Minimal Partitions of a Graph

Thomas Dale Porter[1]
Department of Mathematics
Southern Illinois University
Carbondale, Illinois 62901 USA

Abstract. For a given graph G, we fix s, and partition the vertex set into s classes, so that any given class contains *few* edges. The result gives a partition $(U_1, \cdots, U_s)$, where $e(U_i) \leq \frac{e(G)}{s^2} + 4s\sqrt{e(G)}$ for each $1 \leq i \leq s$. The error term is compared to previous results for $s = 2^P$ [6], and to a result by Bollobás and Scott [1].

1. Introduction.

For our purposes, graphs are finite and simple. We use the standard notation as in [2]. For a given graph G, let $(U_1, \cdots, U_s)$ denote a partition of $V(G)$ into s-classes; we also refer to $(U_1, \cdots, U_s)$ as an s-coloring of $V(G)$, where U_j denotes the vertices colored j. Let $e(U_i)$ denote the number of edges in the induced subgraph $G[U_i]$ and let $\gamma_s(U_1, \cdots, U_s) = \max_{1 \leq i \leq s} \{e(U_i)\}$. The problem is to minimize γ_s over all partitions $(U_1, \cdots, U_s)$; define $\gamma_s(G) = \min_{(U_1, \cdots, U_s)} \gamma_s(U_1, \cdots, U_s)$.

Paul Erdös conjectured [3] $\gamma_2(G) \leq \frac{e(G)}{4} + O\sqrt{e(G)}$, where $e(G)$ denotes the number of edges in G. In [4], the present author verified the conjecture and showed it was best possible. Roger Entringer posed the problem to find $\gamma_2(G)$ and proposed a related matrix discrepancy problem. The solution of the matrix problem, Porter and Székely [5], gives a bound asymptotic to $\gamma_s(G)$, however it did not lead to a solution of the partition problem. In [6], the present author gives an upper bound for $\gamma_s(G)$ when s is a power of 2, i.e., $s = 2^P$. In [1], Bollobás and Scott, using a probabilistic technique, give various upper bounds on $\gamma_s(G)$ for any s. In this paper we use a non-constructive, non-probabilistic technique that gives an upper bound for $\gamma_s(G)$ that depends on the size of G. In [7], Shahrokhi and Székely show that the computation of γ_s is NP-hard.

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Let $e[U_1, \cdots, U_s] = |\{x_i x_j | x_i x_j \in E(G), x_i \in U_i, x_j \in U_j\}|$, and define $M_s(U_1, \cdots, U_s) = e[U_1, \cdots, U_s]$, and $M_s(G) = \max_{(U_1, \cdots, U_s)} M_s(U_1, \cdots, U_s)$. We refer to $M_s(G)$ as the max $s-cut\,of\,G$. Maximum s-cuts give some useful partition properties and we state two results from [6]. For $U \in V(G), H \subset V(G)$ define $d_H(U)$ to be the number of vertices in H adjacent to U.

Lemma A. [6] For a graph G, and a partition $(U_1, \cdots, U_s)$ of $V(G)$ that gives the max s-cut, $M_s(G)$, for any $U_i, U_j \in (U_1, \cdots, U_s), i \neq j, e[U_i, U_j] \geq 2\max\{e(U_i), e(U_j)\}$.

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Let $(U_1, \cdots, U_s)$ be a partition where $M_s(U_1, \cdots, U_s) = M_s(G)$, by Lemma A then $e[U_1, \bigcup_{j=2}^{s-1} U_j] \geq 2(s-1)e(U_1)$. We now define a partition.

Definition 1. Let $(A, X_1, \cdots, X_{s-1})$ denote a partition with $e(A) \geq e(X_i), 1 \leq i \leq s-1$, and $e[A, \bigcup_{j=1}^{s-1} X_j] \geq 2(s-1)e(A)$, and over all such partitions $e(A)$ is minimal.

The following proof is done by induction, the ground case $s = 2$ is provided in [6], and we insert that here.

Theorem 1. [4] For any graph G, there exists a bipartition $(A, \bar{A})$ of $V(G)$, so that $\gamma_2(A, \bar{A}) \leq \frac{1}{4}\left(e(G) + \sqrt{2e(G)}\right)$.

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Theorem 2. For a given graph G, there exists a partition of $V(G)$ into s-classes $(A, A_1, \cdots, A_{s-1})$ where $\gamma_s(A, A_1, \cdots, A_{s-1}) \leq \frac{e(G)}{s^2} + 4s\sqrt{e(G)}$.

Proof. The proof is by induction on s. The ground case $s = 2$ is given by Theorem 1.

Let $(A, X_1, \cdots, X_{s-1})$ be given by definition 1. Now $\bigcup_{j=1}^{s-1} X_j = V(G-A)$; by the induction hypothesis there is some $(s-1)$-coloring $(A_1, \cdots, A_{s-1})$ of $V(G-A)$ with $\gamma_{s-1}(A_1, A_2, \cdots, A_{s-1}) \leq \frac{e(G-A)}{(s-1)^2} + 4(s-1)\sqrt{e(G-A)}$.

Select such a partition $(A_1, \cdots, A_{s-1})$; W.L.O.G. let $e(A_1) \geq \cdots \geq e(A_{s-1})$. We then have a partition $(A, A_1, \cdots, A_{s-1})$ of $V(G)$ where $e[A, \bigcup_{j=1}^{s-1} A_j] \geq 2(s-1)e(A)$, the last inequality since $\bigcup_{j=1}^{s-1} A_j = \bigcup_{j=1}^{s-1} X_j$, and $e(A_1) = \gamma_{s-1}(A_1, \cdots, A_{s-1}) \leq \frac{e(G-A)}{(s-1)^2} + 4(s-1)\sqrt{e(G-A)}$. Now, notice

$$\begin{aligned} e(G) &= e(G-A) + e(A) + e[A, \bigcup_{j=1}^{s-1} A_j] \\ &\geq e(G-A) + (2s-1)e(A). \end{aligned}$$

Define $\Delta_j = e(A) - e(A_j), 1 \le j \le s-1$. We now consider two cases on Δ_1.

Case 1. $\Delta_1 \le 4s\sqrt{e(A)}$.

Suppose $\Delta_1 \le 4s\sqrt{e(A)}$, we have by the inductive hypothesis that

(1) $e(A_1) = e(A) - \Delta_1 \le \frac{e(G-A)}{(s-1)^2} + 4(s-1)\sqrt{e(G-A)}$, hence we need to show,

(2) $e(A) \le \frac{e(G-A)+(2s-1)e(A)}{s^2} + 4s\sqrt{e(G-A)+(2s-1)e(A)}$
$\le \frac{e(G)}{s^2} + 4s\sqrt{e(G)}$, to establish the inductive step.

We have from (1) that

$$\begin{aligned} e(A) &\le \Delta_1 + \frac{e(G-A)}{(s-1)^2} + 4(s-1)\sqrt{e(G-A)} \\ &\le 4s\sqrt{e(A)} + \frac{e(G-A)}{(s-1)^2} + 4(s-1)\sqrt{e(G-A)}, \end{aligned}$$

we then need to show,

$$\begin{aligned} &\left(4s\sqrt{e(A)} + \frac{e(G-A)}{(s-1)^2} + 4(s-1)\sqrt{e(G-A)}\right) = P \\ &\le \left(\frac{e(G-A)+(2s-1)e(A)}{s^2} + 4s\sqrt{e(G-A)+(2s-1)e(A)}\right) = Q \end{aligned}$$

We may assume that $\frac{e(G-A)}{(s-1)^2} \le \frac{e(G-A)+(2s-1)e(A)}{s^2}$, i.e.,

$$\frac{1}{(s-1)^2} \le \frac{e(A)}{e(G-A)}, \tag{3}$$

otherwise we are done, i.e., if $\frac{e(A)}{e(G-A)} < \frac{1}{(s-1)^2}$, then $\frac{e(A)}{e(G)} \le \frac{e(A)}{e(G-A)+(2s-1)e(A)} \le \frac{1}{s^2}$, and we have Theorem 2. Hence to establish $P \le Q$ we show,

$$4s\sqrt{e(G-A)+(2s-1)e(A)} - 4(s-1)\sqrt{e(G-A)} \ge 4s\sqrt{e(A)},$$

i.e.,

$$s\sqrt{e(G-A)+(2s-1)e(A)} - (s-1)\sqrt{e(G-A)} \ge s\sqrt{e(A)},$$

so it is sufficient to show $s\sqrt{e(G-A)+(2s-1)e(A)} - s\sqrt{e(G-A)} \ge s\sqrt{e(A)}$, i.e., $\sqrt{e(G-A)+(2s-1)e(A)} \ge \sqrt{e(G-A)} + \sqrt{e(A)}$, squaring both sides we then need, $2(s-1)e(A) \geq 2\sqrt{e(A)} \cdot \sqrt{e(G-A)}$, i.e., $(s-1)^2 e(A) \geq e(G-A)$ and the last inequality follows from (3).

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Case 2. $\Delta_1 > 4s\sqrt{e(A)}$.

Define $X \cup Y \subset A$ as follows:

$$X = \{x \in A | d_{A_j}(x) > 4s\sqrt{e(A)} \text{ for at least } \lceil \frac{s-2}{2} \rceil \text{ sets } A_j, 1 \leq j < s-1\}$$

$$Y = \{y \in A | d_A(y) \neq 0 \text{ and } d_{A_j}(y) \leq 4s\sqrt{e(A)} \text{ for at least } \lceil \frac{s+1}{2} \rceil \text{ sets } A_j, 1 \leq j \leq s-1\}$$

Note $X \cap Y = \phi$, and $e(A) = e(X) + e(Y) + e[X, Y]$. We may assume that for any $y \in Y$ and any $A_j, 1 \leq j \leq s-1$, where $d_{A_j}(y) \leq 4s\sqrt{e(A)}$ that $\gamma_s(A-y, A_1, \cdots, A_j+y, \cdots, A_{s-1}) = \max\{e(A) - d_A(y), e(A_1), e(A_j) + d_{A_j}(y)\} = e(A) - d_A(y)$, otherwise we are back to case 1 and done, i.e., if

$$\begin{aligned}
&\gamma_s(A-y, A_1, \cdots, A_j+y, \cdots, A_{s-1}) = \max\{e(A_1), e(A_j) + d_{A_j}(y)\} \\
&\leq e(A_1) + 4s\sqrt{e(A)} \leq \frac{e(G)}{s^2} + 4s\sqrt{e(G)},
\end{aligned}$$

the last inequality by case 1, i.e., $e(A_1) + \Delta \leq \frac{e(G)}{s^2} + 4s\sqrt{e(G)}$ whenever $\Delta \leq 4s\sqrt{e(A)}$. We now have the following Lemma.

Lemma 1. If $y \in Y$ and A_j is such that $d_{A_j}(y) \leq 4s\sqrt{e(A)}$ then $d_{A_j}(y) > (2s-1)d_A(y)$.

Proof. Assume to the contrary, i.e., there is a $y \in Y$ and A_j with $d_{A_j}(y) \leq (2s-1)d_A(y)$. Now, sending y to A_j will contradict definition 1. Consider $(A-y, A_1, \cdots, A_j+y, \cdots, A_{s-1})$, recall by definition $e[A, \overset{s-1}{\cup} A_i] \geq 2(s-1)e(A)$ and $e(A)$ is minimal. Then

$$\begin{aligned}
e[A-y, \left(\bigcup_{i=1}^{s-1} A_i\right) + y] &= e[A, \bigcup_{i=1}^{s-1} A_i] - (d_{A_j}(y) - d_A(y)) \\
&\geq e[A, \bigcup_{i-1}^{s-1} A_i] - 2(s-1)d_A(y) \\
&\geq 2(s-1)e(A) - 2(s-1)d_A(y) = 2(s-1)(e(A) - d_A(y)) \\
&= 2(s-1)e(A-y) = 2(s-1)\gamma_s(A-y, A_1, \cdots, A_j+y, \cdots, A_{s-1}).
\end{aligned}$$

But then, write $(A-y, A_1, \cdots, A_j+y, \cdots, A_s) = (A-y, \bar{X}_1, \bar{X}_2, \cdots, \bar{X}_{s-1})$ and we have from above, $e[A-y, \bigcup_{i=1}^{s-1} \bar{X}_i] \geq 2(s-1)$

$e(A-y)$ and $e(A-y) < e(A)$, contradicting the definition of partition $(A, X_1, \cdots, X_{s-1})$.

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From the definitions of X and Y, and Lemma 1. we state the following corollaries.

Corollary 1. $e[Y, \bigcup_{j=1}^{s-1} A_j] > (2s-1)\lceil \frac{s+1}{2} \rceil \sum_{y \in Y} d_A(y)$.

Corollary 2. $e[X, \bigcup_{j=1}^{s-1} A_j] \geq 4s\sqrt{e(A)}\lceil \frac{s-2}{2} \rceil |X|$.

Define ξ, by $\xi \sum_{y \in Y} d_A(y) = e(Y) + e[X, Y]$. Then, $\xi = 0$ or $\frac{1}{2} \leq \xi \leq 1$, where the extreme cases $\xi = \frac{1}{2}, 1$ indicate $e[X, Y] = 0$, resp., $e(Y) = 0$, and $\xi = 0$ if and only if $Y = \phi$. We then have

$$\frac{e(A)}{e(G)} \leq \frac{e(X) + \xi \sum_{y \in Y} d_A(y)}{e(X) + \xi \sum_{y \in Y} d_A(y) + e[X, \bigcup_{i=1}^{s-1} A_i] + e[Y, \bigcup_{i=1}^{s-1} A_i] + e(G-A)}$$

$$\leq \left(\frac{e(X) + \xi \sum_{y \in Y} d_A(y)}{e(X) + 4s\sqrt{e(A)}\lceil \frac{s-2}{2} \rceil |X| + \xi \sum_{y \in Y} d_A(y) + (2s-1)\lceil \frac{s+1}{2} \rceil \sum_{y \in Y} d_A(y)} \right)$$

$$= H.$$

We finish case 2 by showing $H \leq \frac{1}{s^2}$.

Lemma 2. $\frac{e(X)}{e(X) + 4s\sqrt{e(A)}\lceil \frac{s-2}{2} \rceil |X|} \leq \frac{1}{s^2}$.

Proof. We have $e(X) = c|X|^2$, for some c, $0 \leq c < \frac{1}{2}$, and $e(X) \leq e(A)$, hence $\frac{e(X)}{e(X) + 4s\sqrt{e(A)}\lceil \frac{s-2}{2} \rceil |X|} \leq \frac{1}{1 + \frac{2s(s-2)}{\sqrt{c}}} \leq \frac{1}{s^2}$ for $s \geq 3$.

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Lemma 3. $\frac{\xi \sum_{y \in Y} d_A(y)}{\xi \sum_{y \in Y} d_A(y) + (2s-1)\lceil \frac{s+1}{2} \rceil \sum_{y \in Y} d_A(y)} \leq \frac{1}{s^2}$.

Proof. We have $\frac{\xi \sum_{y \in Y} d_A(y)}{\xi \sum_{y \in Y} d_A(y) + (2s-1)\lceil \frac{s+1}{2} \rceil \sum_{y \in Y} d_A(y)} \leq \frac{1}{s^2 + \frac{s}{2} + \frac{1}{2}} < \frac{1}{s^2}$, the first inequality since $\xi \leq 1$.

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Combining Lemma 2 and Lemma 3 yields $H \leq \frac{1}{s^2}$ and completes case 2, and the proof of Theorem 2.

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Conclusions.

We summarize the known results. For any graph G, there is a partition $(U_1, \cdots, U_s)$ of $V(G)$ so that $e(U_i) \leq \frac{e(G)}{s^2} + R, 1 \leq i \leq s$.
For $s = 2^P$:
(Porter [6]) $R = \sqrt{\frac{e(G)}{s}}$.
For general s:
(Bollobás. Scott [1]) $R = \min\{(\Delta e(G) log\, s)^{\frac{1}{2}}, (4e(G))^{\frac{4}{5}}(log\, s)^{\frac{2}{5}}\}$ where Δ denotes the largest degree in G.
(Porter) $R = 4s\sqrt{e(G)}$.

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