The Application of Determining Sets to Projective Configurations

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Abstract

An n_3-configuration in the real projective plane is a configuration consisting of n points and n lines such that every point is on three lines and every line contains three points. Determining sets are used to construct drawings of arbitrary n_3-configurations in the plane, such that one line is represented as a circle. It is proved that the required determining set always exists, and that such a drawing is always possible. This is applied to the problem of deciding when a particular configuration is coordinatizable.

1. Determining Sets

Determining sets were introduced in [3] in order to animate computer representations of configurations in the real projective plane. In order to calculate dynamically the coordinates of the points and lines in a configuration, as a point or line is moved, a determining set is necessary. In this article, we apply determining sets to two related problems in projective plane geometry.

1. Coordinatization of projective configurations.
2. Drawing non-coordinatizable configurations.

We begin with some definitions from [3]. A projective configuration consists of a set Σ of points and lines, and an incidence relation Π. We denote this by (Σ, Π). For example, a triangle with points A, B, C and lines a, b, c can be represented by the pair $(\{A, B, C, a, b, c\}, \{Ab, Ac, Ba, Bc, Ca, Cb\})$. A configuration (Σ, Π) can also be viewed as a bipartite incidence graph of points versus lines.

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If we want to draw the construction, we need to assign coordinates to the lines and points. The coordinates must be assigned so that the incidences in Π are maintained. We assign homogeneous coordinates $[x, y, z]$ to each point and line. A point P and a line ℓ are incident if and only if $P \cdot \ell = 0$. The line joining points P and Q has coordinates $P \times Q$, and so forth.

A determining set is a minimal set Δ of objects (points and/or lines) such that if coordinates are assigned to each object in Δ, then the coordinates of every object in Σ are thereby determined. More formally,

1.1 Definition. Let (Σ, Π) be a projective configuration, $S \subseteq \Sigma$ be a set of points and/or lines. Let $S_0 = S$. For each $i \geq 1$, we define

$$S_i = S_{i-1} \cup \{P \mid P \text{ is incident with exactly 2 lines of } S_{i-1}\} \\ \cup \{\ell \mid \ell \text{ is incident with exactly 2 points of } S_{i-1}\}$$

Let $S^* = S_j$, where $j = \min\{i \mid S_i = S_{i+1}\}$. All objects $o \in S^*$ are said to be *determined* by S. The *rank* of an object $o \in S^*$ with respect to S is $r(o) = \min\{i \mid o \in S_i\}$. If an object has rank i, then it is said to be determined by S_{i-1}.

1.2 Definition. Given a configuration (Σ, Π), a *determining set* for this configuration, denoted by $\Delta(\Sigma, \Pi)$, is defined as a minimal subset of Σ that determines *all* of the objects in Σ (that is, $\Delta^* = \Sigma$), such that *no two objects of the same rank are incident*; that is, $\Delta(\Sigma, \Pi)$ determines all of Σ, but no proper subset of Δ does.

Thus if the positions of the points and lines in Δ are known, then the positions of all objects in Σ are known. Some configurations do not have determining sets. For example, a configuration consisting of a single point P incident on a single line ℓ has no determining set. Once the coordinates of P have been chosen, ℓ is constrained by the relation $P \cdot \ell = 0$, but is otherwise free. In order to deal with situations like this, augmented determining sets are defined.

1.3 Definition. An *augmented determining set* Δ for a configuration (Σ, Π) consists of a set Δ_0 and a set Δ_1 such that:

i) Δ_0 is a determining set for an induced configuration (Σ', Π');
ii) $(\Delta_0^* \cup \Delta_1)^* = \Sigma$;
iii) each $o \in \Delta_1$ is incident with exactly one object in Σ';
iv) Δ_1 is a minimal set with properties (ii) and (iii).

The objects of Δ_0 are called *free* objects, those of Δ_1 are called *constrained* objects. All others are called *determined* objects.

An algorithm for finding an augmented determining set is presented in [3]. It was developed in order to allow a computer user to animate projective configurations on a computer screen. Once an augmented determining set has been found, coordinates can be assigned to the objects of Δ. The coordinates of the remaining objects are then determined in a sequence, called a *construction sequence* after Sturmfels [6,7]. It is based on the sequence $(S_0, S_1, S_2, \ldots)$ of objects sorted by rank. The coordinates of each determined object o are determined by two previously determined objects, called the *antecedants* of o. We write $a_1(o)$ and $a_2(o)$ for the two antecedants of o. The coordinates of o are then given by $o = a_1(o) \times a_2(o)$. When a point P is dragged on a computer screen, an augmented determining set containing P is found. As the coordinates of P change, the construction sequence is used to update the coordinates of all other objects, in real time. We write $a_1(o) \rightarrow o$ to indicate that $a_1(o)$ constrains o. This relation defines a special acyclic orientation of the incidence graph.

1.4 Lemma. Let (Σ, Π) be a configuration whose incidence graph has maximum degree 3, and minimum degree < 3. Then (Σ, Π) has an augmented determining set.

Proof. By induction on n, the number of objects. When $n = 1$ or $n = 2$ it is easy to verify. Assume it holds when $n \leq k$ and consider $n = k + 1$. Pick an object $o \in \Sigma$ with at most 2 incidences. Remove o from Σ, and remove its incidences from Π. This results in a sub-configuration (Σ', Π') with $n-1$ objects. If oA is one of the incidences removed from Π, then object A now has at most 2 incidences. Therefore (Σ', Π') satisfies the conditions of the lemma, and therefore has an augmented determining set $\Delta = (\Delta_0, \Delta_1)$. Either o was incident on 2 objects, A and B, or else on one object only. In the first case, o is now detemined as $A \times B$, so that Δ is an augmented determining set for (Σ, Π). In the second case, o is incident only on A. We can therefore add o to Δ_1 to obtain an augmented determining set for (Σ, Π).

1.5 Definition. An n_3-configuration is a configuration of n points and n lines such that every line contains 3 points and every point is incident on 3 lines.

1.6 Lemma. An n_3-configuration (Σ, Π) does not have an augmented determining set. If a single incidence $P\ell$ is removed from Π, then the resulting configuration does have an augmented determining set.

Proof. Suppose that (Σ, Π) did have an augmented determining set. Let o be the last object determined. Since o has 3 incidences oA, oB, oC, each of A, B and C were determined before o. But each object has only 2 antecedants, a contradiction. Therefore no augmented determining set exists. If a single incidence $P\ell$ is removed from Π, then the conditions of Lemma 1.4 are applicable, so that a determining set will exist.

1.7 Definition. The *dimension* of an augmented determining set (Δ_0, Δ_1) is $|\Delta_0| + |\Delta_1|/2$.

It was proved in [3] that $|\Delta_0| + |\Delta_1|/2$ equals $n - E/2$ where $n = |\Sigma|$, the number of objects, and $E = |\Pi|$, the number of incidences. Therefore all augmented determining sets for (Σ, Π) have the same dimension.

Associated with every projective configuration is its bipartite incidence graph. Let G be any simple connected bipartite graph. Without loss of generality, we can call the two partite classes points and lines. If P is a point and ℓ a line, we write $P \in \ell$ to indicate that P and ℓ are adjacent, even though G may not be the incidence graph of a projective configuration, for 2 points in G need not determine a unique line, and 2 lines may not determine a unique point. However, definitions 1.1 to 1.3 are still applicable to G, so that we can consider determining sets in arbitrary bipartite graphs. In particular, there will be no determining set in a 3-regular bipartite graph. Let G be a 3-regular bipartite graph with $2n$ vertices, with one edge removed. The number of edges in G is $E = 3n - 1$. If an augmented determining set exists, its dimension is $2n - E/2 = (n + 1)/2$.

1.8 Theorem. Let G be a graph formed from a 3-regular connected bipartite simple graph by removing an edge $P\ell$. Then G has an augmented determining set containing P but not ℓ.

Proof. By induction on n the number of points. The smallest connected simple 3-regular bipartite graph is $K_{3,3}$. It is easy to verify that $K_{3,3}$ less an edge has an augmented determining set. We write $K_{3,3} - e$ to indicate a graph isomorphic to $K_{3,3}$ less an edge. Most cases in the induction will delete a point and line from G, then add several edges to produce a similar graph H with $n-1$ points. If G has dimension d, this will give H dimension $d - 1/2$. We will find an augmented determining set Δ for H and then modify Δ slightly to obtain a set for G.

Consider now a graph $G \neq K_{3,3} - e$. In $G + P\ell$ there are 3 lines adjacent to P, and 3 points adjacent to ℓ. Choose an edge $P_1\ell_1$ such that $P_1 \notin \ell$ and $P \notin \ell_1$. This is possible since $G \neq K_{3,3} - e$. There are several cases to consider.

Case 1. $P_1\ell_1$ is not contained in a subgraph $K_{2,2}$.

Let the three points contained in ℓ_1 be P_1, P_2, and P_3. Let the three lines containing P_1 be ℓ_1, ℓ_2 and ℓ_3. See Figure 1. We know that $P_2 \notin \ell_2$ and $P_3 \notin \ell_3$, since $P_1\ell_1$ is not contained in a subgraph $K_{2,2}$. Delete P_1 and ℓ_1 and add the edges $P_2\ell_2$ and $P_3\ell_3$. The result is a simple connected bipartite graph H in which P and ℓ have degree 2, and all other vertices have degree 3. By the induction hypothesis, H has an augmented determining set containing P but not ℓ. In fact, ℓ must be the last vertex in the construction sequence.

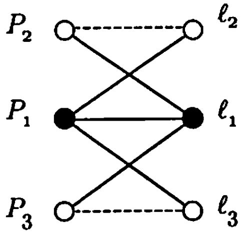

Figure 1 Case 1

Let $\Delta = (\Delta_0, \Delta_1)$ be an augmented determining set for H. There are several possible arrangements for the edges $P_2\ell_2$ and $P_3\ell_3$.

Case 1a. $P_2 \to \ell_2$ and $P_3 \to \ell_3$.

One of P_2 and P_3 comes first in the construction sequence. Wlog, assume it is P_2. We can modify Δ and the construction sequence for H as shown in Figure 2. Remove the edges $P_2\ell_2$ and $P_3\ell_3$. Add the edges $P_2\ell_1$ and $P_3\ell_1$. Take $\ell_1 \to P_1$ and place P_1 in Δ_1. Take $P_1 \to \ell_2, \ell_3$. The result is an augmented determining set and construction sequence for G.

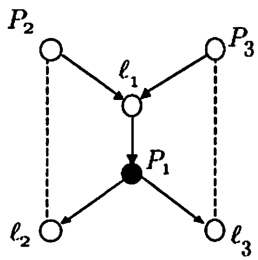

Figure 2 Case 1a

Case 1b. $P_2 \to \ell_2$ and $\ell_3 \to P_3$.

Wlog, assume that P_2 occurs before ℓ_3 in the construction sequence. Take $P_2 \to \ell_1$ and place ℓ_1 in Δ_1. The remaining edges are as indicated in Figure 3. Again we obtain an augmented determining set and construction sequence for G.

Notice that these operations require that $P_2 \notin \ell_3$ and that $P_3 \notin \ell_2$. This is guaranteed since $P_1\ell_1$ is not contained in a subgraph $K_{2,2}$.

Cases 1c ($\ell_2 \to P_2$ and $P_3 \to \ell_3$) and 1d ($\ell_2 \to P_2$ and $\ell_3 \to P_3$) are equivalent to 1a and 1b.

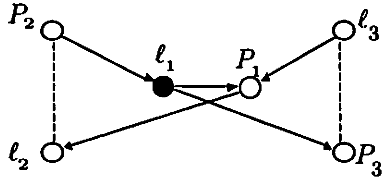

Figure 3 Case 1b

Case 2. $P_1\ell_1$ is contained in a $K_{2,2}$ but not in a $K_{2,3}$.
Let a $K_{2,2}$ be induced by P_1, P_2, ℓ_1, ℓ_2. Transform G by identifying P_1 and P_2 into a single vertex P', and identifying ℓ_1 and ℓ_2 into a single vertex ℓ'. Call the result H. $H + P\ell$ is 3-regular since $P_1\ell_1$ is not contained in a $K_{2,3}$. Choose an augmented determining set $\Delta = (\Delta_0, \Delta_1)$ in H. There are two equivalent subcases, either $P' \to \ell'$ or $\ell' \to P'$. Wlog, assume that $P' \to \ell'$.

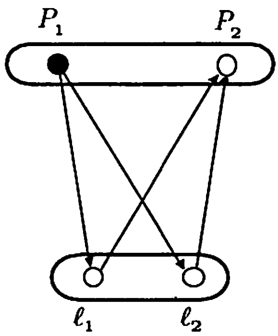

Figure 4 Case 2

Case 2a. $P' \notin \Delta_0, \Delta_1$.
In H, P' has 2 antecedants. When H is transformed back into G, one of them becomes $a_1(P_1)$, and the other becomes $a_1(P_2)$. Wlog, add P_1 to Δ_1. In H we have $P' \to \ell'$. Either $\ell' \in \Delta_1$, or else ℓ' has another antecedant Q. If $\ell' \in \Delta_1$, place $\ell_1 \in \Delta_1$ and take $\ell_1 \to P_2$. We now have two antecedants for P_2. Take $P_1, P_2 \to \ell_2$. This gives two antecedants for ℓ_2, and completes the determining set for G.

Otherwise $\ell' \notin \Delta_1$ so that ℓ' has another antecedant Q. Either $Q \to \ell_1$ or $Q \to \ell_2$. Wlog, assume it is ℓ_1. We then proceed as above and once again obtain two antecedants for P_2 and ℓ_2, thereby completing the construction sequence for G.

Case 2b. $P' \in \Delta_0$.
Place $P_1 \in \Delta_0$ in G. In H, we have $P' \to \ell'$. If ℓ' has another antecedant

Q, either $Q \to \ell_1$ or $Q \to \ell_2$. Wlog assume it is ℓ_1. In G the antecedants of ℓ_1 are P_1 and Q. Take $\ell_1 \to P_2$. Place P_2 in Δ_1 and take $P_2 \to \ell_2$. This give two antecedants for ℓ_2 and completes the construction sequence for G.

Otherwise ℓ' does not have another antecedant in H, so that $\ell' \in \Delta_1$. Place ℓ_1 and P_2 in Δ_1 in G. Proceed as above to complete the construction sequence for G.

Case 2c. $P' \in \Delta_1$.

P' has a single antecedant m. Either $m \to P_1$ or $m \to P_2$. Wlog assume it is P_1. Place P_1 and P_2 in Δ_1 in G. If ℓ' has another antecedant Q in H, we can assume as above that $Q \to \ell_1$. Take $\ell_1 \to P_2$. We then get antecedants P_1 and P_2 for ℓ_2. Otherwise ℓ' does not have another antecedant in H, so that $\ell' \in \Delta_1$. Wlog, place $\ell_1 \in \Delta_1$ and proceed as above to complete the construction sequence for G.

Case 3. $P_1\ell_1$ is contained in a $K_{2,3}$ but not in a $K_{3,3} - e$.

The $K_{2,3}$ containing $P_1\ell_1$ is induced either by $\{P_1, P_2, P_3, \ell_1, \ell_2\}$ or $\{P_1, P_2, \ell_1, \ell_2, \ell_3\}$. Assume first that the former is the case. The second possibility will turn out to be equivalent. Transform G by deleting ℓ_1 and ℓ_2, and identifying P_1, P_2, P_3 into a single vertex P'. Since $P_1\ell_1$ is not contained in a $K_{3,3} - e$ the result is a graph H satisfying the conditions of the theorem. See Figure 5.

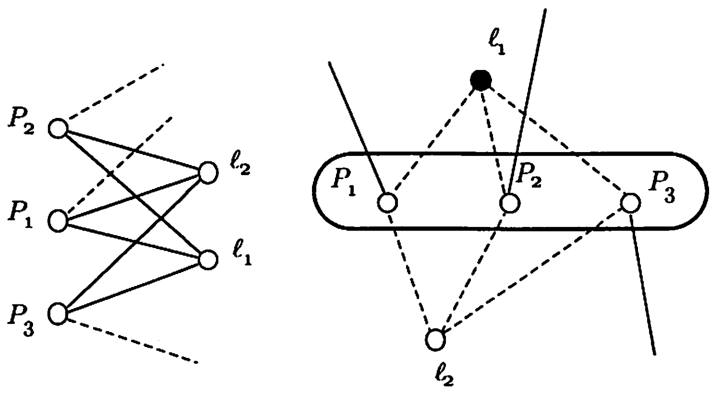

Figure 5 Case 3

If P' has two antecedants m_1 and m_2 in H, wlog we can take $m_1 \to P_1$ and $m_2 \to P_2$. Place ℓ_1 into Δ_0. ℓ_1 becomes the second antecedant of both P_1 and P_2. We then have $P_1, P_2 \to \ell_2$ and $\ell_1, \ell_2 \to P_3$, thereby completing the construction sequence for G.

If P' has only one antecedant m in H, wlog we can take $m \to P_1$. Place ℓ_1 in Δ_0. We then have $m, \ell_1 \to P_1$ and $P_1 \to \ell_2$. Place ℓ_2 in Δ_1. This now gives $\ell_1, \ell_2 \to P_2, P_3$ which thereby completes the construction sequence for G.

If P' has no antecedants in H then $P' \in \Delta_0$. Place ℓ_1 and ℓ_2 both in Δ_0. This determines P_1, P_2 and P_3 and completes the construction sequence in G.

Case 4. $P_1\ell_1$ is contained in a $K_{3,3} - e$.

Wlog, assume that $P_2 \notin \ell_2$, as in Figure 6. Let m be the additional line adjacent to P_2. It is possible that $m = \ell$.

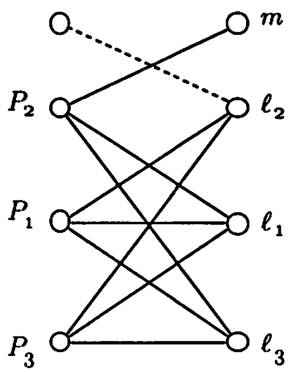

Figure 6 Case 4

Case 4a. $m \neq \ell$.

The edge P_2m is not contained in a subgraph $K_{2,2}$. Therefore this case is identical to Case 1.

Case 4b. $m = \ell$.

Let m be adjacent to P_2 and Q. Transform G into a graph H by deleting P_2 and m and adding the edge $Q\ell_3$. ℓ_1 has degree 2 in H. Let Δ be an augmented determining set for H containing P but not ℓ_1. ℓ_1 is the last vertex in the construction sequence. Therefore $P_1, P_3 \to \ell_1$. One of P_1 and P_3 is the penultimate vertex of the construction sequence. Wlog assume it is P_1. We also have $\ell_3 \to P_1$. There are four subcases.

Case 4bi. $\ell_3 \to P_3$ and $\ell_3 \to Q$.

This is only possible if $\ell_3 \in \Delta_0$. Remove the edge $Q\ell_3$ and place Q in Δ_1. Take $\ell_3, \ell_1 \to P_2$ and take $Q, P_2 \to \ell$. This completes the construction sequence.

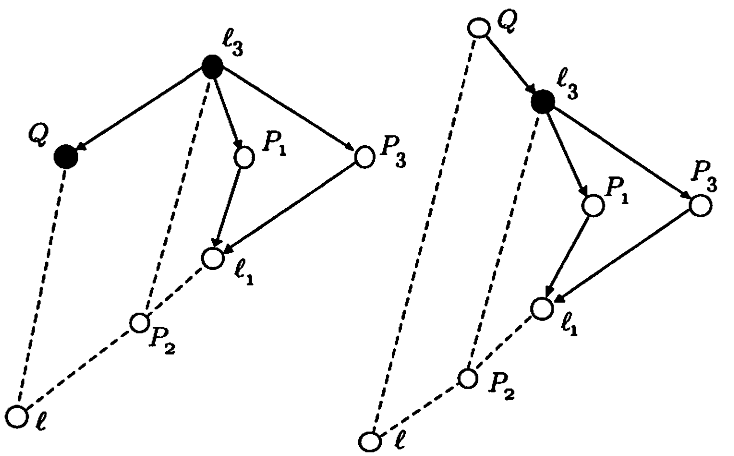

Figure 7 Cases 4bi and 4bii

Case 4bii. $\ell_3 \to P_3$ and $Q \to \ell_3$.
This is possible only if $\ell_3 \in \Delta_1$. Remove the edge $Q\ell_3$ and place ℓ_3 in Δ_0 instead of Δ_1. Take $\ell_3, \ell_1 \to P_2$ and $Q, P_2 \to \ell$ to complete the construction sequence.

Case 4biii. $P_3 \to \ell_3$ and $\ell_3 \to Q$.
This is only possible if $\ell_3 \in \Delta_1$. Remove the edge $Q\ell_3$. If $Q \in \Delta_1$ then move Q to Δ_0. Otherwise place Q in Δ_1 and take $Q \to \ell$. Take $\ell_3, \ell_1 \to P_2$ and $P_2 \to \ell$ to complete the construction sequence for G.

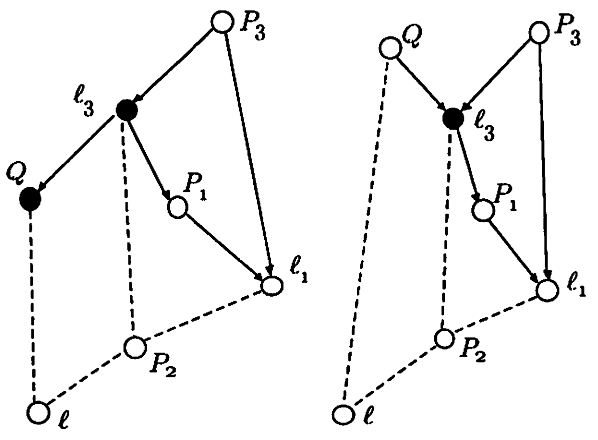

Figure 8 Cases 4biii and 4biv

Case 4biv. $P_3 \to \ell_3$ and $Q_3 \to \ell_3$.
Remove the egde $Q\ell_3$ and place ℓ_3 in Δ_1. Take $\ell_3, \ell_1 \to P_2$ and $Q, P_2 \to \ell$ to complete the construction sequence.

This completes the proof of the theorem. The following corollary will be used to construct drawings of n_3-configurations and to determine coordinatizations of them.

1.9 Corollary. Every n_3-configuration from which one incidence $P\ell$ has been removed has an augmented determining set containing P but not ℓ, and an augmented determining set containing ℓ but not P.
Proof. The incidence graph is a 3-regular simple connected bipartite graph from which an edge has been removed.

2. Coordinatizations

Determining sets were developed in order to allow projective configurations to be animated on a computer screen. It was later realized that they have application to problems of coordinatizability of projective configurations. The book [1] by Bokowski and Sturmfels and the paper [4] by Sturmfels describe an algorithm for deciding whether a given configuration is coordinatizable. It is based on symbolic computation by computer in the Grassmann algebra, where the bracket notation $[PQR]$ is used to denote the 3×3 determinant $P \cdot (Q \times R)$ of the homogeneous coordinates of points P, Q and R. It is a property of the Grassmann algebra that the coordinatizability of a configuration is completely determined by these brackets, for all points in the configuration [1,4,5,8]. The product $[PQR]$ must be zero if P, Q and R are collinear, and non-zero otherwise.

In [4] Sturmfels suggests that research into developing a faster algorithm be attempted. In this section we point out that determinig sets provide a faster algorithm for constructing the coordinatization polynomial. For example, the unique 8_3-configuration can be represented by the table of Figure 9 which gives the lines as triples of points. This configuration is not coordinatizable over the reals.

123 145 167 246 258 357 368 478

Figure 9 The 8_3-configuration

If we temporarily remove the incidence between the point 8 and the line 478, we obtain a reduced configuration for which an augmenting determining set is given by $\Delta = (\{1, 2, 4, 8\}, \{368\})$. It has a line 47 instead of 478. If we now replace the line representing 47 by the unique circle determined by points 4, 7, and 8, we can represent the 8_3-configuration as in Figure 10. Here the points of the determining set are shaded black, and the line 368 of the determining set is drawn thicker than the others.

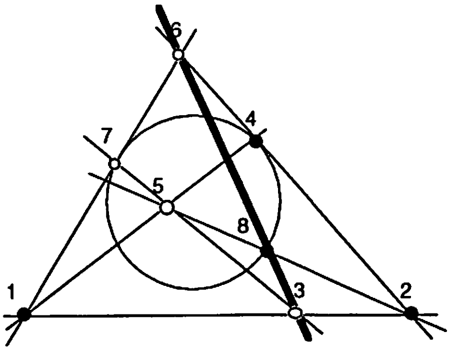

Figure 10 The 8_3 configuration

Let P_i denote the coordinates of point i in Fig. 2, where $i = 1..8$, and let ℓ denote the coordinates of the line 368, which is part of the augmented determining set. Referring to Figure 10, we can write

$P_3 = (P_1 \times P_2) \times \ell$

$P_5 = (P_1 \times P_4) \times (P_2 \times P_8)$

$P_6 = (P_2 \times P_4) \times \ell$

$P_7 = (P_1 \times P_6) \times (P_3 \times P_5)$

The incidence that was removed to find the determining set was between P_8 and the line 478. The line 47 has coordinates $P_4 \times P_7$. P_8 will be on this line if and only if

$P_8 \cdot P_4 \times P_7 = 0.$

This is the equation which determines the coordinatizability of the configuration. We only need to substitute values for the objects in the determining set in order to evaluate it. Without loss of generality we can take $P_1 = [1,0,0], P_2 = [0,1,0], P_8 = [0,0,1], P_4 = [x,y,z], \ell = [u,v,0]$. The third coordinate of ℓ must be zero, since ℓ is constrained to be incident on P_8. Substituting these values into the previous equations results in

$$z(u^2x^2 + v^2y^2 + uvxy) = 0$$

The variables u, v, x, y, z cannot be 0, or unwanted incidences will result. Dividing by u^2v^2z the equation reduces to $w^2+w+1 = 0$, where $w = xy/uv$. This equation is easily seen to have no real roots. This proof is many times shorter than the proof resulting from Sturmfels general algebraic algorithm [1,5]. It is essentially a more direct method of obtaining the polynomial characterizing the coordinatizability. The triples of the Grassmann algebra must still be all evaluated, and the roots of the polynomial must be determined exactly as in Sturmfels' method.

Coordinatizing a configuration consists of assigning homogeneous coordinates to the points such that:

i) all collinearities $[PQR] = 0$ are satisfied; and

ii) all non-collinearities $[PQR] \neq 0$ are satisfied.

For some configurations it is not possible to satisfy both requirements. In [8] White describes configurations for which unwanted collinearities $[PQR] = 0$ must always occur. Although these cannot be coordinatized, for the purposes of computer animation of a configuration, they do have coordinatizations; however every coordinatization has unwanted collinearities – two or more points or lines will have equivalent coordinates. Every drawing of them on the computer screen will be a kind of homomorphic image of the actual configuration. The drawings that will be important are those that satisfy all collinearities and the maximum possible number of non-collinearities.

2.1 Definition. A *near coordinatization* of a configuration is an assignment of coordinates to the points such that all collinearities and the maximum possible number of non-collinearities are satisfied.

2.2 Lemma. Every configuration which has an augmented determining set has a near-coordinatization.

Proof. Whenever an augmented determining set exists, coordinates can be assigned to the objects of the augmented determining set. This will determine coordinates of all remaining objects such that all collinearities are satisfied. As the coordinates of the objects in the determining set are varied, all possible assignments of coordinates to the remaining points will be obtained, including those with the maximum possible number of non-collinearities. Thus a near-coordinatization is always possible.

Note that a near-coordinatization satisfies the maximum possible number of non-collinearities. Thus it will nearly always be a coordinatization satisfying all non-collinearities, except for exceptional configurations.

2.3 Corollary. Every n_3-configuration can be drawn in the real plane so that one line is represented as a circle.

Proof. Remove an arbitrary incidence $P\ell$ from an n_3-configuration. The resulting reduced configuration has an augmented determining set containing P but not ℓ. Therefore it has a near-coordinatization, and so can be drawn in the real plane (possibly with some unwanted collinearities). ℓ will be the last object in the construction sequence. If A and B are the two points which determine ℓ, we can represent ℓ as the unique circle determined by P, A and B.

2.4 Lemma. A polynomial determining the near-coordinatizability of any n_3-configuration can be expressed as $P \cdot A \times B = 0$, where P, A, B are points of the configuration.

Proof. Remove an arbitrary incidence $P\ell$ and find an augmented determining set containing P but not ℓ. If A and B are the antecedants of ℓ, the equation $P \cdot A \times B = 0$ determines the near-coordinatizability. If the polynomial has roots, a near-coordinatization is always possible.

In this way all n_3-configurations can be drawn in the plane (or computer screen), using a circle to represent at most one line. A determining set allows the drawing to be animated as points and lines are dragged on the computer screen. There is an advantage in drawing one of the lines as a circle, namely the drawing animates very smoothly when points or lines are dragged; and non-coordinatizable configurations can be drawn. For example, the Fano configuration is well-known to be non-coordinatizable except over a field of characteristic 2. It has the well known drawing of Figure 11 in which one line is drawn as a circle. A determining set is given by points $\{4,5,6,7\}$. If P_i denotes the coordinates of point i, where $i = 1..7$, we can choose $P_4 = [1,0,0], P_5 = [0,1,0], P_6 = [0,0,1]$ and $P_7 = [x,y,z]$. The remaining points and lines are thereby determined. We leave it to the reader to use the determining set to construct the equation which proves the non-coordinatizability of the Fano configuration over the reals.

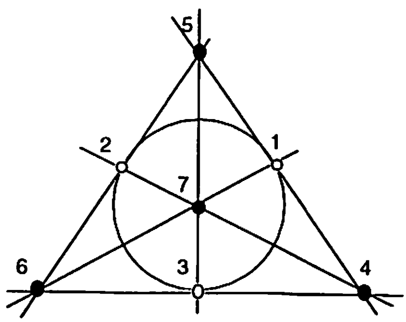

Figure 11 The Fano configuration

It is found that as a diagram containing a circle is animated in this way, the circle will change in size rather dramatically. Many different drawings of a given configuration can be constructed in this way. This allows interesting drawings to be constructed for a great many configurations. This is how the drawing of Figure 10 was constructed, using the *Groups & Graphs** software [2].

The determining set also allows an equation determining the near-coordinatizability of the configuration to be constructed automatically. It

* Groups & Graphs is available on the world wide web, at URL http://bkocay.cs.umanitoba.ca/G&G/G&G.html.

is found that choosing different determining sets will produce different coordinatization polynomials. In practice this is most easily done by constructing a determining set using the algorithm of [3], and then using the computer language Maple to substitute into the vector cross products to produce the coordinatization polynomial. Maple is often able to factor the polynomial, making it easier to find roots. In [6] Sturmfels and White describe techniques for finding rational roots of these polynomials.

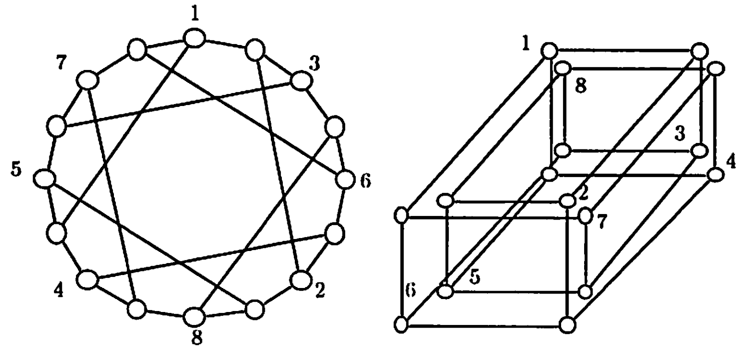

Figure 12 Incidence graph of the 8_3 configuration

The incidence graph of the 8_3-configuration is shown in Figure 12. The group of automorphisms of the incidence graph has order 96. As it is vertex-transitive, the 8_3-configuration is self-dual. In fact, the incidence graph is a double cover of the cube, as can be seen from the second graph of Figure 12 which is drawn to resemble a cube. Although the cube is not a valid incidence graph of any projective configuration, if each vertex and edge is doubled, the result is a valid incidence graph. The cube itself is a double cover of the complete graph K_4. Thus the unique 8_3-configuration is a quadruple cover of K_4.

If an incidence $P\ell$ is removed from an n_3-configuration, it is found that determining sets are plentiful. Our experience shows that if several points are selected, almost arbitrarily, then there is an augmented determining set containing these points. This allows great flexibility in manipulating these drawings on the computer screen.

Question How many points, no three collinear, can be selected arbitrarily from such a configuration, so that an augmented determining set containing them will always exist?

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