Isomorphic Star Decompositions of Multicrowns and the Power of Cycles

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ABSTRACT. For positive integers $k \leq n$, the crown $C_{n,k}$ is the graph with vertex set $\{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$ and edge set $\{a_ib_j : 1 \leq i \leq n, j = i+1, i+2, \cdots, i+k \pmod n\}$. For any positive integer λ, the multicrown $\lambda C_{n,k}$ is the multiple graph obtained from the crown $C_{n,k}$ by replacing each edge e by λ edges with the same end vertices as e. A star S_l is the complete bipartite graph $K_{1,l}$. If the edges of a graph G can be decomposed into subgraphs isomorphic to a graph H, then we say that G has an H-decomposition. In this paper, we prove that $\lambda C_{n,k}$ has an S_l-decomposition if and only if $l \leq k$ and $\lambda nk \equiv 0 \pmod l$. Thus, in particular, $C_{n,k}$ has an S_l-decomposition if and only if $l \leq k$ and $nk \equiv 0 \pmod l$. As a consequence, we show that if $n \geq 3$, $k < \frac{n}{2}$, then C_n^k, the k-th power of the cycle C_n, has an S_l-decomposition if and only if $l \leq k+1$ and $nk \equiv 0 \pmod l$.

1 Introduction and preliminaries

For positive integers $k \leq n$, the *crown* $C_{n,k}$ is the graph with vertex set $\{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$ and edge set $\{a_ib_j : 1 \leq i \leq n, j = i+1, i+2, \cdots, i+k \pmod n\}$. For any positive integer λ, let $\lambda C_{n,k}$ denote the multiple graph obtained from the crown $C_{n,k}$ by replacing each edge e by λ edges with the same end vertices as e. We call $\lambda C_{n,k}$ a *multicrown*. In this paper, we consider the edge decomposition of multicrowns into isomorphic stars. Edge decompositions of crowns have been investigated for isomorphic paths [4] and isomorphic complete bipartite graphs [2].

Let $(p_1, p_2, \cdots, p_m)$, $(q_1, q_2, \cdots q_n)$ be two sequences of nonnegative integers. We say that the pair $((p_1, p_2, \cdots, p_m), (q_1, q_2, \cdots q_n))$ is *bigraphical* if there exists a bipartite graph G with bipartition $(\{x_1, x_2, \cdots, x_m\}, \{y_1, y_2, \cdots y_n\})$ such that

$$\begin{aligned} \deg_G x_i &= p_i \quad \text{for } 1 \leq i \leq m \quad \text{and} \\ \deg_G y_j &= q_j \quad \text{for } 1 \leq j \leq n. \end{aligned}$$

We need the following propositions for our discussions.

Proposition 1.1 [3, 6] *Suppose that $l, m, r_1, r_2, \cdots, r_n$ are nonnegative integers such that $r_i \leq m$ for $i = 1, 2, \cdots, n$, and $r_1+r_2+\cdots+r_n = ml$. Then $((l, l, \cdots, l), (r_1, r_2, \cdots, r_n))$ is bigraphical, where $(l, l, \cdots, l)$ is a sequence of m terms.* □

For a vertex x in a multiple graph G, let $\deg_G x$, the *degree* of x in G, denote the number of edges incident with x.

A *star* S_l is the complete bipartite graph $K_{1,l}$. A *multiple star* is a star with multiple edges allowed. We use $S(r_1, r_2, \cdots, r_n)$ to denote the multiple star with respective edge multiplicities $r_1, r_2, \cdots, r_n$. As an illustration, the multiple star $S(2, 3, 3)$ is displayed in Figure 1. The vertex of degree $\sum_{i=1}^{n} r_i$ in $S(r_1, r_2, \cdots, r_n)$ is called the center of the multiple star and the other vertices are called the terminal vertices of the multiple star.

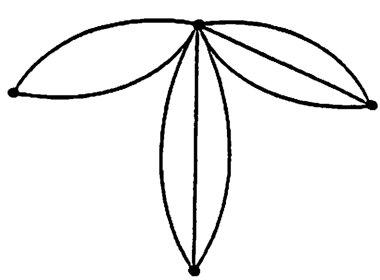

Figure 1: Multiple Star $S(2, 3, 3)$

Proposition 1.2 *The edges of the multiple star $S(r_1, r_2, \cdots, r_n)$ can be decomposed into stars $S_{t_1}, S_{t_2}, \cdots, S_{t_m}$ if and only if $((t_1, t_2, \cdots, t_m), (r_1, r_2, \cdots, r_n))$ is bigraphical.*

Proof: (*Necessity*) Let u be the center and $v_1, v_2, \cdots, v_n$ the terminal vertices of the multiple star $S(r_1, r_2, \cdots, r_n)$. Suppose that the edges of $S(r_1, r_2, \cdots, r_n)$ is decomposed into subgraphs $G_1, G_2, \cdots, G_m$ where each G_i is isomorphic to the star S_{t_i}, and G_i has center u and terminal vertices $v_{i_1}, v_{i_2}, \cdots, v_{i_{t_i}}$. Let G be a bipartite graph with bipartition (X, Y), where $X = \{x_1, x_2, \cdots, x_m\}$, $Y = \{y_1, y_2, \cdots, y_n\}$ such that $E(G) = \{x_i y_j : 1 \leq i \leq m, j = i_1, i_2, \cdots, i_{t_i}\}$. Clearly,

$$\begin{aligned} \deg_G x_i &= t_i, \quad 1 \leq i \leq m, \\ \deg_G y_j &= r_j, \quad 1 \leq j \leq n. \end{aligned}$$

Thus $((t_1, t_2, \cdots, t_m), (r_1, r_2, \cdots, r_n))$ is bigraphical.
(*Sufficiency*) Let G be a bipartite graph with bipartition $(\{x_1, x_2, \cdots, x_m\}, \{y_1, y_2, \cdots, y_n\})$ such that $\deg x_i = t_i$, $1 \leq i \leq m$ and $\deg y_j = r_j$, $1 \leq j \leq n$. Let H be the multiple graph obtained from G by identifying the vertices $x_1, x_2, \cdots, x_m$. Obviously H is the multiple star $S(r_1, r_2, \cdots, r_n)$. Since G can be decomposed into stars $S_{t_1}, S_{t_2}, \cdots, S_{t_m}$, the multiple star $S(r_1, r_2, \cdots, r_n)$ can also be decomposed into stars $S_{t_1}, S_{t_2}, \cdots, S_{t_m}$. □

Suppose that G and H are graphs and that the edges of G can be decomposed into subgraphs isomorphic to H. Then we say that G has an *H-decomposition*. The following proposition follows immediately from Propositions 1.1 and 1.2.

Proposition 1.3 *Suppose that $l, m, r_1, r_2, \cdots r_n$ are nonnegative integers such that $r_i \leq m$ for $i = 1, 2 \cdots, n$, and $r_1 + r_2 + \cdots + r_n = ml$. Then the multiple star $S(r_1, r_2, \cdots r_n)$ has an S_l-decomposition.* □

2 Main result: isomorphic star decomposition of the multicrown

In this section we give a necessary and sufficient condition for the multicrown $\lambda C_{n,k}$ to have an S_l-decomposition. We begin with the following lemma.

Lemma 2.1 *Let $\{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$ be the vertex set of the multicrown $\lambda C_{n,k}$. Suppose that q, l are positive integers such that $q < l \leq k$ and $\lambda nq \equiv 0 \pmod{l}$. Then there exists a spanning subgraph G of $\lambda C_{n,k}$ such that $\deg_G b_j = \lambda q$ for $1 \leq j \leq n$ and G has an S_l-decomposition.*

Proof: Let $g = \text{g.c.d.}(\lambda n, l)$. Then $\lambda n = gb$, $l = gc$ for some positive integers b, c with $\text{g.c.d.}(b, c) = 1$. Since $\lambda nq \equiv 0 \pmod{l}$, we have $bq \equiv 0 \pmod{c}$, which implies that $q \equiv 0 \pmod{c}$. Let d be the integer with $q = cd$. Since $cd = q < l = gc$, we have $d < g$.

Let B be a subset of $\{1, 2, \cdots, \lambda n\}$ such that $B = \{i_1 g + i_2 : i_1, i_2$ are integers, $0 \le i_1 \le b - 1, 1 \le i_2 \le d\}$ (note that $\lambda n = gb$, $d < g$). Let G_1 be the bipartite graph with bipartition $(\{x_1, x_2, \cdots, x_{\lambda n}\}, \{y_1, y_2, \cdots, y_{\lambda n}\})$ and edge set $\{x_i y_j : i \in B, j = i + 1, i + 2, \cdots, i + l \pmod{\lambda n}\}$ (note that $l = gc$). For $i \in B$, let $G_{1,i}$ be the subgraph of G_1 induced by the vertices $x_i, y_{i+1}, y_{i+2}, \cdots, y_{i+l}$. Obviously G_1 is decomposed into subgraphs $G_{1,i}$ $(i \in B)$, and each $G_{1,i}$ is isomorphic to the star S_l. Also obviously, $\deg_{G_1} x_i = 0$ or l for $i = 1, 2, \cdots, \lambda n$. Since $l \equiv 0 \pmod{g}$, it is not hard to see that $\deg_{G_1} y_j = (l/g)d = q$ for $j = 1, 2, \cdots, \lambda n$.

Now we define the required spanning subgraph G of $\lambda C_{n,k}$ as follows. First, define a graph G_2 from G_1. For each j $(j = 1, 2, \cdots, n)$ we identify the vertices y_j, y_{j+n}, y_{j+2n}, $\cdots$, $y_{j+(\lambda-1)n}$, and let b_j be the new vertex obtained. Let G_2 be the graph thus obtained from G_1. Since G_1 is a simple graph and $l \le n$, we can see that G_2 is also a simple graph. Since $\deg_{G_1} y_j = q$ for $1 \le j \le \lambda n$, we have $\deg_{G_2} b_j = \lambda q$ for $1 \le j \le n$.

Next we define a graph G from G_2. For each i $(i = 1, 2, \cdots, n)$ we identify the vertices x_i, x_{i+n}, x_{i+2n}, $\cdots$, $x_{i+(\lambda-1)n}$ and let a_i be the new vertex obtained. Let G be the graph thus obtained from G_2.

Now we show that G can be viewed as a spanning subgraph of $\lambda C_{n,k}$. Since G_2 is a simple graph, each edge in G has multiplicity $\le \lambda$. Each edge $x_i y_{i+t}$ in G_1, where $i \in B$, $1 \le t \le l$, and the index of y is taken modulo λn, is converted into an edge joining $a_{i'}$ and $b_{i'+t}$ in G, where $i' = i - vn$ for some integer v such that $1 \le i' \le n$ and the index of b is taken modulo n. Hence, from the facts that $V(G) = \{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$, each edge in G joins $a_{i'}$ and $b_{i'+t}$ for some $1 \le i' \le n$, $1 \le t \le l$ where $l \le k$ and each edge in G has multiplicity $\le \lambda$, we can view G as a spanning subgraph of $\lambda C_{n,k}$.

We show that G has the required properties about decomposition and degrees. Recall that G_1 is decomposed into $G_{1,i}$ $(i \in B)$. Since $l \le n$, we see that $G_{1,i}$, which is a star S_l in G_1, is converted into a star S_l in G. Thus G has an S_l-decomposition. Furthermore, since $\deg_{G_2} b_j = \lambda q$ for $1 \le j \le n$, we have $\deg_G b_j = \lambda q$ for $1 \le j \le n$. □

We are now ready to prove the main result of this section. For vertices x, y in a multiple graph G, we use $m_G(x, y)$ to denote the number of edges joining x and y.

Theorem 2.2 *The multicrown $\lambda C_{n,k}$ has an S_l-decomposition if and only if $l \le k$ and $\lambda nk \equiv 0 \pmod{l}$.*

Proof: (*Necessity*) The fact that S_l is a subgraph of $\lambda C_{n,k}$ implies $l \leq k$. Since $\lambda C_{n,k}$ has an S_l-decomposition, the size of $\lambda C_{n,k}$ is a multiple of the size of S_l. Thus $\lambda nk \equiv 0 \pmod{l}$.

(*Sufficiency*) Suppose that p and q are nonnegative integers such that $k = pl + q$ where $0 \leq q < l$. Since $l \leq k$, we have $p \geq 1$. Consider two cases.
Case 1. $q = 0$.
Since $k \equiv 0 \pmod{l}$, it is trivial that $C_{n,k}$ has an S_l-decomposition. Hence so does $\lambda C_{n,k}$.
Case 2. $0 < q < l$.
It follows from $\lambda nk \equiv 0 \pmod{l}$ and $k = pl + q$ that $\lambda nq \equiv 0 \pmod{l}$. By Lemma 2.1, there exists a spanning subgraph G of $\lambda C_{n,k}$ such that $\deg_G b_j = \lambda q$ for $1 \leq j \leq n$, and G has an S_l-decomposition. Let G' be the graph $\lambda C_{n,k} - E(G)$. Then $\deg_{G'} b_j = \lambda k - \lambda q = \lambda pl$ and $m_{G'}(b_j, a_i) \leq \lambda$ for $1 \leq i, j \leq n$. For $1 \leq j \leq n$, let $a_{j_1}, a_{j_2}, \cdots, a_{j_t}$ be the vertices adjacent to b_j, and let G'_j be the subgraph of G' induced by the vertices $b_j, a_{j_1}, a_{j_2}, \cdots, a_{j_t}$. We see that G'_j is a multiple star with center b_j, terminal vertices $a_{j_1}, a_{j_2}, \cdots, a_{j_t}$. and edge multiplicities $m_{G'_j}(b_j, a_{j_v})$ $(v = 1, 2, \cdots, t)$. Now we apply Proposition 1.3 to the multiple star G'_j. We have $\sum_{v=1}^{t} m_{G'_j}(b_j, a_{j_v}) = \sum_{v=1}^{t} m_{G'}(b_j, a_{j_v}) = \deg_{G'} b_j = \lambda pl$, and $m_{G'_j}(b_j, a_{j_v}) = m_{G'}(b_j, a_{j_v}) \leq \lambda \leq \lambda p$ for $v = 1, 2, \cdots, t$. By Proposition 1.3, each G'_j has an S_l-decomposition, which implies that G' has an S_l-decomposition. Since both G and G' have S_l-decompositions, $\lambda C_{n,k}$ has an S_l-decomposition. □

3 Isomorphic star decomposition of the power of a cycle

For a graph G and a positive integer k, let G^k denote the graph with $V(G^k) = V(G)$ and $E(G^k) = \{uv : u, v \in V(G), u \neq v, d_G(u,v) \leq k\}$ where $d_G(u,v)$ denotes the distance between u and v in G. We call G^k the k-th power of G (for example, see [5, p.228]) Let C_n denote the cycle on n vertices. In this section we will obtain a necessary and sufficient condition for C_n^k, the k-th power of C_n, to have an S_l-decomposition where $k < \frac{n}{2}$. A *multiple bipartite graph* is a bipartite graph with multiple edges allowed. The following lemma is trivial.

Lemma 3.1 *Suppose that G is a multiple bipartite graph with bipartition $(\{a_1, a_2, \cdots, a_n\}, \{b_1, b_2, \cdots, b_n\})$ such that a_i is not adjacent to b_i for $i =$*

$1, 2, \cdots, n$. *Let H be the multiple graph obtained from G by identifying a_i and b_i for each i. If G has an S_l-decomposition then so does H.* □

For any positive integer λ, let λC_n^k denote the multiple graph obtained from C_n^k by replacing each edge e by λ edges with the same end vertices as e. Let $\{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$ be the vertex set of the multicrown $\lambda C_{n,k}$ as defined in the beginning of Section 1. Assume that $n \geq 3$, $k < \frac{n}{2}$. The multiple graph obtained from $\lambda C_{n,k}$ by identifying a_i and b_i ($i = 1, 2, \cdots, n$) is λC_n^k. Combining Theorem 2.2 with Lemma 3.1, we have the following result.

Corollary 3.2 *Suppose that λ, n, k, l are positive integers such that $n \geq 3$, $l \leq k < \frac{n}{2}$ and $\lambda nk \equiv 0 \pmod{l}$. Then λC_n^k has an S_l-decomposition.* □

Given a graph G, we use $s(G)$ to denote the minimum number of stars (not necessarily isomorphic) needed to decompose the edges of G. A subset S of $V(G)$ is called an *independent set* of G if no two vertices in S are adjacent in G. Let $\alpha(G)$ denote the number of vertices in a maximum independent set of G. A subset K of $V(G)$ is called a *covering* of G if every edge of G has at least one end in K. Let $\beta(G)$ denote the number of vertices in a minimum covering of G. It is well known [1] that $\alpha(G) + \beta(G) = |V(G)|$. Also it is easy to see that $\beta(G) = s(G)$ if G is a simple graph. Now we determine $s(C_n^k)$. For the following discussions, we let $V(C_n) = \{x_1, x_2, \cdots, x_n\}$ and $E(C_n) = \{x_i x_{i+1} : i = 1, 2, \cdots, n\}$ where $x_{n+1} = x_1$.

Lemma 3.3 *Suppose that n, k are positive integers such that $n \geq 3$ and $k < \frac{n}{2}$. Then $\alpha(C_n^k) = \left\lfloor \frac{n}{k+1} \right\rfloor$ and $s(C_n^k) = \beta(C_n^k) = \left\lceil \frac{nk}{k+1} \right\rceil$.*

Proof: First determine $\alpha(C_n^k)$. Let

$$U = \{x_{i(k+1)} : i = 1, 2, \cdots, t\} \text{ where } t = \left\lfloor \frac{n}{k+1} \right\rfloor.$$

It is easy to see that U is an independent set of C_n^k. Thus $\alpha(C_n^k) \geq |U| = \left\lfloor \frac{n}{k+1} \right\rfloor$. Now we show the reverse inequality. Suppose $S = \{x_{i_1}, x_{i_2}, \cdots, x_{i_{|S|}}\}$ is an independent set of C_n^k where $1 \leq i_1 < i_2 < \cdots < i_{|S|} \leq n$. For $1 \leq j \leq |S|$, we have $i_{j+1} - i_j \geq k+1$ where $i_{|S|+1} = i_1 + n$.

Thus $|S| \leq \left\lfloor \frac{n}{k+1} \right\rfloor$, which implies $\alpha(C_n^k) \leq \left\lfloor \frac{n}{k+1} \right\rfloor$. Therefore $\alpha(C_n^k) = \left\lfloor \frac{n}{k+1} \right\rfloor$ and hence $s(C_n^k) = \beta(C_n^k) = n - \alpha(C_n^k) = n - \left\lfloor \frac{n}{k+1} \right\rfloor = \left\lceil \frac{nk}{k+1} \right\rceil$. □

We conclude the paper with a necessary and sufficient condition for isomorphic star decompositions of C_n^k.

Theorem 3.4 *Let n, k be positive integers such that $n \geq 3$, $k < \frac{n}{2}$. Then C_n^k has an S_l-decomposition if and only if $l \leq k+1$ and $nk \equiv 0 \pmod{l}$.*

Proof: (*Necessity*) Let $\mathscr{D}$ be an S_l-decomposition of C_n^k. Since $|E(C_n^k)| = nk$, $|E(S_l)| = l$, we have $nk \equiv 0 \pmod{l}$ and $|\mathscr{D}| = \frac{nk}{l}$. By Lemma 3.3, $s(C_n^k) = \left\lceil \frac{nk}{k+1} \right\rceil$. Since $|\mathscr{D}| \geq s(C_n^k)$, we have $l \leq k+1$.

(*Sufficiency*) We consider two cases.

Case 1. $l \leq k$.

By Corollary 3.2 (with $\lambda = 1$), C_n^k has an S_l-decomposition.

Case 2. $l = k+1$.

In this case, $nk \equiv 0 \pmod{k+1}$, which implies $n \equiv 0 \pmod{k+1}$. Let

$$\begin{aligned} n_1 &= \frac{n}{k+1}, \\ A &= \{j \text{ is an integer} : i(k+1) < j < (i+1)(k+1), \\ &\qquad \text{for some integer } i,\ i = 0, 1, 2 \cdots, n_1 - 1\}. \end{aligned}$$

For each $j \in A$ with $i(k+1) < j < (i+1)(k+1)$ for some $0 \leq i \leq n_1 - 1$, let G_j be the star with center x_j and terminal vertices $x_{i(k+1)}, x_{j+1}, x_{j+2}, x_{j+3}, \cdots, x_{j+k}$ where the indices are taken modulo n. Obviously, G_j is a star with $k+1$ edges and it is not hard to see that C_n^k is decomposed into stars G_j ($j \in A$). Thus C_n^k has an S_l-decomposition. □

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