A note on orders with level diagrams

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ABSTRACT. A finite ordered set is upper levellable iff it has a diagram in which, for each element, all upper covers of the element are on the same horizontal level. In this note we give a method for computing a canonical upper levelling, should one exist.

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1 Introduction

A finite ordered set is *upper levellable* iff there exists a diagram of the order in which, for each element, all the upper covers of the element are on the same horizontal level. The ordered set is *lower levellable* iff its dual is upper levellable. These notions were introduced in [1] where it was shown that a finite ordered set is upper levellable iff it contains no alternating cover cycle. Since alternating cover cycles are self dual it follows that upper levellable and lower levellable are equivalent conditions. As the authors remark in [1], this equivalence does not seem obvious without the alternating cover cycle theorem. The purpose of this note is to present a (greedy) algorithm which computes an upper levelling of an ordered set should one exist and which makes this self duality fairly clear and does not rely on the alternating cover cycle theorem. However, we do use the alternating cover cycle theorem to give a stopping condition should the ordered set fail to be upper levellable.

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2 Results

Let P be a finite ordered set and N_0 the natural numbers with 0. An isotone function $f : P \mapsto N_0$ is an *upper levelling* of P iff $p \prec q, r$ ($\prec$ is the covering relation) implies $f(p) < f(q) = f(r)$, for all $p, q, r \in P$. Dually, an isotone function $g : P \mapsto N_0$ is a *lower levelling* of P iff $q, r \prec p$ implies $g(p) > g(q) = g(r)$, for all $p, q, r \in P$. P is *upper (lower) levellable* iff it possesses an upper (lower) levelling.

Define $h_0 : P \mapsto N_0$ by,

$$h_0(x) = \begin{cases} 0 & \text{if } x \text{ is minimal,} \\ sup\{h_0(y) + 1 | y \prec x\} & \text{otherwise.} \end{cases}$$

and $h_{i+1} : P \mapsto N_0$ by,

$$h_{i+1}(x) = sup(\{h_{i+1}(y) + 1 | y \prec x\} \cup \{h_i(y) | x \equiv_l y\}),$$

where $x \equiv_l y$ iff $x = y$ or there is a $w \in P$ with $w \prec x$ and $w \prec y$. We'll also use the dual notion; $x \equiv_u y$ iff $x = y$ or there is a $w \in P$ with $x \prec w$ and $y \prec w$.

We observe,

Lemma 2.1 *For $i \leq j$, $h_i \leq h_j$, $x < y$ implies $h_i(x) < h_i(y)$, and $h_i(x) = 0$ iff x is minimal.*

If the sequence h_i of functions is bounded then we set $h = sup\{h_i | i \in N_0\}$ and call this the *up-function* for P. By considering the existence of an up-function for the dual of P we obtain the increasing sequence of anti-isotone functions $d_i : P \mapsto N_0$ and, if these are bounded, the *down-function*, d, of P.

Let f be an upper levelling of P and let m be the maximum value that f attains. For each $i \in N_0$, set $d_i^* = m - d_i$.

Lemma 2.2 *For each $i \in N_0$, $h_i \leq f$ and $f \leq d_i^*$.*

Proof. We'll prove $h_i \leq f$ by contradiction. Let i be minimal with $h_{i+1} \not\leq f$ and let x be minimal with $h_{i+1}(x) > f(x)$.

If there exists $y \prec x$ with $h_{i+1}(x) = h_{i+1}(y)+1$ then $f(y) < f(x) < h_{i+1}(x) = h_{i+1}(y) + 1$ and thus $f(y) < h_{i+1}(y)$, contradicting the minimality of x.

Otherwise, there exists y with $x \equiv_l y$ and $h_{i+1}(x) = h_i(y)$. Since x and y have a common lower cover, $f(x) = f(y)$, and $f(y) = f(x) < h_{i+1}(x) = h_i(y)$, contradicting the minimality of i.

The proof of $d_i^* \geq f$ is very similar, but more complicated. Let i be minimal with $d_i^* \not\geq f$, and let x be maximal with $d_i^*(x) < f(x)$.

We define sequences $(y_j)_{j=0}^n$ and $(w_j)_{j=0}^n$, where $n \in \mathbb{N}_0$ is defined below, as:

(0) $y_0 = x$.

(1) If there exists w so that $y_j \prec w$ and $d_{i-j}^*(y_j) = d_{i-j}^*(w) - 1$, then set $w_j = w$ and $j = n$.

(2) Otherwise, there exist w_j, y_{j+1} with $y_j \prec w_j$, $y_{j+1} \prec w_j$ so that $y_j \equiv_u y_{j+1}$, and $d_{i-j}^*(y_j) = d_{(i-j)-1}^*(y_{j+1})$.

Each time we apply (2) we are working with a smaller value of $i - j$ until, at the latest, we are calculating with d_0^* and then only (1) can apply. In particular, $n \leq i$.

If $n = 0$ then $d_i^*(x) = d_i^*(w_0) - 1$, $d_i^*(x) < f(x)$ and $f(x) < f(w)$. Thus, $d_i^*(w_0) = d_i^*(x) + 1 \leq f(x) < f(w_0)$, which contradicts the maximality of x.

Assume $n > 0$ and observe that $f(w_0) = f(w_1) = \ldots = f(w_n)$. Since f is an upper levelling of P, we also have $d_i^*(x) = d_{i-1}^*(y_1) = \ldots = d_{i-n}^*(y_n)$ by (2), and hence, $d_i^*(x) = d_{i-n}^*(y_n)$. Now, $d_i^*(x) < f(x) < f(w_0) = f(w_n)$. The first inequality by assumption, the second because f is an upper levelling function. Hence, $d_i^*(x) + 1 < f(w_n)$. Finally, combining these obsevations, we have, $d_{i-n}^*(w_n) = d_{i-n}^*(y_n) + 1 = d_i^*(x) + 1 < f(w_n)$. This contradicts the minimality of i and completes the proof of the lemma.

It is perhaps worth elaborating on the comment made in the proof. The existence of an upper levelling function guarantees the existence of a canonical upper levelling, or up function, in a simple direct manner. It also guarantees the existence of a canonical lower levelling, but the reasons for this, I believe, are intrinsically more complicated.

Proposition 2.3 *For a finite ordered set P the following are equivalent:*

1. *P is upper levellable.*
2. *h exists.*
3. *d exists.*
4. *P is lower levellable.*

Proof. 2 implies 1 and 3 implies 4 are trivial. The first part of lemma 2.2 gives 1 implies 2, and its dual gives 4 implies 3. The last part of lemma 2.2 gives 1 implies 3 and its dual gives 4 implies 2.

Propositions 2.3 tells us how to calculate an upper (lower) levelling should one exist, but it doesn't tell us when to stop if one doesn't. It is here that we make direct use of the main result of [1].

An *alternating cover cycle* of length n in an ordered set P is a sequence $(c_0, a_0, c_1, a_1, ..., c_{n-1}, a_{n-1})$ so that $c_i \succ a_i$, $a_i \leq c_{i+1}$, for all i, where indices are computed modulo n, and there exists $x \in P$ with $a_{n-1} < x < c_0$.

Lemma 2.4 *Let $y_0, ..., y_{n-1}$ be a sequence in P so that either $y_{i+1} \prec y_i$ or $y_i \equiv_l y_{i+1}$, and so that, for at least one i, $y_{i+1} \prec y_i$ (again indices are computed modulo n). Then P contains an alternating cover cycle.*

Proof. For convenience let us assume $y_0 \succ y_1$. Let $c_0, c_{k-1}, c_{k-2}, ..., c_1$, be the subsequence of the sequence $y_0, ..., y_{n-1}$ consisting of those elements y_i such that $y_i \equiv_l y_{i-1}$ but $y_i \neq y_{i-1}$. The sequence $c_0, c_{k-1}, c_{k-2}, ..., c_1$ is non-empty because $y_0, ..., y_{n-1}$ cannot be a chain, remember we are working modulo n. Again for convenience, let $c_0 = y_i$ where $y_i = y_0$ or $y_i \succ y_{i+1} \succ ... \succ y_0$. With $c_j = y_i$, set a_j to be a common lower cover of y_{i-1} and y_i. Now set $x = y_1$ to obtain that $(c_0, a_0, ..., c_{k-1}, a_{n-1})$ is an alternating cover cycle.

Proposition 2.5 *Let P be an upper levellable ordered set. Then, for each $i \in \mathrm{N}_0$, $h_i \leq |P|$.*

Proof. By proposition 2.3, h exists. Let $x \in P$, and $i \in \mathrm{N}_0$. The calculation of $h_i(x)$ determines a sequence (y_j) as, $y_0 = x$ and $y_{j+1} \prec y_j$, if $h_{i-(j+1)}(y_{j+1}) + 1 = h_{i-j}(y_j)$, or, $y_{j+1} \equiv_l y_j$, if $h_{i-(j+1)}(y_{j+1}) = h_{i-j}(y_j)$ Notice that the value of $h_{i-j}(y_j)$ in this calculation only drops in the first case, ie. $y_{j+1} \prec y_j$. Thus, if $h_i(x) > |P|$ then there must be some element $a \in P$ which is repeated as the top element of two such covers. Let $a = y_k$ be the first time this happens and let the next occurence be $a = y_l$. Then $y_{k+1} \prec y_k$ and it follows from lemma 2.4 applied to the sequence $(y_k, ..., y_l)$ that P contains an alternating cover cycle, contradicting corollary 2 of [1].

3 Concluding Remarks

An algorithm to compute the canonical upper levelling h of an ordered set is provided by recursively computing the h_i's until they cease to change. Proposition 2.5 provides a simple stopping condition should the ordered set fail to be upper levellable. The time complexity of the algorithm is at most $O(|P|^3)$. This does not compare well to the algorithm outlined in [1], which is $O(|P|+m)$, where m is the number of covers (which in most instances will be the length of the input file). On the other hand, our algorithm has the advantage of being extremely easy to implement and in practice it seems to run quite quickly. A more detailed analysis might provide a better bound on the time complexity of the algorithm. Implementations of the algorithm are available from the author.

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References

[1] A. Pelc and I. Rival, Orders with level diagrams, *European J. Combin.* **12** (1991), 61-68.

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