Gracefulness of Replicated Paths and Cycles

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ABSTRACT. We investigate whether *replicated paths* and *replicated cycles* are graceful. We also investigate the number of different graceful labelings of the complete bipartite graph.

1 Introduction

Let G be a graph (without loops and without multiple edges) with set of vertices $V(G) = \{v_1, \ldots, v_n\}$ and set of edges $E(G)$ where $|E(G)| = e$. Let P_n and C_n be a path and a cycle with n edges respectively and let K_n be the complete graph on n vertices.

A *labeling* (or *valuation*) of a graph G is an assignment f of labels to the vertices of G that induces for each edge $\{u, v\}$ a label depending on the vertex label $f(u)$ and $f(v)$.

Let f be an injection from the vertices of G to the set $\{0, \ldots, e\}$. A.Rosa [5] called f a *β- valuation* of G if, when we assign to each edge $\{u, v\}$ the label $|f(u) - f(v)|$, the resulting edge labels are distinct (and thus, the edge labels are $1, 2, \ldots, e$). Golomb [3] subsequently called such labeling *graceful* and this terminology is now the most commonly used. The graph G is said to be *graceful* if it has a graceful labeling.

A *caterpillar* is a tree with the property that the removal of its endpoints leaves a path while a *lobst* is a tree with the property that the removal of its endpoints leaves a caterpillar. It has been shown that all caterpillars are graceful [5], and J.C.Bermond [1] conjectured that lobsters are graceful; see [2] not only for an extensive survey but also for open problems and conjectures.

We are interested in the following construction. Given a graph G with n vertices $v_1, \ldots, v_n$ and a vector $\mathbf{x}= (x_1, \ldots, x_n)$ of positive integers define the corresponding *replicated graph* of G, $R_{\mathbf{x}}(G)$, as follows: For each $v_i \in V(G)$ form a stable set S_i consisting of x_i new vertices $i = 1, \ldots, n$ (recall that a stable set S consists of a set of vertices such that there is not edge $\{v_i, v_j\}$ for all pair $v_i, v_j \in S$); two stable sets S_i, S_j $i \neq j$ form a complete bipartite graph if the edge $\{v_i, v_j\} \in E(G)$ and otherwise there are no edges between S_i and S_j.

In section 2, we show that the replicated path $R_{\mathbf{x}}(P_n)$ is graceful for all x and $n \geq 1$. In on 3, w prove that some replicated cycles are graceful. Among other results we show that $R_{\mathbf{x}}(C_n)$ is graceful for:

1) $x = (2, \ldots, 2)$ with n even,

2) $x = (m, 1, \ldots, 1)$ with $n \equiv 0 \ (mod\ 4)$ and $m \geq 1$,

3) $x = (2, 1, \ldots, 1)$ for all $n \geq 8$ and

4) $x = (2, 2, 1, \ldots, 1)$ with $n \equiv 0 \ (mod\ 4)$ and $n \geq 12$.

Finally, in section 4 we study the number of different graceful labelings of the complete bipartite graph $K_{m,n}$.

2 Replicated Paths

Let $R_{\mathbf{x}}(P_n)$ be a replicated graph of the path with $n+1$ vertices. $R_{\mathbf{x}}(P_1)$ with $x = (m_1, m_2)$ is just the complete bipartite graph K_{m_1,m_2} which is known to be graceful [3]; we may extend this result as follows:

We say that G is a *consecutively orderable bipartite graph* if $G = (S, T; E)$ is a bipartite graph such that the sets S and T (with $|S| = l$ and $|T| = m$) may be ordered $s_0, s_1, \ldots, s_{l-1}$ and $t_1, \ldots, t_m$ so that for each vertex t in T the neighbours $N(t)$ of t, are consecutive, (this is $N(t) = \{s_i, s_{i+1}, \ldots, s_{i+r}\}$ for some i and r, $i+r \leq l-1$) with $N(t_{i-1}) \cap N(t_i) \neq \emptyset$ for each $i = 2, \ldots, m$ and $s_0 \in N(t_1)$ and $s_{l-1} \in N(t_m)$ (see for example figure 2).

Define a graph H on vertex set $\{t_1, \ldots, t_m\}$ by letting t_i and t_j $i \neq j$ be adjacent if they have a common neighbour in G. Then for G to be consecutively orderable, we insist that H is an *interval* graph, with $t_1, \ldots, t_m$ giving a Hamilton path, and that say t_1 corresponds to a *leftmost* interval and t_m to *rightmost*.

Lemma 2.1 *Let $G = (S, T; E)$ be a consecutively orderable bipartite graph. Then G is graceful.*

Proof: Let $e = |E(G)|$, $r_i = |N(t_i)|$ and $v_i = \min\{j | s_j \in N(t_i)\}$ for each $i = 1, \ldots, m$.

Label s_i with number i for each $i = 0, \ldots, l-1$. Number t_1 with label $l(t_1) = e$ and t_i with label $l(t_i) = e - \sum_{j=1}^{i-1} r_j + v_i$ for each $i = 2, \ldots, m$.

Since $N(t_{i-1}) \cap N(t_i) \neq \emptyset$ then $v_{i+1} < v_i + r_i$ so $l(t_{i+1}) = e - \sum_{j=1}^{i} r_j + v_{i+1} < e - \sum_{j=1}^{i-1} r_j + v_i = l(t_i)$. Hence, the labels $l(t_i)$ are strictly decreasing with $l(t_m) = e - \sum_{j=1}^{m-1} r_j + v_m = r_m + v_m = l$. Thus all the vertex labels are distinct and are in $\{0, 1, \ldots, e\}$. Further, it is easy to see that the edge labels are precisely $\{1, 2, \ldots, e\}$. □

Note that this labeling . .kes $O(e)$ steps.

Lemma 2.2 *$R_x($. consecutively orderable bipartite gragh for all $n \geq 1$ and all x.*

Proof: This is (· by figure 1 (whether the path has an even or odd number of vertices,. □

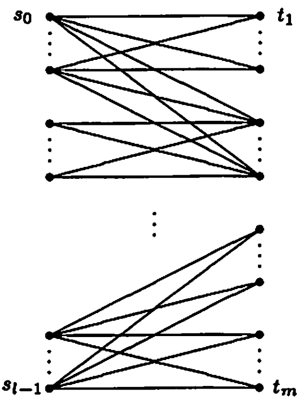

Figure 1

Theorem 2.3 *$R_x(P_n)$ is graceful for all $n \geq 1$ and x.*

Proof: By lemmas 2.1 and 2.2. □

Example 1: We illustrate in figure 2 how the labeling given in lemma 2.1 works for the replicated path $R_x(P_6)$ with $x = (3, 3, 2, 4, 2, 1, 5)$.

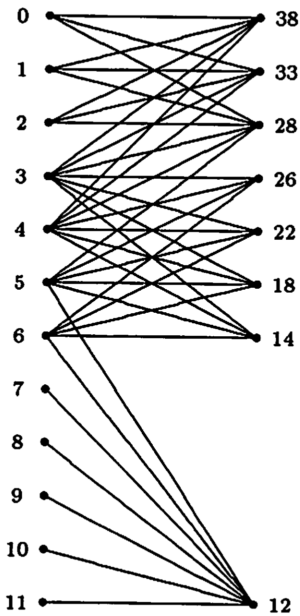

Figure 2

Lemma 2.4 *All caterpillars are consecutively orderable bipartite graphs.*

Proof: The *natural* plane bipartite drawing yields orderings as required, see figure 3. □

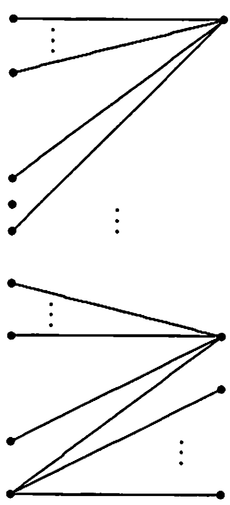

Figure 3

Theorem 2.5 *All caterpillars are graceful.*

Proof: By lemmas 2.1 and 2.4. □

3 Replicated cycles

Let $R_x(C_n)$ be the replicated graph of a cycle with n vertices. It is known [5] that $R_1(C_n)$ (that is, C_n) is graceful if and only if $n \equiv 0$ or 3 (mod 4); we partially extend this result.

Theorem 3.1 *Let n be any positive integer $n \geq 2$ and let $x = (2, \dots, 2)$. Then $R_x(C_{2n})$ is graceful.*

Proof: Case 1. n even. Consider the vertex labeling given in figure 4.

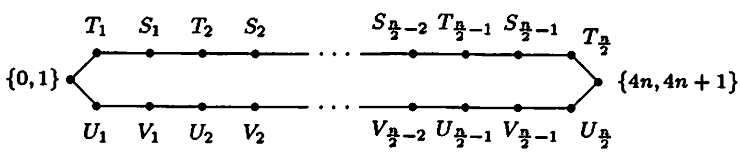

Figure 4

where $U_i = \{8(n-i)+8, 8(n-i)+6\}$ for $i = 1, \dots, \frac{n}{2}$,
$T_i = \{8(n-i)+4, 8(n-i)+2\}$ for $i = 1, \dots, \frac{n}{2}$,
$S_i = \{8i-4, 8i\}$ for $i = 1, \dots, \frac{n}{2}$ and
$V_i = \{8i+1, 8i+5\}$ for $i = 1, \dots, \frac{n}{2}$.

Note that all the vertex labels are different. The pairs of vertex labels $(\{0,1\}, T_1)$ and $(\{0,1\}, U_1)$ form the edge labels $\{8n, 8n-1, \dots, 8n-7\}$ and the pairs of vertex labels $(\{4n, 4n+1\}, T_{\frac{n}{2}})$ and $(\{4n, 4n+1\}, U_{\frac{n}{2}})$ form the edge labels $\{1, 2, \dots, 8\}$; hence, just remain the edge labels $\{9, 10, \dots, 8n-9, 8n-8\}$.

Consider each triple $E_i = (T_i, S_i, T_{i+1})$ for $1 \leq i \leq \frac{n}{2}-1$. The edge labels in each E_i are the numbers $8n - 16i + p$ with $p = -6, -4, -2, 0, 2, 4, 6, 8$; note that, the minimal label in E_i is strictly bigger than the maximal label in E_{i+1}. Hence there are not repeated edge labels among the triples E_i. Moreover, these edge labels are al the even numbers between 10 (with $i = \frac{n}{2} - 1$ and $p = -6$) and $8n - 8$ (with $i = 1$ and $p = 8$).

Similarly, consider each triple $O_i = (U_i, V_i, U_{i+1})$ for $1 \leq i \leq \frac{n}{2} - 1$. The edge labels in each O_i are the numbers $8n - 16i + p$ with $p = -7, -5, -3, -1, 1, 3, 5, 7$; again, the minimal label in O_i is strictly bigger than the maximal label in O_{i+1}. Hence there are not repeated edge labels among the triples O_i. Moreover, these edge labels are all the odd numbers between 9 (with $i = \frac{n}{2} - 1$ and $p = -7$) and $8n - 9$ (with $i = 1$ and $p = 7$).

Case 2. n odd. Consider the vertex labelings given in figure 5 and 6.
For $n = 3$

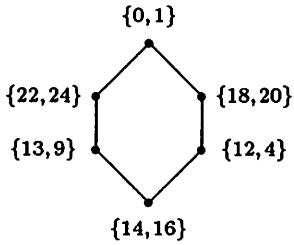

Figure 5

For $n \geq 5$

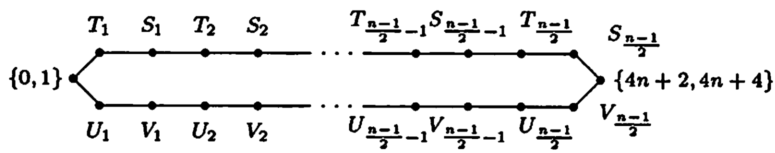

Figure 6

where $U_i = \{8(n-i)+8, 8(n-i)+6\}$ for $i = 1, \ldots, \frac{n-1}{2}$,
$T_i = \{8(n-i)+4, 8(n-i)+2\}$ for $i = 1, \ldots, \frac{n-1}{2}$,
$S_i = \{8i-4, 8i\}$ for $i = 1, \ldots, \frac{n-1}{2} - 1$,
$S_{\frac{n-1}{2}} = \{4n-8, 4n\}$ and
$V_i = \{8i+1, 8i+5\}$ for $i = 1, \ldots, \frac{n-1}{2}$.

The triples $(T_{\frac{n-1}{2}}, S_{\frac{n-1}{2}}, \{4n+2, 4n+4\})$ and $(U_{\frac{n-1}{2}}, V_{\frac{n-1}{2}}, \{4n+2, 4n+4\})$ form the edge labels $\{1, 2, \ldots, 16\}$.

The remaining edge labels $\{17, 18, \ldots, 8n\}$ are formed by the rest of the triples as the case before. □

Theorem 3.2 *Let m, n be positive integers with $m \geq 1$, $n \equiv 0 \pmod 4$ and let $x = (m, 1, \ldots, 1)$. Then $R_x(C_n)$ is graceful*

Proof: Let $v' = |V(R_x(C_n))| = n + m - 1$ and $e' = |E(R_x(C_n))| = n + 2(m-1)$ where $x = (m, 1, \ldots, 1)$; consider the following labeling.

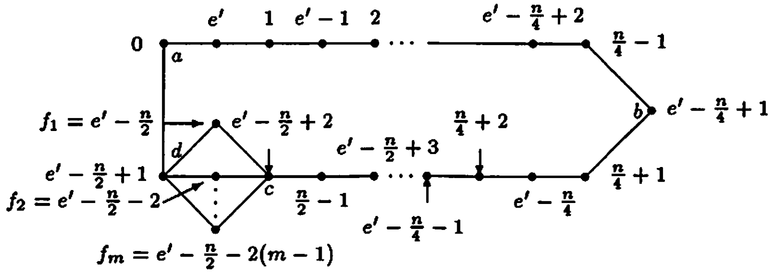

Figure 7

The vextex labels are formed by the sets $\{e', e'-1, \ldots, e'-\frac{n}{2}+1\}$, $\{0, 1, \ldots, \frac{n}{2}-1\}\backslash\{\frac{n}{4}\}$ and the numbers $e'-\frac{n}{2}-2(i-1)$ for each $i = 1, \ldots, m$. Since $e'-\frac{n}{2}-2(m-1) = \frac{n}{2} > \frac{n}{2}-1$ then all the vertex labels are different. The paths P_{ab} and P_{bc} contain the edge labels $\{e', e'-1, \ldots, e'-\frac{n}{2}+2\}$ and $\{e'-\frac{n}{2}, \ldots, e'-n+3 = 2m+1\}$ respectively; and the edge $\{a, d\}$ has the label $e'-\frac{n}{2}+1$. Finally, it is easy to check that the edge labels formed by the vertices c, d and f_i for $i = 1, \ldots, m$ are the numbers $\{1, \ldots, 2m\}$. □

Example 2: A graceful labeling of $R_x(C_{12})$ with $x = (3, 1, \ldots, 1)$ given by Theorem 3.2 is shown in figure 8.

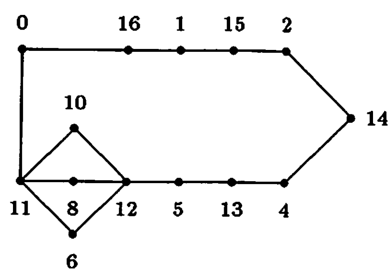

Figure 8

Theorem 3.3 *Let n be a positive integer $n \geq 8$ and let $x = (2, 1, \ldots, 1)$. Then $R_x(C_n)$ is graceful.*

Proof: Let $v' = |V(R_x(C_n))| = n+1$ and $e' = |E(R_x(C_n))| = n+2$ where $x = (2, 1, \ldots, 1)$. By theorem 3.2, it remains to prove the following three cases.

[I] $n \equiv 1 \ (mod\ 4)$ where $n \geq 9$. Consider the following labeling.

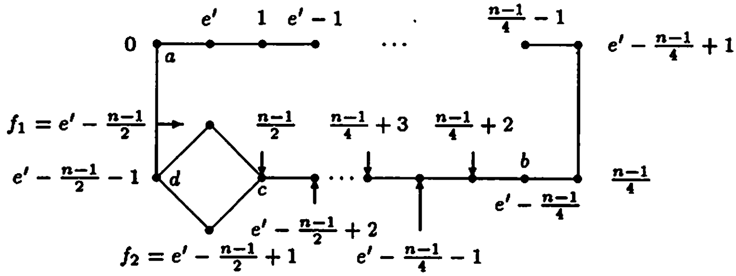

Figure 9

The vextex labels are formed by the sets $\{e', e'-1, \ldots, e'-(\frac{n-1}{2})-1\}$ and $\{0, 1, \ldots, \frac{n-1}{2}\} \backslash \{\frac{n-1}{4}+1\}$. Since $e'-(\frac{n-1}{2})-1 = \frac{n+1}{2}+1 > \frac{n-1}{2}$ then all the vertex labels are different. The paths P_{ab} and P_{bc} contain the edge labels $\{e', e'-1, \ldots, e'-(\frac{n-1}{2})\}$ and $\{e'-(\frac{n-1}{2})-2, \ldots, e'-(\frac{n-1}{2})+2-(\frac{n-1}{2}) = 5\}$ respectively; and the edge $\{a, d\}$ has the label $e'-(\frac{n-1}{2})-1$. Finally, the edge labels formed by the vertices c, d and f_i for $i = 1, 2$ are the numbers $\{1, \ldots, 4\}$.

[II] $n \equiv 2 \ (mod\ 4)$ where $n \geq 6$. For $n = 6$ take the following labeling.

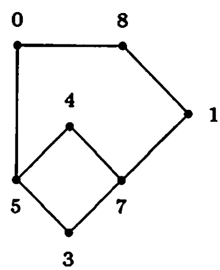

Figure 10

For $n \geq 10$ consider the following labeling.

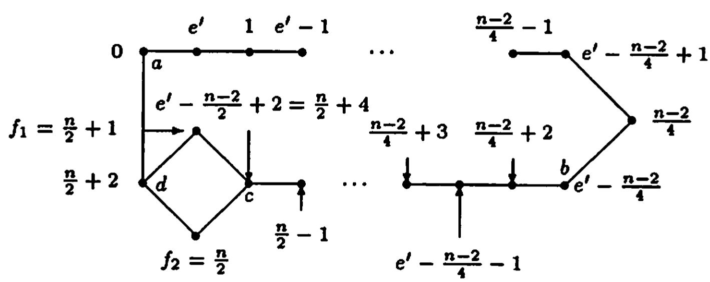

Figure 11

The vextex labels are formed by the sets $\{e', e'-1, \dots, e'-\frac{n}{2}+2\}$ and $\{0,1,\dots,\frac{n}{2}+2\}\setminus\{\frac{n-2}{4}+1\}$. Since $e'-\frac{n}{2}+2=\frac{n}{2}+4>\frac{n}{2}+2$ then all the vertex labels are different. The paths P_{ab} and P_{bc} contain the edge labels $\{e',e'-1,\dots,e'-\frac{n}{2}+1=\frac{n}{2}+3\}$ and $\{e'-\frac{n}{2}-1=\frac{n}{2}+1,\dots,e'-\frac{n}{2}+2-(\frac{n}{2}-1)=5\}$ respectively; and the edge $\{a,d\}$ has the label $\frac{n}{2}+2$. Finally, the edge labels formed by the vertices c, d and f_i for $i=1,2$ are the numbers $\{1,\dots,4\}$.

[III] $n \equiv 3 \ (mod\ 4)$ where $n \geq 11$. For $n=11$ take the following labeling.

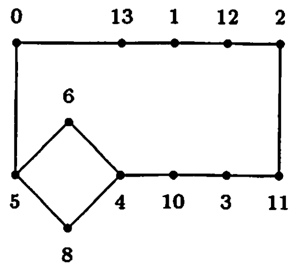

Figure 12

For $n \geq 15$ consider the following labeling.

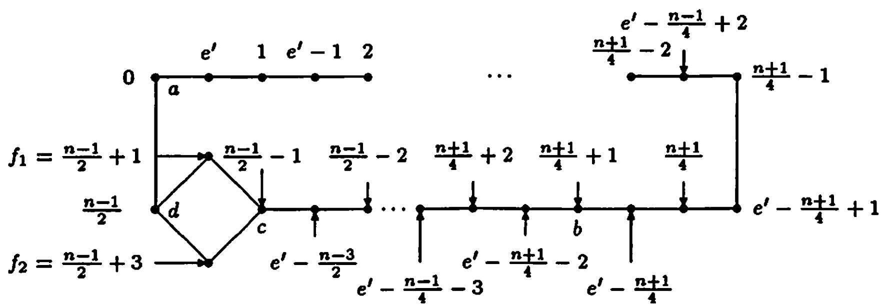

Figure 13

The vextex labels are formed by the sets $\{e', e'-1, \dots, e'-(\frac{n-3}{2})\}\setminus\{e'-(\frac{n+1}{4})-1\}$ and $\{0,1,\dots,\frac{n-1}{2}+3\}\setminus\{\frac{n-1}{2}+2\}$. Since $e'-(\frac{n-3}{2})=\frac{n+7}{2}=\frac{n-1}{2}+4>\frac{n-1}{2}+3$ then all thes are different. The paths P_{ab} and P_{bc} contain the edge labels $\{e',e'-1,\dots,\frac{n+1}{2}\}$ and $\{\frac{n+1}{2}-2,\dots,e'-(\frac{n-3}{2})-(\frac{n-1}{2}-1)=5\}$ respectively; and the edge $\{a,d\}$ has the label $\frac{n-1}{2}=\frac{n+1}{2}-1$. Finally, the edge labels formed by the vertices c, d and f_i for $i=1,2$ are the numbers $\{1,\dots,4\}$. □

We already proved that $R_{\mathrm{x}}(C_{2n})$ is graceful for $x=(2,2,\dots,2)$ and n even; we present now a close result.

Theorem 3.4 *Let n be a positive integer $n \equiv 0 \pmod 4$, $n \geq 12$ and let $x = (2, 2, 1, \ldots, 1)$. Then $R_x(C_n)$ is graceful.*

Proof: Let $v' = |V(R_x(C_n))| = n+2$ and $e' = |E(R_x(C_n))| = n+5$ where $x = (2, 2, 1, \ldots, 1)$. Consider the following labeling.

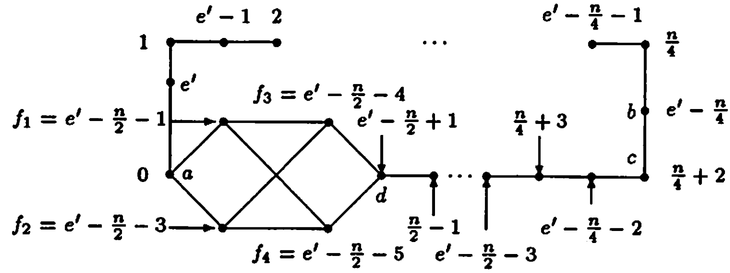

Figure 14

The vextex labels are formed by the sets $\{e', e'-1, \ldots, e'-\frac{n}{2}-5\}\backslash\{e'-\frac{n}{4}-1, e'-\frac{n}{2}, e'-\frac{n}{2}-2\}$ and $\{0, 1, \ldots, \frac{n}{2}-1\}\backslash\{\frac{n}{4}+1\}$. Since $e'-\frac{n}{2}-5 = \frac{n}{2} > \frac{n}{2}-1$ then all the vertex labels are different. The paths P_{ab} and P_{cd} contain the edge labels $\{e', e'-1, \ldots, e'-\frac{n}{2}\}$ and $\{e'-\frac{n}{2}-4, \ldots, e'-\frac{n}{2}+1-(\frac{n}{2}-1) = 7\}$ respectively; and the edges $\{a, f_1\}$, $\{b, c\}$, $\{a, f_2\}$, $\{d, f_3\}$ and $\{d, f_4\}$ have the labels $e'-\frac{n}{2}-1$, $e'-\frac{n}{2}-2$, $e'-\frac{n}{2}-3$, 5 and 6 respectively. Finally the edge labels formed by the vertices f_i for $i = 1, \ldots, 4$ are the numbers $\{1, \ldots, 4\}$. □

It is clear that $R_x(C_4)$ with x= $(x_1, \ldots, x_4)$ is always graceful since it is equivalent to $K_{x_1+x_3, x_2+x_4}$.

Given a graph G consider the join graph $G + \bar{K}_t$ (called the t point suspension of G) which consist of the graph G, t independent vertices and all the edges between the vertices of G and $\bar{K}_t$. It is known [4] that if the tree T_n is graceful then $T_n + \bar{K}_t$ is also graceful (just consider the graceful labeling of T_n with vertex labels $0, 1, \ldots, n$ and label the vertices of $\bar{K}_t$ by $\{2n+1, 3n+2, \ldots, (t+1)n+t\}$ and an easy computation shows that the labeling is graceful). Hence $R_{1,1,n}(C_3)$ is graceful since it is equivalent to $P_1 + \bar{K}_n$.

Theorem 3.5 *Let G be a graceful graph with $|V(G)| = |E(G)| = e$, and suppose that there is a graceful labeling such that either label 1 or $e-1$ does not appear in its vertex labels (we say that G is* graceful balanced*). Then $G + \bar{K}_t$ is graceful.*

Proof: Note that $|E(G + \bar{K}_t)| = e(t+1)$ and suppose that label $e-1$ does not appear. Let $l_1 = 0, l_2 = 1, \ldots, l_{e-1} = e-2, l_e = e$ be the vertex labels of G . Relabel the vertices of G with numbers $l_i(t+1)$ for all i and label the vertices of $\bar{K}_t$ by $\{e(t+1)-1, e(t+1)-2, \ldots, e(t+1)-t\}$.

Clearly, all the new vertex labels of G in $G+\bar{K}_t$ are different. The vertex label $e(t+1)$ together with all the vertex labels of $\bar{K}_t$ produce the edge labels $\{1, \ldots, t\}$ which are unique since $l_i \neq e-1$ for all i.

Each vertex label $l_i(t+1)$ of G form t different edge labels with the vertices of $\bar{K}_t$ and no two vertex labels $l_k(t+1)$ and $l_m(t+1)$, $k \neq m$ of G produce the same edge label, otherwise there exist i, j, $1 \leq i, j \leq t$ such that $e(t+1)-i-l_k(t+1) = e(t+1)-j-l_m(t+1)$ with $e-1 > l_k > l_m \geq 0$ then $(t+1)(l_k-l_m) = j-i$ so $j-i \geq t+1$ which is impossible. Hence $G+\bar{K}_t$ is graceful.

Finally, for the case when $l_i \neq 1$ for all i, we change the vertex labels of $\bar{K}_t$ to $\{1, \ldots, t\}$. □

$K_{2,2}$ is graceful balanced, then by theorem 3.5 the graph $K_{2,2}+\bar{K}_n$ is graceful, hence $R_x(C_3)$ with $x=(2,2,n)$ is also graceful.

4 Distinct graceful labeling

In this section we discuss whether or not a graceful labeling for some graphs G is unique. It is clear that given a graceful labeling $l_1, \ldots, l_k$ of any graph G a new graceful labeling is given by $e-l_i$ for all i with $e=|E(G)|$ (name it its *complement*).

Two graceful labelings f_1 and f_2 of G are isomorphic if there exists an isomorphism $\psi : V(G) \longrightarrow V(G)$ such that $f_1(v) = f_2(\psi(v))$. We say that two graceful labelings f_1 and f_2 are *equivalent* if they are isomorphic or one is isomorphic to the complement of the other.

We focus our attention in finding the number of distinct (this is, non-equivale) graceful labelings of K_{m_1,m_2} with $m_1, m_2 \geq 2$.

Let $g(G)$ be the number of distinct graceful labelings of G and let $\varphi(m)$ be the number of different factors of the integer m.

Theorem 4.1 *For the complete bipartite graph K_{m_1,m_2},*

$$g(K_{m_1,m_2}) \geq \begin{cases} 2(\varphi(m_1)+\varphi(m_2)-2) & \text{if } m_1 \neq m_2 \\ 2\varphi(m_1) & \text{otherwise.} \end{cases}$$

Proof: Let s be a positive integer such that $s|m_2$ (that is, exists an integer p such that $sp = m_2$). For each s we give a graceful labeling L and its complement L' in figure 15.

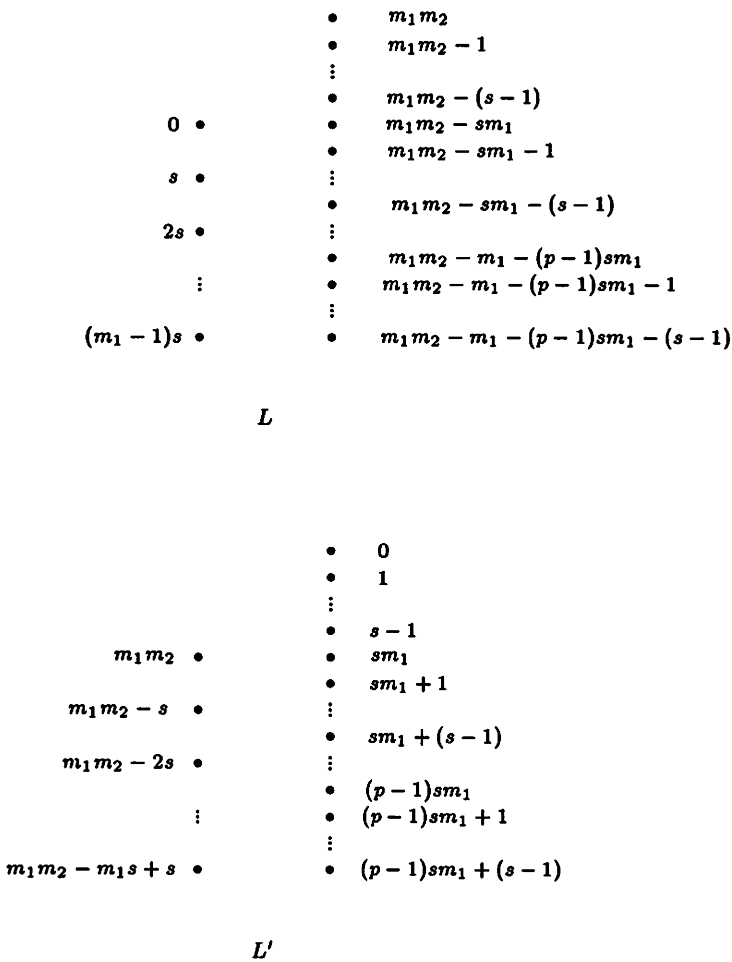

Figure 15

We have also similar labelings for each t with $t|m_1$. Note that in the case $m_1 \neq m_2$ all labelings L and L' for each s and t are distinct except when $s = m_2$ the corresponding labelings L and L' are the same as the labelings L' and L when $t = 1$ respectively and the same when $s = 1$ and $t = m_1$. □

There are *strange* graceful labeling of K_{m_1,m_2} other than those considered in the proof of theorem 4.1, as is shown in figure 16.

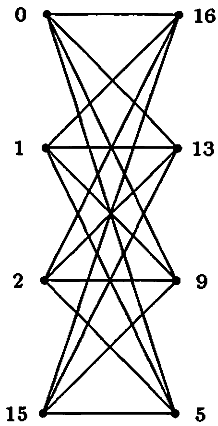

Figure 16

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References

[1] J.C. Bermond, Graceful graphs, radio antennae and French windmills, *Graph Theory and Combinatorics*, Pitman, London (1979), 13–37.

[2] J.A. Gallian, A survey: Recent results, conjectures, and open problems in labeling graphs, *Journal of Graph Theory* **13** No.4 (1989), 491–504.

[3] S.W. Golomb, How to number a graph, *Graph Theory and Computing*, Academic Press (1972), 23–37.

[4] T. Grace, On sequential labelings of graphs, *Journal of Graph Theory* .7 (1983), 195–201.

[5] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs*, (International Symposium, Rome July 1966), Gordon and Breach, New York, N.Y. and Dunod Paris (1967), 349–355.