# THE EDGE COVERING NUMBER OF ORDERED SETS

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ABSTRACT. The *edge covering number* $e(P)$ of an ordered set $P$ is the minimum number of suborders of $P$ of dimension at most two so that every covering edge of $P$ is included in one of the suborders. Unlike other familiar decompositions we can reconstruct the ordered set $P$ from its components. In this paper we find some familiar ordered sets of edge covering number two and then show that $e(\mathbf{2}^n) \to \infty$ as $n$ gets large .

## 1. Introduction

In this paper we introduce a new parameter, edge covering number, for finite ordered sets related to their decompositions. We begin with basic definitions and notations for ordered sets.

The elements of an ordered set $P$ are called *vertices* and an ordered pair $(a, b)$ of elements of $P$ is called an *edge* if $b$ covers $a$, i.e., $a < b$ and there no vertex $x$ such that $a < x < b$. In fact, vertices and edges are just the vertices and directed edges of the (Hasse) diagram of $P$ as a directed graph. An ordered set is called a *tree* if its diagram has no cycle as a (undirected) graph. For an ordered set $P$, a *suborder* is a subset $Q$ together with a subrelation of the ordering of $P$ which is itself an ordering of $Q$. An *induced suborder* is a subset $Q$ with the restriction of the ordering of $P$ to the set $Q$ as its ordering. A *linear extension* $L$ of $P$ is a linearly ordered set with the same underlying set as $P$ such that $x < y$ in $P$ implies $x < y$ in $L$.

Let $[n] = \{1, 2, \cdots, n\}$. Then $\mathbf{n}$ and $\underline{n}$ denote the chain and the antichain, respectively, on $[n]$. For ordered sets $P_1, ..., P_n$ , the *product* $P_1 \times \cdots \times P_n$ is the ordered set defined by the condition that $(a_1, ..., a_n) \leq (b_1, ..., b_n) \Leftrightarrow a_1 \leq b_1, ..., a_n \leq b_n$ for any elements $a_1, b_1 \in P_1, ..., a_n, b_n \in P_n$. Given an ordered set $P$, the notation $P^n$ is used as shorthand for the $n$-fold product $P \times \cdots \times P$. If $\mathcal{P}([n])$ is the power set of $[n]$, then $(\mathcal{P}([n]), \subseteq) \cong \mathbf{2}^n$.

For an antichain $A$, we define the *disjoint (cardinal) sum*, denoted by $\sum(P_i \mid i \in A)$, of a family $\{P_i \mid i \in A\}$ of disjoint ordered sets to be the set $\bigcup(P_i \mid i \in A)$ on which no new order relations are added, that is, $x \leq y \Longleftrightarrow x \leq y$ in some $P_i$ . When $A = \{1, 2, \cdots, n\}$, we write $\sum(P_i \mid i \in A) = P_1 + P_2 + \cdots + P_n$. For a chain $C$, the *linear (ordinal) sum*, denoted by $\bigoplus(P_i \mid i \in C)$, of a family $\{P_i \mid i \in C\}$ of

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disjoint ordered sets is defined to be the set $\bigcup(P_i \mid i \in C)$ on which $x \leq y \iff x \leq y$ in some $P_i$ or $x \in P_i$ and $y \in P_j$ with $i < j$ in $C$. When $C = \{1 < 2 < \cdots < n\}$, we write $\bigoplus(P_i \mid i \in C) = P_1 \oplus P_2 \oplus \cdots \oplus P_n$. An ordered set is an *interval order* if no 4-vertex induced order is a disjoint sum of two length-2 chains, and is a *semiorder* if no 4-vertex induced order is a disjoint sum of two nonempty chains.

Ordered sets are assumed to be finite throughout this article. We now define some familiar parameters of an ordered set $P$ as follows (cf.[6]) :

The *width* $w(P)$ of $P$ is the maximum size of an induced antichain of $P$.

The *length* $l(P)$ of $P$ is the maximum size of an induced chain of $P$.

The *dimension* $d(P)$ of $P$ is the minimum number of linear extensions the intersection of whose orderings is the ordering of $P$ itself.

**Examples.** $w(\mathbf{n}) = 1$, $l(\mathbf{n}) = n$, $d(\mathbf{n}) = 1$, $w(\underline{n}) = n$, $l(\underline{n}) = 1$, $d(\underline{n}) = 2$, $w(\mathbf{2}^n) = \binom{n}{\lceil n/2 \rceil}$, $l(\mathbf{2}^n) = n + 1$ and $d(\mathbf{2}^n) = n$.

In 1950, Dilworth [2] proved the celebrated chain decomposition theorem that the width $w(P)$ of an ordered set $P$ is the minimum number of chains whose vertices cover $P$. Motivated by this result, Fishburn [3] recently defined another parameter of describing the structure and complexity of an ordered set. To do this he considered relatively simple ordered sets, called *semichains*, which are ordered sets of dimension at most 2. For example, chains, antichains, upward rooted trees (connected ordered sets with no induced suborder of the form $\underline{2} \oplus \mathbf{1}$ ), downward rooted trees (connected ordered sets with no induced suborder of the form $\mathbf{1} \oplus \underline{2}$ ), products of two chains, complete bipartite ordered sets ($\underline{m} \oplus \underline{n}$, for some $m, n \geq 1$) and ordered sets of width 2 are semichains. For more information refer to [3] and [4]. The following is Fishburn's new parameter.

The *thickness* $t(P)$ of an ordered set $P$ is the minimum number of induced semichains of $P$ whose vertices cover all vertices of $P$.

The following are some of the results in [3].

1. $t(P) \leq l(P)$ for any ordered set $P$.
2. $t(P) \leq (w(P) + 1)/2$ for any ordered set $P$.
3. $t(P) \leq 2$ for any semiorder $P$.
4. $t(\mathbf{2}^n) \to \infty$ as $n$ gets large.
5. $t(\mathbf{n}^3) \to \infty$ as $n$ gets large.

Unfortunately, in this decomposition we cannot reconstruct the original ordered set $P$ from its semichain components. For example, any bipartite ordered set can be decomposed into two antichains, maximal vertices and minimal vertices, and so it has thickness two. But we cannot see its original ordering from the antichains. Here we introduce our new parameter of an ordered set $P$ related to a decomposition of edges in which we can reconstruct $P$ from its components. In fact, the ordering of $P$ is the transitive closure of the union of the orderings of the components.

The *edge covering number* $e(P)$ of an ordered set $P$ is the minimum number of suborders of $P$ of dimension at most two (semichains) so that every edge of $P$ is included in one of the suborders.

## 2. Examples of Edge Covering Number 2

In this section we exhibit some familiar ordered sets of edge covering number at most 2.

**Proposition 2.1.** *Every ordered set can be embedded into an ordered set of edge covering number at most 2.*

*Proof.* Let $P$ be an ordered set. Put a newvertex on each edge of $P$ to obtain an ordered set $P'$. Then the downward edges and the upward edges in $P'$, respectively, from the original vertices of $P$ form two semichains which in fact are disjoint sums of downward rooted trees and upward rooted trees, respectively, of length 2. □

For a vertex $x$ in an ordered set $P$, let $\delta(x, y) = \min\{n : x = z_0, z_1, \cdots, z_n = y,$ and $(z_i, z_{i+1})$ or $(z_{i+1}, z_i)$ is an edge of $P$ for $0 \leq i \leq n-1\}$.

**Proposition 2.2.** *Every tree has edge covering number at most 2.*

*Proof.* Pick a vertex $x$ in a tree $T$. Now, $E_1 = \{(y, z) : (y, z)$ is an edge of $T$ and $\{\delta(x, y), \delta(x, z)\} = \{2i, 2i+1\}$ for some $i \geq 0\}$ and $E_2 = \{(y, z) : (y, z)$ is an edge of $T$ and $\{\delta(x, y), \delta(x, z)\} = \{2i-1, 2i\}$ for some $i \geq 1\}$ form semichains which realize $T$. In fact, $E_1$ and $E_2$ are disjoint sums of ordered sets of the form $\underline{m} \oplus \underline{1} \oplus \underline{n}$, for some $m, n \geq 0$. □

For a natural number $n > 2$, consider $S_n = (\{A \in \mathcal{P}([n]) : |A| \in \{1, n-1\}\}, \subseteq)$, which is usually called the *standard* ordered set (of degree $n$).

**Proposition 2.3.** *Every standard ordered set has edge covering number 2.*

*Proof.* In the standard ordered set $S_n, n > 2$, let $a_i = \{i\}$ and $b_i = [n] - \{i\}$, the complement of the set $\{i\}$, for each $i \in [n]$. Then the sets of edges $E_1 = \{(a_i, b_j) : 1 \leq i \leq n-1$ and $i < j \leq n\}$ and $E_2 = \{(a_i, b_j) : 1 \leq j \leq n-1$ and $j < i \leq n\}$ form semichains which realize $S_n$. In fact, $E_1$ is represented by two linear extensions $\{a_{n-1} < a_{n-2} < \cdots < a_1 < b_n < b_{n-1} < \cdots < b_2\}$ and $\{a_1 < b_2 < a_2 < \cdots < b_{n-1} < a_{n-1} < b_n\}$ and $E_2$ by two linear extensions $\{a_2 < a_3 < \cdots < a_n < b_1 < b_2 < \cdots < b_{n-1}\}$ and $\{a_n < b_{n-1} < a_{n-1} < \cdots < b_2 < a_2 < b_1\}$. On the other hand, we know that $d(S_n) = n > 2$ (cf. [4]), whence $e(S_n) = 2$. □

Bogart, Rabinovitch and Trotter [1] has shown that there is no upper bound on the dimension of interval orders. But the edge covering number does better for certain interval orders. An ordered set $P$ is called *ranked* if all maximal chains in $\underline{1} \oplus P$ have the same finite length. For a subset $S$ of an ordered set $P$, we denote by min $S$ the set of all minimal elements of $S$.

**Proposition 2.4.** *Every ranked interval order has edge covering number at most 2.*

*Proof.* Let $P$ be a ranked interval order. Set $P_1 = \min P$ and, for $k > 1, P_k = \min (P - \bigcup_{i<k} P_i)$. Since $P$ is finite, $P_n \neq \emptyset$ and $P_{n+1} = \emptyset$ for some $n$. Observe that, for $1 \leq k \leq n-1$, the induced suborder $A_k$ on $P_k \cup P_{k+1}$ is of length 2. Rabinovitch showed implicitly in [5] that an interval order with length at most 2 has dimension at most 2. Hence, the disjoint sums $A_1 + A_3 + \cdots$ and $A_2 + A_4 + \cdots$ form two semichains which realize $P$. □

## 3. Main Theorem

In this section we prove that there is no bound on the edge covering number of the Boolean lattice $\mathbf{2}^n$. First we consider an induced suborder of $\mathbf{2}^n$. For an odd number $n$, consider $M_n = (\{A \in \mathcal{P}([n]) : \frac{n-1}{2} \leq |A| \leq \frac{n+1}{2}\}, \subseteq)$. In fact, this is just the middle two layers of $\mathbf{2}^n$.

**Lemma 3.1.** *$e(M_n) > \frac{n+1}{4}$ for any odd number $n$.*

*Proof.* Let $v_n = \binom{n}{(n+1)/2}$. Then the number of all vertices of $M_n$ is $2v_n$ and so the number of all edges is $e_n = \frac{n+1}{2}v_n$. Suppose that the edges of $M_n$ are decomposed into at most $k$ suborders. Then there exist at least $\lceil \frac{e_n}{k} \rceil$ edges in one of the suborders, say $A$. A simple fact in graph theory is that if a graph has no cycles then it has fewer edges than vertices. Now $\frac{e_n}{k} = \frac{n+1}{2k}v_n$ and we have

$$\frac{n+1}{2k}v_n \geq 2v_n \Longleftrightarrow \frac{n+1}{2k} \geq 2 \Longleftrightarrow \frac{n+1}{4} \geq k$$

Hence we conclude that if $\frac{n+1}{4} \geq k$ then $A$ as a graph has a cycle and thus $A$ contains an induced suborder of the form $C_p$ (Figure 1). But $M_n$ clearly does not contain $C_1$ and each $C_p$ with $p > 1$ is well known to be of dimension 3 (cf. [4]). Therefore $A$ is not a semichain, whence $e(M_n) > k$, which implies that $e(M_n) > \frac{n+1}{4}$. □

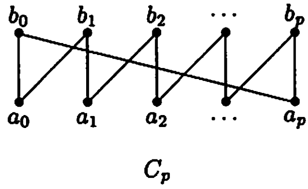



Figure 1.

An induced suborder $Q$ of an ordered set $P$ is said to be *convex* if $a, b \in Q$ and $a \leq x \leq b$ in $P$ imply that $x \in Q$. Now we have a simple but useful observation.

**Lemma 3.2.** *If $Q$ is a convex induced suborder of an ordered set $P$, then $e(Q) \leq e(P)$.*

**Theorem 3.3.** *$e(\mathbf{2}^n) \to \infty$ as $n$ gets large .*

*Proof.* Since $M_n$ is a convex induced suborder of $\mathbf{2}^n$, the proof follows from Lemmas 3.1 and 3.2.

## 4. Concluding Remarks

In this final section we consider the edge covering number of products of chains and apply this to compute $e(\mathbf{2}^k)$ for some $k$. We begin with an observation on the edge covering number of products of ordered sets.

**Proposition 4.1.** *For ordered sets $P$ and $Q$, $e(P \times Q) \leq e(P) + e(Q)$.*

*Proof.* Let $e(P) = m$ and $e(Q) = n$. Suppose that we use $m$ colors for the edges of $P$ and other $n$ colors for the edges of $Q$ to realize their respective edge covering numbers. We now want to show that these $m + n$ colors are enough to realize the edge covering number of $P \times Q$. Observing that $((a,b),(a',b'))$ is an edge of $P \times Q$ if and only if either $(a,a')$ is an edge of $P$ and $b = b'$ or $(b,b')$ is an edge of $Q$ and $a = a'$, color the edge $((a,b),(a',b'))$ of $P \times Q$ according to the color of $(a,a')$ or $(b,b')$. Then the suborder of $P \times Q$ with the edges of one color is in fact the disjoint sum of the copies of the semichain of $P$ or $Q$ with the edges of the same color, indexed by the elements of the other order. Hence $e(P \times Q) \leq m + n$. □

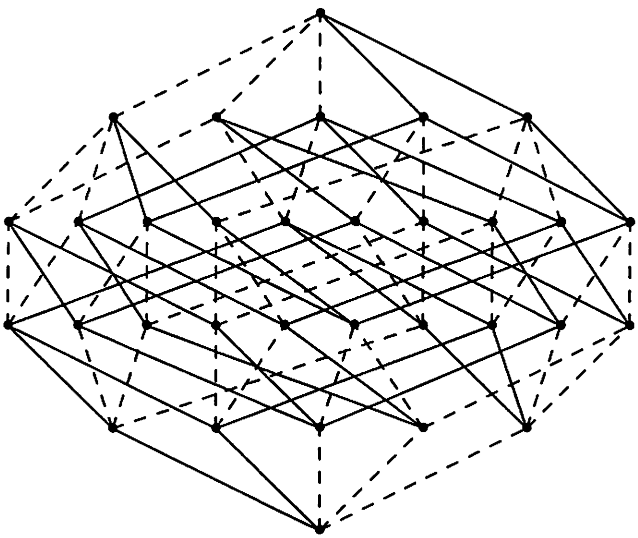
Since the product of two chains is a semichain, we immediately have the following corollary.

**Corollary 4.2.** *For natural numbers $k, n_1, n_2, \cdots, n_k$,*

$$e(\mathbf{n}_1 \times \mathbf{n}_2 \times \cdots \times \mathbf{n}_k) \leq \frac{k+1}{2}.$$

*Especially, $e(\mathbf{n}^k) \leq \frac{k+1}{2}$ for natural numbers $n$ and $k$.*

We first show that $e(\mathbf{2}^5) = 2$. In fact, the set of all edges of $\mathbf{2}^5$ is divided into two semichains which are represented by solid and broken line segments, respectively, in Figure 2.



$\mathbf{2}^5$

Figure 2.

We now can compute $e(\mathbf{2}^7)$. By the preceding result and Proposition 4.1, we have $e(\mathbf{2}^7) = e(\mathbf{2}^5 \times \mathbf{2}^2) \leq e(\mathbf{2}^5) + e(\mathbf{2}^2) = 3$. On the other hand, $M_7$ is a convex induced subset of $\mathbf{2}^7$ and then it follows from Lemma 3.1 that $e(\mathbf{2}^7) > 2$. Consequently, $e(\mathbf{2}^7) = 3$. But $e(\mathbf{2}^6)$ remains to be unknown at the moment. However, we know that $3 \leq e(\mathbf{2}^i) \leq 4$ for $i = 8, 9, 10$ and $4 \leq e(\mathbf{2}^i) \leq 6$ for $i = 11, 12, 13, 14, 15$.

There is no simple relation between the parameters $e$ and $t$ since

$$e(\mathbf{n}^3) = 2 \text{ and } t(\mathbf{n}^3) \to \infty \text{ as } n \text{ gets large;}$$

$$e(M_n) \to \infty \text{ as } n \text{ gets large and } t(M_n) \leq l(M_n) = 2.$$

However, we compare the parameters $d$, $t$ and $e$ for some familiar ordered sets in the following table which remains the symbols ? to be determined.

| | $d$ | $t$ | $e$ |
|---|---|---|---|
| $\mathbf{2}^n$ | $n$ | $\to \infty$ | $\to \infty$ |
| $S_n$ | $n$ | 2 | 2 |
| $\mathbf{2}^2$ | 2 | 1 | 1 |
| $\mathbf{2}^3$ | 3 | 2 | 2 |
| $\mathbf{2}^4$ | 4 | 2 | 2 |
| $\mathbf{2}^5$ | 5 | ? | 2 |
| $\mathbf{2}^6$ | 6 | ? | ? |
| $\mathbf{2}^7$ | 7 | ? | 3 |
| $\mathbf{3}^3$ | 3 | 2 | 2 |
| $\mathbf{3}^4$ | 4 | ? | 2 |
| $\mathbf{4}^3$ | 3 | ? | 2 |

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