Card Sorting by Dispersions and Fractal Sequences

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We start with two simple card sorting problems. Cards numbered $1, 2, 3, \ldots, n$ are arranged in a certain order and held face up in your hand. Then the top card is placed face up on the bottom of the deck and the next card is placed face up on a table. This process is continued until all n cards are face up on the table.

Problem 1A. If the cards on the table are in order from n on the bottom to 1 on top, what was the original order of the cards in your hand?

Problem 2A. What is the number, $t(n)$, of the top card in the original deck of n cards, for $n = 1, 2, 3, \ldots$?

Concerning the solution of Problem 2A, it is easy to check that the sequence t begins with

$$1, 1, 2, 1, 3, 2, 4, 1, 5, 3, 6, 2, 7, 4, 8, 1, 9, 5, 10, 3 \qquad (1)$$

and that this sequence contains the solutions of Problem 1A. For $n = 5$, for example, start at the front of sequence t, and go down to the first term after the first 5. From there read back 5 terms, obtaining $3, 5, 1, 4, 2$ as the required initial ordering of the 5-card deck. Continuing to read backwards yields the *extended order* of the cards: $3, 5, 1, 4, 2, 3, 1, 2, 1, 1$. Similarly, for $n = 6$, we find from (1) the initial ordering $2, 6, 3, 5, 1, 4$, and for $n = 7$, the initial ordering $4, 7, 2, 6, 3, 5, 1$.

Before deriving (1) or proving anything, we generalize the two problems: instead of selecting the 2nd, 4th, 6th, ... cards for placement on the table, let $\sigma = (\sigma(1), \sigma(2), \sigma(3), \ldots)$ be any increasing sequence of positive integers. We shall call such a sequence a *selection sequence.* (Throughout this paper, the word "sequence" will mean a finite sequence or infinite sequence, depending on context.) The two problems can now be stated in terms of σ.

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Problem 1. What initial ordering of the n cards results in the reverse ordering, $n, n-1, n-2, \ldots, 1$, if cards in positions $\sigma(1), \sigma(2), \sigma(3), \ldots$ are placed on the table and the others are retained as in Problem 1A?

Problem 2. What is the number $t(n)$ of the top card in the original deck of n cards?

Dispersions

In order to prepare for a proof, we consider another example, in which the selection sequence

$$\sigma = (2, 3, 6, 8, 9, 12, 14, 15, 18, 20, 21, 24, 26, 27, 30, 32, \ldots)$$

is determined (after inserting an initial 0) by its periodic sequence of first differences,

$$2, 1, 3, 2, 1, 3, 2, 1, 3, \ldots.$$

We call this the *gap sequence of* σ and derive from it three *reversal sequences*:

$$\begin{array}{l} 3, 1, 2, 3, 1, 2, 3, 1, 2, \ldots \\ 2, 3, 1, 2, 3, 1, 2, 3, 1, \ldots \\ 1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots. \end{array}$$

Each of these sequences, we shall see, yields a sequence comparable to (1), from which some of the solutions (one-third of them, asymptotically speaking) of the two problems can be obtained.

Write the first of these gap sequences as $g(1), g(2), g(3), \ldots$, and generate a sequence c of numbers $c(i)$ as follows:

$$\begin{aligned} c(1) &= 1 \\ c(2) &= c(1) + g(1) = 1 + 3 = 4 \\ c(3) &= c(2) + g(2) = 4 + 1 = 5 \\ &\vdots \\ c(i) &= c(i-1) + g(i-1) \end{aligned}$$

Let c' be the sequence obtained by ranking in increasing order the complement of c; e.g., the first ten terms of c' are $2, 3, 6, 8, 9, 12, 14, 15, 18, 20$.

Let A_0 be the dispersion, as defined in [Kim1], of c'. That is, A_0 is an array consisting of all the positive integers, each occurring exactly once, with row 1 consisting of

$$a(1,1) = 1, \quad a(1,2) = c'(1), \quad a(1,3) = c'(a(1,2)), \ldots,$$
$$a(1,j) = c'(a(1,j-1)), \ \ldots,$$

row 2 consisting of

$$a(2,1) = \text{least positive integer not in row 1},$$
$$a(2,2) = c'(a(2,1)),$$
$$a(2,3) = c'(a(2,2)), \ldots,$$

row 3 consisting of

$$a(3,1) = \text{least positive integer not in row 1 or row 2},$$
$$a(3,2) = c'(a(3,1)),$$
$$a(3,3) = c'(a(3,2)), \ldots,$$

and so on. A helpful way to write out terms of a dispersion is to write the numbers from 1 to 30 or more, and beneath each, the matching term of c', like this:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	6	8	9	12	14	15	18	20	21	24	26	27	30

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
32	33	36	38	39	42	44	45	48	50	51	54	56	57	60

From this arrangement, read $1 \to 2 \to 3 \to 6 \to 12 \to 24 \to 48 \to \cdots$; then come back to the least unassigned number, 4, and read $4 \to 8 \to 15 \to 30 \to 60 \to \cdots$; return again to find $5 \to 9 \to 18 \to 36 \to \cdots$; and so on. The numbers in successive rows of A_0 are thereby identified. A few terms in the northwest corner of A_0 are shown here:

1	2	3	6	12	48
4	8	15	30	60	
5	9	18	36		
7	14	27	54		
10	20	39			
11	21	42			
13	26	51			

16	32
17	33
19	38

Fractal sequences

Now let s be the *fractal sequence of* A_0, defined by

$$s(k) = \text{the number of the row of } A_0 \text{ that contains } k.$$

The first twenty terms of s are

$$1, 1, 1, 2, 3, 1, 4, 2, 3, 5, 6, 1, 7, 4, 2, 8, 9, 3, 10, 5.$$

For $n = 3, 6, 9, \ldots$, the sequence s provides solutions to Problem 1. For example, the initial deck order for $n = 6$ is $1, 6, 5, 3, 2, 4$ and the extended order is $1, 6, 5, 3, 2, 4, 1, 3, 2, 1, 1, 1$. Clearly, the given selection sequence $2, 3, 6, 8, 9, 12$, applied to the extended order, does, in fact, leave cards $6, 5, 4, 3, 2, 1$ on the table, in this order, as desired. For $n = 3, 6, 9, \ldots$, certain terms of s also provide a subsequence of the top-card sequence of Problem 2. Indeed, taking the first number after the first occurrence of $3h$, for $h = 1, 2, 3, \ldots$ yields $t(3) = 1$, $t(6) = 1$, $t(9) = 3$, ...

One may anticipate that the other two reversal sequences lead to solutions for the remaining values of n. Explicitly, the second reversal sequence generates a dispersion A_1 having fractal sequence

$$1, 1, 2, 1, 2, 3, 4, 1, 5, 2, 3, 6, 7, 4, 8, 1, 5, 9, 10, 2, \ldots,$$

and the third generates A_2, with fractal sequence

$$1, 2, 1, 3, 2, 1, 4, 5, 3, 6, 2, 1, 7, 8, 4, 9, 5, 3, 10, 11, 1, \ldots.$$

It is easy to check that these fractal sequences provide solutions to the two problems for $n \equiv 1 \bmod 3$ and $n \equiv 2 \bmod 3$, respectively. We turn now to lemmas concerning fractal sequences.

Lemma 1. *Suppose* $A = A(i, j)$ *is the dispersion of a fractal sequence* s, *and* $i \geq 1, j \geq 1$. *Let*

$$r = a(i, j + 1) - a(i, j).$$

If $1 \le h \le r$, *then there exists exactly one* $k \ge 1$ *such that*

$$a(i,j) \le a(h,k) < a(i,j+1).$$

In other words, in s, between consecutive appearances of i *(including only the first), the greatest number that occurs is* r, *and each of the numbers* $1, 2, \ldots, r$ *occurs exactly once.*

Proof. (This lemma is essentially Theorem 1 of [Kim2], where a proof is given.) □

Lemma 2. *Suppose* $n \ge 2$ *and*

$$s(m+1), s(m+2), \ldots, s(m+n), \tag{2}$$

where $m \ge 0$, *are* n *consecutive terms of a fractal sequence* s. *Suppose that one of the numbers in* (2) *is* n *itself, and that*

$$s(k) \ne n \qquad \textit{for } k = 1, 2, \ldots, m \tag{3}$$

and

$$s(k) \le n \qquad \textit{for } k = 1, 2, \ldots, m+n. \tag{4}$$

Then the numbers in (2) *are, in some order,* $1, 2, \ldots, n$.

Proof. We use induction on n. For $n = 2$, segment (2) cannot be $2, 2$, by Lemma 1, and so it must be $1, 2$ or $2, 1$, as desired. Suppose $n' \ge 2$, and assume that the statement of Lemma 2 is valid when n' is written for n. Further, suppose that

$$s(m'+1), s(m'+2), \ldots, s(m'+n'+1), \tag{5}$$

are $n'+1$ consecutive terms of s, one of which is $n'+1$, and that

$$s(k) \ne n'+1 \qquad \text{for } k = 1, 2, \ldots, m'$$

and

$$s(k) \le n'+1 \qquad \text{for } k = 1, 2, \ldots, m+n'+1.$$

Let $q < = n'+1$ be a number satisfying $s(m'+q) = n'+1$. The segment

$$s(m'+q-n'), s(m'+q-n'+1), \ldots, s(m'+q-1)$$

of length n' satisfies the induction hypothesis and so consists of the numbers $1, 2, \ldots, n'$ in some order. Consequently, the segment

$$s(m' + q - n'),\ s(m' + q - n' + 1), \ldots, s(m' + q - 1), s(m' + q)$$

consists of the numbers $1, 2, \ldots, n', n' + 1$ in some order. Thus, if $m' + q = m' + n' + 1$, then (5) consists of the numbers $1, 2, \ldots, n', n' + 1$ in some order, as desired. On the other hand, if $m' + q < m' + n' + 1$, then, we shall show,

$$s(m' + q + 1) = s(m' + q - n') \qquad (6.1)$$
$$s(m' + q + 2) = s(m' + q - n' + 1) \qquad (6.2)$$
$$\vdots$$
$$s(m' + n' + 1) = s(m' + 1), \qquad (6.u)$$

To establish these, we first abbreviate $s(m' + q - n')$ as x. If $s(m' + q + 1) \neq x$, then some $z < n$ occupies position $m' + q + 1$ in s, and the next occurrence of x comes after this occurrence of z. But z has already occurs between positions $m' + q - n$ and $m' + q + 1$, so that there are two occurrences of z between consecutive occurrences of x. This contradicts Lemma 1. With (6.1) now established, (6.2), ..., (6.u), where $u = n' - q + 1$, follow inductively. Therefore (5) consists of the numbers $1, 2, \ldots, n' + 1$ in some order, and the induction on n is now complete. □

Lemma 3. *Suppose $s(1), s(2), \ldots, s(m), s(m + 1), \ldots, s(m + q)$ is the initial segment of length $m + q$ of a fractal sequence s, that the numbers in the segment*

$$s' = (s(m + 1), \ldots, s(m + q)) \qquad (7)$$

are, in some order, the numbers $1, 2, \ldots, q$, and that the number u of numbers in s' that are not in the segment $(s(1), s(2), \ldots, s(m))$ satisfies $u \geq 1$. If these u numbers are removed from s', then the remaining numbers form the segment

$$(s(m - q + u + 1), \ldots, s(m)) \qquad (8)$$

of s.

Proof. As a fractal sequence, s satisfies the upper self-similarity property [Kim2, Theorem 3]. That is, if the first occurrence of each n is removed from s, then the remaining sequence is s. Moreover, by Lemma 2, the u numbers are necessarily consecutive. Thus, the removal operation maps every segment of the form (7) onto the immediately preceding segment (8). □

Solving the two problems

In order to state Theorem 1, it will be expedient to define $\sigma(0) = 0$.

Theorem 1. *Suppose σ is a selection sequence with periodic gap sequence G. Let*

$$G(1) = \sigma(1), \quad G(2) = \sigma(2) - \sigma(1), \ldots, \quad G(p) = \sigma(p) - \sigma(p-1),$$

where $p \geq 1$, be the fundamental period of G. Define p reversal sequences g_j by

$$(g_j(1), g_j(2), \ldots, g_j(p)) = (G(j+p), G(j-1+p), \ldots, G(j+1)),$$

for $j = 0, 1, \ldots, p-1$, and $g_j(i + hp) = g_j(i)$ for $h = 1, 2, \ldots,$ for $i = 1, 2, \ldots, p$, for $j = 0, 1, \ldots, p-1$. Let c_j be the sequence given by $c_j(1) = 1$ and

$$c_j(i) = c_j(i-1) + g_j(i-1) \tag{9}$$

for $i = 2, 3, \ldots,$ for $j = 0, 1, \ldots, p-1$. Let C_j be the complement of the set of terms of c_j, and let c'_j be the sequence of elements of C_j arranged in increasing order. Let s_j, with terms $s_j(1), s_j(2), \ldots,$ be the fractal sequence of the dispersion, A_j, of c'_j. Suppose $n \geq \sigma(1)$ and $n \equiv j \bmod p$. Then the solution of Problem 1 is the sequence

$$s_j(\sigma(n)), \quad s_j(\sigma(n) - 1), \quad \ldots, \quad s_j(\sigma(n) - n + 1). \tag{10}$$

Using the notation of Theorem 1, we shall state and prove two more lemmas before turning to the main body of proof of Theorem 1.

Lemma 4. *If $0 \leq i \leq n$, then*

$$c_j(n + 1 - i) = \sigma(n) + 1 - \sigma(i). \tag{11}$$

Proof. Starting with $c_j(1)$, the recurrence relation (9) gives

$$c_j(n + 1 - i) = 1 + g_j(1) + g_j(2) + \cdots + g_j(n - i)$$

$$= 1 + \sum_{\tau=1}^{n} g_j(\tau) - \sum_{\tau=n-i+1}^{n} g_j(\tau)$$

$$= 1 + \sum_{t=1}^{n} G(n+1-t) - \sum_{t=1}^{i} G(n+1-t)$$

$$= \sigma(n) + 1 - \sigma(i). \qquad \square$$

Lemma 5. $s_j(\sigma(n)+1) = n+1.$

Proof. $s_j(\sigma(n)+1)$ = (number of the row of A_j that contains $\sigma(n)+1$)
= (number of the row that contains $c_j(n+1)$, by Lemma 4, with $i = 0$)
$= n+1$, since $a_j(n+1,1) = c_j(n+1)$. $\square$

Proof of Theorem 1. By the definition of s_j, we have

$$s_j(c_j(t)) = \text{number of the row of } A_j \text{ that contains } c_j(t),$$

which is t, since $c_j(t)$ is the number in row t and column 1 of A_j. In other words, $c_j(t)$, for $t = 1, 2, \ldots, n$, is the position in s_j of the first occurrence of t. Consequently, by (11), the positions of first occurrences of cards $n, n-1, \ldots, 2, 1$ are as indicated by the following table, for which row 1 gives the card number, row 2 the position of the card in s_j, and row 3 the reversed position:

n	$n-1$	$\ldots$	2	1
$\sigma(n)+1-\sigma(1)$	$\sigma(n)+1-\sigma(2)$	$\ldots$	$\sigma(n)+1-\sigma(n-1)$	$\sigma(n)+1-\sigma(n)$
$\sigma(1)$	$\sigma(2)$	$\ldots$	$\sigma(n-1)$	$\sigma(n)$

Here, column i consists of entries $n-i+1$, $1+\sigma(n)-\sigma(i)$, $\sigma(i)$, so that, starting from position $\sigma(n)+1$ in the fractal sequence s_j and counting back to $s_j(1)$, the number $n-i+1$ is in position $\sigma(i)$.

This shows that the selection sequence $\sigma(1), \sigma(2), \ldots, \sigma(n)$ applied to the sequence

$$\begin{aligned} &s_j(\sigma(n)),\ s_j(\sigma(n)-1),\ \ldots,\ s_j(\sigma(n)-n+1),\\ &\quad s_j(\sigma(n)-n),\ \ldots,\ s_j(1) = 1 \end{aligned} \tag{12}$$

yields the same sequence, $(n, n-1, \ldots, 2, 1)$ as is required for Problem 1.

Next, partition the reversal of (12) into adjoining segments

$$1, \ldots, \; s_j(\sigma(n) - n) \tag{13}$$

and

$$s_j(\sigma(n) - n + 1), \ldots, \; s_j(\sigma(n) - 1), \; s_j(\sigma(n)). \tag{14}$$

Note that (14), as the reversal of (10), has the form (2) with $m = \sigma(n) - n$. We shall show that segment (14) satisfies the hypotheses of Lemma 2. First, from $1 \leq \sigma(1) \leq n$ follows

$$\sigma(n) - n + 1 \leq \sigma(n) + 1 - \sigma(1) \leq \sigma(n),$$

and since $\sigma(n) + 1 - \sigma(1)$ is the position occupied by n in s_j, one of the numbers in (14) is n. Hypothesis (3) holds since $\sigma(n) + 1 - \sigma(1)$, which exceeds m, is the position in s_j of the *first* occurrence of n. That hypothesis (4) holds is a clear implication of the identity $s_j(\sigma(n) + 1) = n + 1$ of Lemma 5. Thus, by Lemma 2, the numbers in (14) are, in some order, $1, 2, \ldots, n$.

Let u be the number of numbers removed from (14) when cards selected by σ are removed from (10). If $u \geq 1$, then by Lemma 3, the numbers remaining form a subsegment

$$s_j(\sigma(n) - 2n + 1 + u), \ldots, s_j(\sigma(n) - n) \tag{15}$$

as in (13). (The possibility that $u = 0$ poses no difficulty in what follows.) Clearly, the card-sorting procedure removes u cards bearing the same numbers as those removed from (14), leaving a hand of cards bearing the same consecutive numbers, in the same order, as in (15). This removal process is repeated until the fractal sequence and matching cards are exhausted by the final selection, $s_j(1) = 1$. Since the successive subsegments of (12) exactly match the successive hands of cards, the extended order of cards – that is, the numbers on the cards in your hand, in order, from the beginning until you have tabled all the cards – is given by (12), and the original hand of n cards is given by (10).□

Corollary 1. *The solution of Problem 2 is given by* $t(n) = s_j(\sigma(n))$ *for* $n = 1, 2, 3, \ldots$.

Proof. This follows immediately from the expression for the first term of the sequence (10). □

Explicit solution of Problems 1A and 2A

As a final consideration, let $c(n,h)$ denote the hth card in a deck of n cards ordered as in the solution of Problem 1. We have already seen that $c(n,h) = s(2n - h + 1)$, where s is the fractal sequence associated with a certain dispersion. For Problems 1A and 2A, this dispersion – call it A with general term $a(i,j)$ – is especially simple. In fact,

$$a(i,j) = (2i-1)2^{j-1}.$$

We seek from this an explicit formula for $c(n,h)$. First, $c(n,2w) = s(2n - 2w + 1) =$ the number of the row of A that contains $2n - 2w + 1$. Since column 1 of A consists of all the oldd positive integers, in increasing order, the required row number is $n + 1 - w$.

Subtler is the case for odd h. We have

$$c(n, 2w-1) = s(2n - (2w-1) + 1) = s(2(n+1-w)),$$

which is the number of the row of A that contains $2(n+1-w)$. We seek, therefore, the number i satisfying

$$(2i-1)2^{j-1} = 2(n-w+1)$$

for some j. Clearly, $2i-1$ must be the largest odd divisor of $n - w + 1$. To summarize, explicity solutions to Problems 1A and 2A are given by

$$c(n,2w) = n + 1 - w,$$
$$c(n,2w-1) = i,$$

where $2i - 1$ is the largest odd divisor of $n - w + 1$.

References

[K1] Clark Kimberling, "Interspersions and dispersions," *Proc. Amer. Math. Soc.* 117 (1993) 313-321.

[K2] Clark Kimberling, "Fractal sequences and interspersions," *Ars Comb.*, 45 (1997) 157-168.