-1)(n-2)/2$.

Let $n$ be even for the remainder of this paper.

A *one-factor* of $K_n$ consists of $n/2$ disjoint edges. A *one-factorization* of $K_n$ consists of $n-1$ disjoint one-factors. That is, a one-factorization partitions the edge set of $K_n$ into $n-1$ one-factors, each of which partitions the vertex set of $K_n$ into $n/2$ edges. A one-factorization of $K_n$ is *perfect* if the union of any two of its one-factors is an $n$-cycle. A perfect one-factorization of $K_n$, $\mathcal{F} = \{F_i \mid 1 \leq i \leq n-1\}$ where each $F_i$ is a one-factor, thereby gives rise to a collection,

$$\mathcal{D} = \{F_i \cup F_j \mid F_i, F_j \in \mathcal{F}, 1 \leq i < j \leq n-1\},$$

of $\binom{n-1}{2}$ $n$-cycles that comprise a $C(n, n, 1)$ design. We shall call such a cycle design a *perfect* $C(n, n, 1)$ design.

The *incidence matrix* $M$ of a $C(n, n, 1)$ design is an $\binom{n-1}{2} \times \binom{n}{2}$ matrix $(m_{ij})$ whose rows are indexed by the $n$-cycles of the design and whose columns are indexed by the edges of $K_n$ where $m_{ij}$ equals 1 if edge $j$ is in $n$-cycle $i$ and 0 otherwise.

A *linear $(m, l)$ code $A$* over the finite field $\mathbb{F}_p$, $p$ a prime, is an $l$-dimensional subspace of $\mathbb{F}_p^m$, the vector space of all $m$-tuples with entries from $\mathbb{F}_p$. The elements of $A$ are called *code words* or *vectors*. For $a \in A$, the *weight* of $a$ is

$$wgt(a) = |\{i \mid a_i \neq 0\}|;$$

that is, the number of nonzero components of the vector $a = (a_1, a_2, \ldots, a_m)$.

Let $M$ be the incidence matrix of a $C(n, n, 1)$ design. The *p-rank* of $M$, denoted by $Rk_p(M)$, is the dimension over $\mathbb{F}_p$ of the rowspace of $M$; that is, the rows of $M$ span an $(m, l) = (\binom{n}{2}, Rk_p(M))$ code over $\mathbb{F}_p$, which we shall denote by $A_p$ or $SP_p(M)$.

## 3 Codes of Perfect $C(n, n, 1)$ Designs

Say $n \geq 4$ is an even integer for which there exists a perfect one-factorization of $K_n$.

**Theorem 1.** *Let $\mathcal{F} = \{F_i \mid 1 \leq i \leq n-1\}$ be a perfect one-factorization of $K_n$ and let $M$ be the incidence matrix of the resulting perfect cycle design.*

(1) *If $p = 2$, then $Rk_2(M) = n-2$ and the $n$-cycles*

$$\mathcal{B} = \{F_1 \cup F_j \mid 2 \leq j \leq n-1\}$$

*form a basis for the code $A_2 = SP_2(M)$.*

(2) *If $p \neq 2$, then $Rk_p(M) = n-1$ and $\mathcal{F}$ forms a basis for the code $A_p = SP_p(M)$.*

**Proof:** The perfect cycle design consists of $\mathcal{D} = \{F_i \cup F_j \mid 1 \leq i < j \leq n-1\}$.

(1) The elements of $\mathcal{B}$ form a linearly independent set of $n-2$ vectors and, since

$$(F_1 \cup F_i) + (F_1 \cup F_j) = F_i \cup F_j \pmod 2,$$

each of the $n$-cycles in $\mathcal{D}$ is obtained.

(2) It suffices to show that $\mathcal{F} \subset A_p$. Well, in $\mathbb{F}_p$, with $p \neq 2$,

$$2^{-1}((F_1 \cup F_2) - (F_2 \cup F_3) + (F_1 \cup F_3)) = 2^{-1}(2F_1) = F_1 \in A_p.$$

Thus

$$(-1)F_1 + (F_1 \cup F_j) = F_j \in A_p,$$

for $2 \leq j \leq n-1$.

**Theorem 2.** *Again, let $M$ be the incidence matrix of a perfect cycle design $\mathcal{D}$. The weights of the vectors in the code $A_2 = SP_2(M)$ consist of $\{jn \mid 0 \leq j \leq \frac{n-2}{2}\}$. Moreover, the number of vectors of weight $jn$ in $A_2$ is $\binom{n-2}{2j-1} + \binom{n-2}{2j}$ for $1 \leq j \leq \frac{n-2}{2}$.*

**Proof:** From Theorem 1, $\mathcal{B} = \{F_1 \cup F_j \mid 2 \leq j \leq n-1\}$ is a basis for $A_2$. Now, $|F_i| = \frac{n}{2}$, for $1 \leq i \leq n-1$. Thus the $\pmod 2$ sum of any $2j-1$ of these basis vectors is of weight $|F_1| + (2j-1)\frac{n}{2} = (2j)\frac{n}{2} = jn$. But $F_1$ does not appear in the $\pmod 2$ sum of an even number of basis vectors, and so the $\pmod 2$ sum of $2j$ of these basis vectors is of weight $(2j)\frac{n}{2} = jn$ as well.

Note that the number of weight-$n$ vectors in $SP_2(M)$ is $\binom{n-2}{1} + \binom{n-2}{2} = \binom{n-1}{2} = |\mathcal{D}|$ and so the $n$-cycles of the perfect $C(n, n, 1)$ design comprise the set of weight-$n$ vectors of the code it spans. Also, the $\binom{n-2}{n-3} + \binom{n-2}{n-2} =$

$\binom{n-1}{n-2} = n-1$ weight-$\frac{(n-2)n}{2}$ vectors of the code are the complements of the one-factors in the perfect one-factorization.

Now $SP_2(M)$ is of codimension 1 in the code over $\mathbb{F}_2$ spanned by the vectors in the perfect one-factorization $\mathcal{F}$. $\{\frac{jn}{2} \mid 0 \leq j \leq n-1\}$ contains the weights of the vectors in this code and the number of weight-$\frac{jn}{2}$ vectors is $\binom{n-1}{j}$ since the (mod 2) sum of $j$ disjoint vectors of weight-$\frac{n}{2}$ yields a vector of weight-$\frac{jn}{2}$. The additional vectors in this larger code consist of the complements of the vectors in $SP_2(M)$. By Theorem 2, the number of weight-$jn$ vectors in $SP_2(M)$ is $\binom{n-2}{2j-1} + \binom{n-2}{2j}$ and this equals $\binom{n-1}{2j}$, naturally obtaining the weight-$jn$ vectors in the larger code as the (mod 2) sum of $2j$ of the one-factors.

## 4 Imperfect $C(n, n, 1)$ Designs

In this section, let $n \geq 6$ be an even integer for which there exists a perfect one-factorization of $K_n$.

**Lemma 1.** *Let $\mathcal{F}$ be a perfect one-factorization of $K_n$. Apply the transposition $(a, b)$ to each one-factor in $\mathcal{F}$ and thereby obtain an isomorphic perfect one-factorization, call it $\mathcal{G}$. Say $(a, b)$ sends $F_i$ to $G_i$, for $1 \leq i \leq n-1$. Then $\mathcal{F}$ and $\mathcal{G}$ have exactly one one-factor in common, call it $F_1 = G_1$.*

**Proof:** The transposition appears as an edge in a single one-factor of $\mathcal{F}$, say $F_1$. $(a, b)$ sends $F_1$ to itself, called $G_1$ in $\mathcal{G}$. Each other one-factor of $\mathcal{F}$ has $a$ and $b$ in separate edges with additional edge(s) containing neither, and so $(a, b)$ sends it to a one-factor different from any of those in $\mathcal{F}$.

**Lemma 2.** *Let $\mathcal{F}$ and $\mathcal{G}$ be as in Lemma 1. The resulting isomorphic perfect $C(n, n, 1)$ designs*

$$\mathcal{D}_{\mathcal{F}} = \{F_i \cup F_j \mid 1 \leq i < j \leq n-1\}$$

*and*

$$\mathcal{D}_{\mathcal{G}} = \{G_i \cup G_j \mid 1 \leq i < j \leq n-1\}$$

*have no $n$-cycles in common.*

**Proof:** If they shared an $n$-cycle, then $(a, b)$ applied to a certain $n$-cycle in $\mathcal{D}_{\mathcal{F}}$ would yield the common $n$-cycle which would therefore be in $\mathcal{D}_{\mathcal{F}}$ as well. If $a$ and $b$ are not opposite vertices in these two $n$-cycles, then the $n$-cycles would have common 2-paths and therefore cannot both be in $\mathcal{D}_{\mathcal{F}}$. If $a$ and $b$ are opposite vertices and $n > 6$, then again they share 2-paths, leading to a contradiction.

In the final case, $n = 6$ and $a$ and $b$ are opposite vertices. The two 6-cycles would be of the form $H_1 = (a, c, d, b, e, f)$ and $H_2 = (a, b)H_1 = (b, c, d, a, e, f)$. For completeness, the argument showing that $H_1$ and $H_2$

cannot both be in $\mathcal{D}_{\mathcal{F}}$ that appeared in Lemma 1 of [5] will be repeated here. If $H_1, H_2 \in \mathcal{D}_{\mathcal{F}}$, the 2-path $a - f - b$ must be part of some other 6-cycle $H$ in $\mathcal{D}_{\mathcal{F}}$. Since 2-path $c - a - f$ is in $H_1$ and $f - b - c$ is in $H_2$, vertex $c$ must be opposite to vertex $f$ in $H$. But this is impossible since vertex $d$ would then occur between either vertices $a$ and $c$, or $b$ and $c$, of $H$, and 2-path $a - d - c$ is already in $H_2$ and 2-path $b - d - c$ is already in $H_1$.

**Lemma 3.** *Let $\mathcal{D}_{\mathcal{F}}$ and $\mathcal{D}_{\mathcal{G}}$ be as in Lemma 2. The 2-paths in*

$$\{F_1 \cup F_j \mid 2 \le j \le n-1\}$$

*are the same as the 2-paths in*

$$\{G_1 \cup G_j \mid 2 \le j \le n-1\}.$$

**Proof:** Recall $F_1 = G_1$. Say 2-path $x - y - z$ occurs in some $F_1 \cup F_j$. Either $\{x, y\}$ or $\{y, z\}$ is in $F_1$. Say it is $\{x, y\}$. Then $\{x, y\}$ is in $G_1$. Since $\mathcal{G}$ is a one-factorization, $\{y, z\}$ is in some $G_k$. Thus $x - y - z$ is in $G_1 \cup G_k$.

**Theorem 3.** *Let $\mathcal{D}_{\mathcal{F}}$ and $\mathcal{D}_{\mathcal{G}}$ be as in Lemma 2. The $n$-cycles in*

$$\{F_1 \cup F_j \mid 2 \le j \le n-1\} \cup \{G_i \cup G_j \mid 2 \le i < j \le n-1\}$$

*form a $C(n, n, 1)$ design which we call an imperfect cycle design denoted by $\mathcal{D}_{\mathcal{F}\mathcal{G}}$.*

**Proof:** $\mathcal{D}_{\mathcal{G}}$ is a $C(n, n, 1)$ design, and by Lemma 3, the 2-paths in

$$\{G_1 \cup G_j \mid 2 \le j \le n-1\}$$

are the same as those in

$$\{F_1 \cup F_j \mid 2 \le j \le n-1\}.$$

Thus this new collection of $n$-cycles still contains each 2-path precisely once.

Recall from Theorem 1 that the $n$-cycles $\{F_1 \cup F_j \mid 2 \le j \le n-1\}$ form a basis for the code over $\mathbb{F}_2$ spanned by the $n$-cycles of $\mathcal{D}_{\mathcal{F}}$. Thus the perfect cycle design $\mathcal{D}_{\mathcal{F}}$ is contained in the code over $\mathbb{F}_2$ spanned by the $n$-cycles of the imperfect design $\mathcal{D}_{\mathcal{F}\mathcal{G}}$.

**Theorem 4.** *Let $M$ be the incidence matrix of an imperfect $C(n, n, 1)$ design $\mathcal{D}_{\mathcal{F}\mathcal{G}}$.*

(1) *If $p = 2$, then $Rk_2(M) = 2n - 6$.*

(2) *If $p$ does not divide $(n-2)$, then $Rk_p(M) = 2n - 4$.*

(3) *If $p \neq 2$ but $p$ divides $(n-2)$, then $Rk_p(M) = 2n-5$.*

**Proof:** We exhibit a basis for the rowspace of $M$ in each of these cases.

**(1)** $\{G_2 \cup G_j \mid 3 \leq j \leq n-1\}$ will generate the other $n$-cycles from $\mathcal{D}_G$ that are in $\mathcal{D}_{\mathcal{F}G}$ since

$$(G_2 \cup G_i) + (G_2 \cup G_j) = G_i \cup G_j \pmod 2.$$

Note that the complement of $F_1 = G_1$ is $\bigcup_{j=2}^{n-1} F_j = \bigcup_{j=2}^{n-1} G_j$ so that

$$\sum_{j=2}^{n-1}(F_1 \cup F_j) + \sum_{j=3}^{n-1}(G_2 \cup G_j) = \bigcup_{j=2}^{n-1} F_j + \bigcup_{j=2}^{n-1} G_j = 0 \pmod 2.$$

Thus a basis consists of

$$\{F_1 \cup F_j \mid 2 \leq j \leq n-1\} \cup \{G_2 \cup G_j \mid 3 \leq j \leq n-2\}$$

and so $Rk_2(M) = (n-2) + (n-4) = 2n-6$.

**(2)** In $\mathbb{F}_p$, where $p$ does not divide $(n-2)$,

$$2^{-1}((G_2 \cup G_3) - (G_3 \cup G_4) + (G_2 \cup G_4)) = 2^{-1}(2G_2) = G_2 \in SP_p(M).$$

Thus

$$(-1)G_2 + (G_2 \cup G_j) = G_j \in SP_p(M),$$

for $3 \leq j \leq n-1$. Also,

$$\sum_{j=2}^{n-1}(F_1 \cup F_j) - \sum_{j=2}^{n-1} G_j = (n-2)F_1 + \bigcup_{j=2}^{n-1} F_j - \bigcup_{j=2}^{n-1} G_j = (n-2)F_1.$$

So $F_1 \in SP_p(M)$ and also

$$(-1)F_1 + (F_1 \cup F_j) = F_j \in SP_p(M),$$

for $2 \leq j \leq n-1$. Thus a basis consists of

$$\{F_j \mid 1 \leq j \leq n-1\} \cup \{G_j \mid 2 \leq j \leq n-2\}$$

and so $Rk_p(M) = (n-1) + (n-3) = 2n-4$.

**(3)** Just as in case (2) we obtain $G_j \in SP_p(M)$, for $2 \leq j \leq n-1$. But now $p$ divides $(n-2)$ and so

$$\sum_{j=2}^{n-1}(F_1 \cup F_j) - \sum_{j=2}^{n-1} G_j = 0 \pmod p.$$

Thus a basis consists of

$$\{F_1 \cup F_j \mid 2 \leq j \leq n-1\} \cup \{G_j \mid 2 \leq j \leq n-2\}$$

and so $Rk_p(M) = (n-2) + (n-3) = 2n-5$.

**References**

[1] L.D. Anderson, Factorizations of Graphs, in C.J. Colbourn and J.H. Dinitz (eds.), *The CRC Handbook of Combinatorial Designs*, CRC Press, New York, 1996, pp. 653–667.

[2] H.E. Dudeney, *Amusements in Mathematics*, Nelson, London, 1917; Dover Reprint, New York, 1970.

[3] C.H. Judson, Problems for Solution (Algebra), *Amer. Math. Monthy* **6** (1899), 92.

[4] M. Kobayashi, Kiyasu-Zen'iti, and G. Nakamura, A Solution of Dudeney's Round Table Problem for an Even Number of People, *J. Combinatorial Theory A* **63** (1993), 26–42.

[5] C.J. Salwach, An Oval in $PG(2,4)$ and $C(6,6,1)$ Designs, *Ars Combinatoria* **36** (1993), 233–239.

[6] W.D. Wallis, One-Factorizations of Complete Graphs, in J.H. Dinitz and D.R. Stinson (eds.), *Contemporary Design Theory: A Collection of Surveys*, John Wiley & Sons, New York, 1992, pp. 593–631.