On the Number of Dependent Sets in a Connected Graph

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Abstract

A set X of vertices of a graph is said to be *dependent* if X is not an independent set. For the graph G, let $P_k(G)$ denote the set of dependent sets of cardinality k.

In this paper, we show that if G is a connected graph on $2n$ vertices where $n \geq 3$ then $|P_n(G)| \geq |P_{n+1}(G)|$. This study is motivated by a conjecture of Lih.

1 Introduction

We assume that the reader is familiar with the basic terminology of graph theory and Sperner theory of partially ordered sets (posets). Any definitions not stated here may be found in [1], [2], or [3].

Let B_n be the Boolean algebra of order n, that is, the poset consisting of all subsets of $[n] = \{1, 2, \ldots, n\}$, ordered by inclusion. In 1928, Sperner [7] showed that the maximum size of an antichain in B_n is $\binom{n}{\lfloor n/2 \rfloor}$. In general, a poset P is said to have the *Sperner property* if the maximum size of an antichain in P equals the size of the largest rank of P.

Let $\mathcal{F}$ be a nonempty collection of subsets of $[n] = \{1, 2, \ldots, n\}$, each having cardinality t. Denote by $P_{\mathcal{F}}$ the poset consisting of all subsets of $[n]$ which contain at least one member of $\mathcal{F}$, ordered by set-theoretic inclusion. Ko-wei Lih [6] showed that if $\mathcal{F}$ is any nonempty collection of singleton subsets of $[n]$ then $P_{\mathcal{F}}$ has the Sperner property. Moreover, Lih made the following conjecture.

Conjecture 1.1 (Lih [6]) *Let $1 \leq t \leq n$ be an integer. If $\mathcal{F}$ is a nonempty collection of t-subsets of $[n]$ then $P_{\mathcal{F}}$ has the Sperner property.*

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This conjecture, while true in the case $t = 1$, is false in general. For counterexamples for $t \geq 4$, see [4] and [8]. Our focus here is on the case $t = 2$.

A subset of vertices in a graph is said to be *dependent* if it is not independent. Now let $\mathcal{F}$ be a nonempty collection of 2-sets of $[n]$. It will be convenient to associate with $\mathcal{F}$, a graph, as follows. Let G be the graph with vertex set $[n]$ and edge set $\mathcal{F}$, and let P_G denote the poset consisting of all dependent sets of $V(G)$, ordered by inclusion. Clearly, the poset P_G is isomorphic to $P_{\mathcal{F}}$.

For the case $t = 2$, Lih's conjecture has been established. Using the graph theoretic viewpoint, we state this result as follows.

Theorem 1.2 *For every graph G having at least one edge, the poset P_G has the Sperner property.*

Theorem 1.2 was proved in the case that G has an odd number of vertices by Zhu [9], and for the case of an even number of vertices by Horrocks [5].

For any graph G with at least one edge, P_G is a ranked poset. Although the minimal elements in P_G are 2-sets, we take the rank of $H \in P_G$ to be $|H|$, as in B_n. The following theorem concerning the unimodality of the rank numbers of P_G was obtained by Zha [8].

Theorem 1.3 *Let G be a graph on n vertices and let $P_2, P_3, \ldots, P_r$ be the ranks of P_G. Then P_G is rank unimodal with largest rank P_{n+1} if $r = 2n+1$, and P_n or P_{n+1} if $r = 2n$.*

In this paper, we focus on the problem of determining the largest rank in P_G where G has an even number of vertices. The following partial results are known.

Let G be a graph on $2n$ vertices.

- If the maximum degree of a vertex in G is at least $n+1$ then $|P_n(G)| \geq |P_{n+1}(G)|$. (Zha [8])
- If $|P_n(G)| \leq \sum_{i=2}^{n} \binom{2i-1}{i}$ then $|P_n(G)| \leq |P_{n+1}(G)|$. (Zha [8])
- If there is a vertex in G which is incident with every edge except possibly one then $|P_n(G)| \leq |P_{n+1}(G)|$. (Horrocks [5])
- If G contains six disjoint edges then $|P_n(G)| \geq |P_{n+1}(G)|$. (Horrocks [5])

In this paper, we prove the following result.

Theorem 1.4 *Let $n \geq 3$ be a positive integer. If G is a connected graph on $2n$ vertices then*

$$|P_n(G)| \geq |P_{n+1}(G)|.$$

2 Spanning Subgraphs and Forests

The main idea in the proof of Theorem 1.4 is to exhibit a spanning subgraph H of G such that $|P_n(H)| \geq |P_{n+1}(H)|$. Provided that H satisfies an additional condition, it may be shown that the desired result follows, namely $|P_n(G)| \geq |P_{n+1}(G)|$.

We require the following lemma found in [5].

Lemma 2.1 *Let G be a graph on $2n$ vertices, and let H be a spanning subgraph of G. If*

1. *for any two isolated vertices x and y of H, $H \setminus \{x, y\}$ has no more than $\sum_{i=2}^{n-1} \binom{2i-1}{i}$ independent sets of size $n-1$, and*

2. $$|P_n(H)| \geq |P_{n+1}(H)| \quad \textit{and,}$$

3. *P_H has the Sperner property,*

then P_G has the Sperner property, and $|P_n(G)| \geq |P_{n+1}(G)|$.

In each connected graph G, it will be convenient to identify a particular spanning subgraph, namely a spanning forest, each component of which is a star graph. We now show that G admits such a spanning forest.

Definition 2.2 *For $n \geq 2$, let S_n be the complete bipartite graph $K_{1,n-1}$. A graph is called a* star graph *if it is isomorphic to S_n for some $n \geq 2$.*

Lemma 2.3 *If G is a connected graph having at least 2 vertices then G has a spanning forest, each component of which is a star graph.*

Proof: The proof is by induction on $n = |V(G)|$. Let T be a spanning tree of G.

For $n = 2$, the tree T is isomorphic to S_2 so the lemma holds.

Suppose now that the lemma is true for all graphs having fewer than n vertices.

Now T has at least two leaves by Corollary 2.2 on page 26 of [2]. Let x be one of these leaves and let y be the vertex adjacent to x. By the induction hypothesis, $T \setminus x$ admits a decomposition $C_1, C_2, \ldots, C_r$ into star graphs. Without loss of generality, let C_1 be the component of the spanning forest of $T \setminus x$ which contains y. Now $C_1 = S_m$ for some $m \geq 2$.

If y is adjacent to every vertex in C_1 then $C_1{}' = C_1 + (x, y) = S_{m+1}$ and $C_1{}', C_2, \ldots, C_r$ is a spanning forest of T.

Otherwise, y is not adjacent to every vertex in C_1 and thus $m \geq 3$. Let C_0 consist of the vertices x and y and the edge (x, y) and let $C_1{}' =$

$C_1 \setminus \{y\}$. Since $C_0 = S_2$ and $C_1' = S_{m-1}$, we have that $C_0, C_1', C_2, \ldots, C_r$ is a spanning forest for T.

In either case, T, and therefore also G, has a spanning forest which is composed of star graphs and the proof is complete. □

3 Proof of Theorem 1.4

This section is devoted to proving our main result, Theorem 1.4.

To this end, let G be a connected graph on $2n$ vertices where $n \geq 3$. By Lemma 2.3, G has a spanning forest F, each component of which is a star graph.

We now consider six cases, according to the number of components of F. In each case, we show that G contains a spanning subgraph H which satisfies the first two conditions of Lemma 2.1. That P_H has the Sperner property follows from Theorem 1.2. Now applying Lemma 2.1, we obtain $|P_n(G)| \geq |P_{n+1}(G)|$ as desired.

In this section, the notation Z_m is used to refer to the graph having m vertices and no edges. The *independent set generating function* for the graph G is defined to be the polynomial $f(G) = \sum_{i \geq 0} a_i x^i$ where a_i is the number of independent sets in G of cardinality i. Note, for example, that $f(S_n) = (1+x)^{n-1} + x$.

3.1 F has exactly one component

In this case, F is isomorphic to S_{2n}. For $k \geq 2$, the number of independent sets of size k in S_{2n} is $\binom{2n-1}{k}$ so

$$|P_n(F)| - |P_{n+1}(F)| = \binom{2n}{n} - \binom{2n-1}{n} - \left\{ \binom{2n}{n+1} - \binom{2n-1}{n+1} \right\} = 0.$$

Moreover, as F has no isolated vertices, it satisfies the first two conditions of Lemma 2.1.

3.2 F has exactly two components

The following lemma will be required.

Lemma 3.1 *Let $m \geq 2$ be an integer. For $t = 1, 2, \ldots, 2m-1$,*

$$\binom{2m}{m} \geq (t+1)\binom{2m-t}{m} + (2m-t+1)\binom{t}{m}.$$

Proof: By symmetry, we need only consider $t = 1, 2, \ldots, m$. We will consider two cases; $t = m$ and $1 \le t < m$.

First, for $t = m$, the right hand side becomes $2(m+1)$. For $m \ge 2$, $\binom{2m}{m} \ge 2(m+1)$ so the result holds.

Secondly, for $t < m$, as $\binom{t}{m} = 0$ it suffices to show that $\binom{2m}{m} \ge (t+1)\binom{2m-t}{m}$. Now

$$\binom{2m}{m} = \frac{2m}{m}\frac{2m-1}{m-1}\cdots\frac{2m-t+1}{m-t+1}\binom{2m-t}{m}$$
$$\ge 2^t\binom{2m-t}{m} \ge (t+1)\binom{2m-t}{m}$$

for $t \ge 1$ which completes the proof. □

In this case, $F \cong S_{r+1} + S_{2n-r-1}$ for some $1 \le r \le 2n-3$. The independent set generating function for F is $f(F) = [(1+x)^r + x][(1+x)^{2n-r-2} + x]$ so $P_k(F) = \binom{2n}{k} - \binom{2n-2}{k} - \binom{r}{k-1} - \binom{2n-r-2}{k-1}$ for $k \ge 3$.

The inequality $|P_n(F)| \ge |P_{n+1}(F)|$ may be seen to be equivalent to

$$\binom{2n-2}{n-1} \ge (r+1)\binom{2n-r-2}{n-1} + (2n-r-1)\binom{r}{n-1}.$$

This latter inequality holds for $n \ge 3$ and $1 \le r \le 2n-3$ by Lemma 3.1. Therefore, $|P_n(F)| \ge |P_{n+1}(F)|$ and F satisfies the first two conditions of Lemma 2.1.

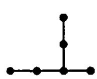
3.3 F has exactly three components

The number of components of F which are isomorphic to S_2 is 0, 1, 2, or 3 so we consider 4 subcases.

3.3.1 F contains 3 copies of S_2

In this subcase, F is isomorphic to $3S_2$. Since G is connected, one of the graphs F_1 or F_2 shown below is a spanning tree for G.

F_1



F_2

Now $f(F_1) = 1 + 6x + 10x^2 + 4x^3$ so $|P_3(F_1)| = \binom{6}{3} - 4 > \binom{6}{4} - 0 = |P_4(F_1)|$, and $f(F_2) = 1 + 6x + 10x^2 + 5x^3$ so $|P_3(F_2)| = \binom{6}{3} - 5 = \binom{6}{4} - 0 = |P_4(F_2)|$. Thus, both F_1 and F_2 satisfy the first two conditions of Lemma 2.1.

3.3.2 F contains 2 copies of S_2

In this case, $F \cong 2S_2 + S_{2n-4}$ where $n \geq 4$. The independent set generating function for F is therefore $f(F) = (1+2x)^2[(1+x)^{2n-5} + x]$ so $P_k(F) = \binom{2n-5}{k} + 4\binom{2n-5}{k-1} + 4\binom{2n-5}{k-2}$ for $k \geq 4$. Therefore

$$|P_n(F)| - |P_{n+1}(F)| =$$
$$\binom{2n}{n} - \binom{2n-5}{n} - 4\binom{2n-5}{n-1} - 4\binom{2n-5}{n-2}$$
$$-\left\{\binom{2n}{n+1} - \binom{2n-5}{n+1} - 4\binom{2n-5}{n} - 4\binom{2n-5}{n-1}\right\}$$
$$= \binom{2n-5}{n-2} - \binom{2n-5}{n-1} > 0$$

and F satisfies the first two conditions of Lemma 2.1.

3.3.3 F contains 1 copy of S_2

If G has 8 vertices then $F \cong S_2 + 2S_3$. The independent set generating function for F is then $f(F) = (1+2x)(1+3x+x^2)^2 = 1 + 8x + 23x^2 + 28x^3 + 13x^4 + 2x^5$ so

$$|P_4(F)| = \binom{8}{4} - 13 > \binom{8}{5} - 2 = |P_5(F)|$$

and thus F satisfies the first two conditions of Lemma 2.1.

Otherwise, G has 10 or more vertices and thus F, and therefore G, contains the subgraph $H = S_2 + S_3 + S_4 + Z_{2n-9}$ where $n \geq 5$. We now show that H satisfies the hypotheses of Lemma 2.1.

First, $f(H) = (1+2x)(1+3x+x^2)(1+4x+3x^2+x^3)(1+x)^{2n-9}$. The inequality $|P_n(H)| \geq |P_{n+1}(H)|$ may be verified to be equivalent to $n^3 + 16n^2 - 107n + 140 \geq 0$ which holds for $n \geq 5$.

To verify the second condition in the lemma, we need to show that

$$\binom{2n-11}{n-1} + 9\binom{2n-11}{n-2} + 30\binom{2n-11}{n-3} + 45\binom{2n-11}{n-4} +$$
$$29\binom{2n-11}{n-5} + 6\binom{2n-11}{n-6} \leq \sum_{i=2}^{n-1}\binom{2i-1}{i}$$

for all $n \geq 6$. This follows from the fact that $120\binom{2n-11}{n-5} \leq \binom{2n-3}{n-1}$ for all $n \geq 6$, and the unimodality of the binomial coefficient.

3.3.4 F contains 0 copies of S_2

In this subcase, F, and therefore G, contains $H = 3S_3 + Z_{2n-9}$ where $n \geq 5$ as a subgraph. We now show that H satisfies the hypotheses of Lemma 2.1.

First, we have $f(H) = (1+3x+x^2)^3(1+x)^{2n-9}$. The inequality $|P_n(H)| \geq |P_{n+1}(H)|$ may be verified to be equivalent to $n^3 + 16n^2 - 107n + 140 \geq 0$ which holds for $n \geq 5$.

To verify the second condition in the lemma, we need to show that

$$\binom{2n-11}{n-1} + 9\binom{2n-11}{n-2} + 30\binom{2n-11}{n-3} + 45\binom{2n-11}{n-4} + 30\binom{2n-11}{n-5} + 9\binom{2n-11}{n-6} + \binom{2n-11}{n-7} \leq \sum_{i=2}^{n-1}\binom{2i-1}{i}$$

for all $n \geq 6$. This follows from the fact that $125\binom{2n-11}{n-5} \leq \binom{2n-3}{n-1}$ for all $n \geq 6$, and the unimodality of the binomial coefficient.

3.4 F has exactly four components

If G has exactly 8 vertices then F is isomorphic to $4S_2$. Since G is connected, the following graph H is a spanning subgraph of G.

H

Now $f(H) = (1+2x)^2(1+4x+3x^2)$ so $|P_4(H)| = \binom{8}{4} - 12 > \binom{8}{5} - 0 = |P_5(H)|$ and H satisfies the first two conditions of Lemma 2.1.

Otherwise, G contains 10 or more vertices and thus either $H_1 = 2S_2 + 2S_3 + Z_{2n-10}$ or $H_2 = 3S_2 + S_4 + Z_{2n-10}$ is a subgraph of G. We now show that both H_1 and H_2 satisfy the hypotheses of Lemma 2.1.

We have that $f(H_1) = (1+2x)^2(1+3x+x^2)^2(1+x)^{2n-10}$ and it may be shown that the inequality $|P_n(H_1)| \geq |P_{n+1}(H_1)|$ is equivalent to $49n^4 + 66n^3 - 3781n^2 + 14586n - 15120 \geq 0$ which holds for $n \geq 5$.

To verify the second condition in the lemma, we need to show that

$$\binom{2n-12}{n-1} + 10\binom{2n-12}{n-2} + 39\binom{2n-12}{n-3} + 74\binom{2n-12}{n-4} + 69\binom{2n-12}{n-5} + 28\binom{2n-12}{n-6} + 4\binom{2n-12}{n-7} \leq \sum_{i=2}^{n-1}\binom{2i-1}{i}$$

for all $n \geq 6$. It is easily verified that the inequality holds for $n = 6$. For $n \geq 7$, it may be shown that $225\binom{2n-12}{n-6} \leq \binom{2n-3}{n-1}$ which implies the truth of the desired inequality.

For H_2, we have $f(H_2) = (1+2x)^3(1+4x+3x^2+x^3)(1+x)^{2n-10}$ and it may be shown that the inequality $|P_n(H_2)| \geq |P_{n+1}(H_2)|$ is equivalent to $5n^4 + 54n^3 - 857n^2 + 2982n - 3024 \geq 0$ which holds for $n \geq 5$.

To verify the second condition in the lemma, we need to show that

$$\binom{2n-12}{n-1} + 10\binom{2n-12}{n-2} + 39\binom{2n-12}{n-3} + 75\binom{2n-12}{n-4} +$$
$$74\binom{2n-12}{n-5} + 36\binom{2n-12}{n-6} + 8\binom{2n-12}{n-7} \leq \sum_{i=2}^{n-1}\binom{2i-1}{i}$$

for all $n \geq 6$. It is easily verified that the inequality holds for $n = 6, 7$. For $n \geq 8$, it may be shown that $243\binom{2n-12}{n-6} \leq \binom{2n-3}{n-1}$. Once again, the desired inequality follows from the unimodality of the binomial coefficient.

3.5 F has exactly five components

If G has exactly 10 vertices then F is isomorphic to $5S_2$. Now

$$|P_5(F)| = \binom{10}{5} - 2^5 > \binom{10}{6} - 0 = |P_6(F)|$$

and so F satisifes the first two conditions of Lemma 2.1.

Otherwise, either $H_1 = 2S_2 + 2S_3 + Z_{2n-10}$ or $H_2 = 3S_2 + S_4 + Z_{2n-10}$ is a subgraph of G. It was verified in Section 3.4 that both H_1 and H_2 satisfy the first two conditions of Lemma 2.1.

3.6 F has six or more components

In this case, F contains $H = 6S_2 + Z_{2n-12}$ as a subgraph. We now show that H satisfies the hypotheses of Lemma 2.1.

First, $f(H) = (1+2x)^6(1+x)^{2n-12}$. It may be verified that the inequality $|P_n(H)| \geq |P_{n+1}(H)|$ is equivalent to $451n^5 - 4276n^4 - 379n^3 + 111664n^2 - 364500n + 332640 \geq 0$ which holds for $n \geq 6$.

To verify the second condition in the lemma, we need to show that

$$\binom{2n-14}{n-1} + 12\binom{2n-14}{n-2} + 60\binom{2n-14}{n-3} + 160\binom{2n-14}{n-4} +$$
$$240\binom{2n-14}{n-5} + 192\binom{2n-14}{n-6} + 64\binom{2n-14}{n-7} \leq \sum_{i=2}^{n-1}\binom{2i-1}{i}$$

for all $n \geq 7$. It is easily verified that the inequality holds for $n = 7$. For $n \geq 8$, it may be shown that $729\binom{2n-14}{n-7} \leq \binom{2n-3}{n-1}$ and the desired inequality follows.

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