Isomorphic factorization of complete bipartite graph into forest

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Abstract

We show that there exists an isomorphic factorization of a complete bipartite graph $K(m,n)$ into forests without isolated vertices if and only if $m+n-c$ divides mn and $m,n \geq c$.

1 Introduction

A subgraph H of G is called a *factor* of G if $V(H) = V(G)$, and a *factorization* of G is a decomposition of G into the edge disjoint factors. If each factor is isomorphic to some graph H, it is called the *isomorphic factorization*, and we say "H divides G" or "G is divisible by H". If H divides G, the number of edges of H divides that of G. This necessary condition is called the *divisibility condition*.

Let $K(m,n)$ be a complete bipartite graph with two partite sets having m and n vertices. Shibata and Seki[6] have investigated the isomorphic factorizations of complete bipartite graphs into trees. Since any spanning tree of $K(m,n)$ has $m+n-1$ edges, the divisibility condition means $m+n-1$ divides mn. They have introduced the notion of the *interlaced graph*, and shown that interlaced trees dividing $K(m,n)$ are constructed if $m+n-1$ divides mn.

Theorem 1.1 (Shibata, Seki[6]) *A complete bipartite graph $K(m,n)$ is divisible by a tree if and only if $m+n-1$ divides mn.*

In that paper, the divisibility of mn by $m+n-1$ has been studied. Shibata, Araki and Kogure[1] have investigated the generalization of the study, which is the divisibility of mn by $am+bn+c$. For integers m,n, (m,n) stands for the greatest common divisor of m and n. Other number theoretic terminology, we refer Shapiro[5].

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Theorem 1.2 ([1]) *For integer a, b, c, if $am+bn+c \neq 0$ or $mn \neq 0$, then*

$$(am+bn+c, mn) = \frac{(m, bn+c)(n, am+c)}{\theta},$$

where

$$\theta = \frac{(d_m, d_n)}{(d_m, d_n, a\alpha+\alpha', b\beta+\beta')},$$

$$\begin{array}{ll} d_m = (m, bn+c), & d_n = (n, am+c), \\ m = d_m\alpha, & n = d_n\beta, \\ bn+c = d_m\alpha' & am+c = d_n\beta'. \end{array}$$

From Theorem 1.2, a necessary and sufficient condition for the divisibility is obtained as follows.

Theorem 1.3 ([1]) *For integer a, b, c and a pair $[m, n]$ such that $am+bn+c \neq 0$, $am+bn+c$ divides mn if and only if*

$$|am+bn+c| = \frac{(m, bn+c)(n, am+c)}{\theta}.$$

Theorem 1.3 enables us to present the necessary and sufficient conditions for $am+bn+c$ dividing mn using parameters defined in Theorem 1.2.

Lemma 1.4 ([1]) *The following three statements are equivalent.*

1. $(am+bn+c)$ *divides* mn,
2. $d_m = \theta|b\beta+\beta'|$,
3. $d_n = \theta|a\alpha+\alpha'|$.

The following lemma is used later in this paper.

Lemma 1.5 *Let $am+bn+c$ divide mn. If $m = m_1\theta$, $n = n_1\theta$ and $c = c_1\theta$, then $m_1+n_1-c_1$ divides m_1n_1 and $\theta_1 = (m_1, bn_1+c_1)(n_1, am_1+c_1)/|am_1+bn_1+c_1| = 1$.*

Proof. Let $d_{m_1} = (m_1, bn_1+c_1)$ and $d_{n_1} = (n_1, am_1+c_1)$. Since $d_m = (\theta m_1, \theta(bn_1+c_1)) = \theta d_{m_1}$ and $d_n = (\theta n_1, \theta(am_1+c_1)) = \theta d_{n_1}$, we have

$$\begin{aligned} d_{m_1}d_{n_1} &= d_m d_n/\theta^2, \\ &= \theta|am+bn+c|/\theta^2, \\ &= |am_1+bn_1+c_1|. \end{aligned}$$

Further we have

$$m_1n_1 = \frac{mn}{\theta^2} = \frac{d_m d_n}{\theta^2}\alpha\beta = d_{m_1}d_{n_1}\alpha\beta.$$

Hence $am_1+bn_1+c_1$ divides mn. θ_1 is equal to $d_{m_1}d_{n_1}/|am_1+bn_1+c_1| = 1$.
∎

By generalizing the form of the number of edges $m+n-1$ to $am+bn+c$, it is expected that the structure of factors can be considered more widely. A forest with $c \geq 1$ components has $m+n-c$ edges. Thus, the divisibility condition for $K(m,n)$ divided by a forest is equivalent to $m+n-c$ dividing mn. The purpose of this paper is to prove the condition is also sufficient, that is, the following theorem holds.

Theorem 1.6 *A complete bipartite graph $K(m,n)$ is divisible by a forest with c components and without isolated vertices if and only if $m+n-c$ divides mn and $m,n \geq c$.*

2 Isomorphic factorizations

2.1 Isomorphic factorizations of complete bipartite graphs

Studies on the existence of isomorphic factorizations have been often considered using specific permutations on the vertex set(e.g., Harary[3]), and we follow this method.

Let the bipartition of $K(m,n)$ be $U \cup V$, where $U = \{u_0, u_1, \ldots, u_{m-1}\}$ and $V = \{v_0, v_1, \ldots, v_{n-1}\}$. For a positive integer c, permutations σ and τ on U and V, respectively, are defined as follows.

$$\sigma = \gamma_0\gamma_1 \cdots \gamma_{d_m-1}, \quad d_m = (m, n-c),$$
$$\tau = \phi_0\phi_1 \cdots \phi_{d_n-1}, \quad d_n = (n, m-c).$$

γ_i's and ϕ_j's are disjoint cyclic permutations having length α and β, respectively. Label the vertices in the cycles as $\gamma_i = (u_i^0, u_i^1, \ldots, u_i^{\alpha-1})$ and $\phi_j = (v_j^0, v_j^1, \ldots, v_j^{\beta-1})$. Let Γ_i and Φ_j be

$$\Gamma_i = \{u_i^0, u_i^1, \ldots, u_i^{\alpha-1}\}, \ (0 \leq i \leq d_m - 1),$$

$$\Phi_j = \{v_j^0, v_j^1, \ldots, v_j^{\beta-1}\}, \ (0 \leq j \leq d_n - 1).$$

For a bipartite graph G with bipartition $U \cup V$ and edge set $E(G)$, let G_{ij} be a bipartite graph with partite sets U and V and edge set

$$E(G_{ij}) = \left\{\sigma^i(u)\tau^j(v) | uv \in E(G), \ u \in U, \ v \in V\right\}.$$

Then we have $G \cong G_{ij}$ for $(0 \leq i \leq \alpha - 1, \ 0 \leq j \leq \beta - 1)$ and $G_{00} = G$. If $\cup_{ij} E(G_{ij})$ is a partition of $E(K(m,n))$, then G is an isomorphic factor of

$K(m,n)$ under σ and τ, and we say "G divides $K(m,n)$ under σ and τ". Let $E_{ij} = \{uv | u \in \Gamma_i,\ u \in \Phi_j\}$, $(0 \le i \le d_m - 1,\ 0 \le j \le d_n - 1)$. Then $\cup_{i,j} E_{ij}$ is a partition of $E(K(m,n))$.

Lemma 2.1 *A bipartite graph G with bipartition $U \cup V$ divides $K(m,n)$ under σ and τ if and only if $|E(G) \cap E_{ij}| = 1$ for all i,j $0 \le i \le d_m - 1$, $0 \le j \le d_n - 1$. If G divides $K(m,n)$ under σ and τ, then $|E(G)| = d_m d_n = \theta(am + bn + c)$, where d_m, d_n and θ are defined in Theorem 1.2.*

Proof. If G divides $K(m,n)$ under σ and τ, G has just one edge of E_{ij} in common. The converse holds obviously. Since $K(m,n)$ has d_m Γ_i's and d_n Φ_j's, G has $d_m d_n$ edges. ∎

Corollary 2.2 *If G divides $K(m,n)$ under σ and τ, then*

$$\sum_{u \in \Gamma_i} \deg(u) = d_n, \quad (0 \le i \le d_m - 1),$$

$$\sum_{v \in \Phi_j} \deg(v) = d_m, \quad (0 \le j \le d_n - 1).$$

2.2 Isomorphic forest factors

Let us consider an isomorphic factorization of $K(m,n)$ such that the isomorphic factors are forest. Let G be a forest with c components and without isolated vertices. If G divides $K(m,n)$, then $m, n \ge c$ and $|E(G)| = m+n-c$ divides mn. Hence, by Theorem 1.3, we have $m+n-c = (m, n-c)(n, m-c)/\theta$, where θ is a divisor of c. Assume that the pair $[m,n]$ satisfies $\theta > 1$. By Lemma 1.5, putting $m = m_1\theta$, $n = n_1\theta$, $c = c_1\theta$, we obtain a new pair $[m_1, n_1]$ such that $m_1 + n_1 - c_1$ divides $m_1 n_1$ and $\theta_1 = 1$. Hence, if $K(m_1, n_1)$ has an isomorphic forest factor G with c_1 components and without isolated vertices, then a graph $\overbrace{G \cup \cdots \cup G}^{\theta \text{ times}}$ is a forest which has $m+n-c$ edges and c components, and it divides $K(m_1\theta, n_1\theta) \cong K(m,n)$. Thus, if we can show that there exists isomorphic forest factors dividing $K(m,n)$ when $\theta = 1$, the proof of Theorem 1.6 is completed.

Shibata and Seki[6] have introduced a notion of the *interlaced graph.* $N_G(v)$ is a set of vertices adjacent to v in G.

Definition 2.3 *Let G divides $K(m,n)$ under σ and τ, and let $[m,n]$ satisfies $\theta = 1$. G is called interlaced if*

$$N_G(\Phi_j) = \cup_{v \in \Phi_j} N_G(v) = \{u_i^j | 0 \le i \le d_m - 1\}, \quad 0 \le j \le \alpha - 1.$$

So, if G is an interlaced graph, then $\cup_{0\leq j\leq \alpha-1} N_G(\Phi_j) = \cup_{0\leq i\leq d_m-1}\Gamma_i = U$.

Assume that $G = (U \cup V, E)$ is an interlaced forest. We now construct a bipartite graph G_1 from G as follows. Let the bipartition of $V(G_1)$ be $U_1 = \cup_{0\leq j\leq\alpha-1}\Phi_j$ and $V_1 = \cup_{\alpha\leq j\leq d_n-1}\Phi_j$. ($U_1 \cup V_1$ is a partition of V.) A vertex $u \in U_1$ and $v \in V_1$ are adjacent if and only if $N_G(u) \cap N_G(v) \neq \emptyset$ in G.

Lemma 2.4 *If G is an interlaced forest, then G_1 is a forest such that*

$$\sum_{v\in\Phi_j} \deg_{G_1}(v) = d_m, \quad \alpha \leq j \leq d_n - 1.$$

Moreover, the number of connected components of G is equal to that of G_1.

Proof. Since G is interlaced, a vertex v in V_1 is adjacent to some vertex u in U_1. If $|N_G(v)\cap N_G(u)| \geq 2$, then G has a cycle. Thus, we have $|N_G(v)\cap N_G(u)| = 1$. Hence $\deg_G(v) = \deg_{G_1}(v)$, and we obtain $\sum_{v\in\Phi_j}\deg_{G_1}(v) = d_m$, for $\alpha \leq j \leq d_n - 1$.

By the definition of the interlaced graph, if u and v are adjacent in G_1, there exists a path from u to v in G (since $N_G(u) \cap N_G(v) \neq \emptyset$). Thus, if G_1 has a cycle, then G also has a cycle. This contradicts that G is a forest. Hence, G_1 is a forest.

Finally, we show the number of connected components of G is equal to that of G_1. It is sufficient to prove that if G has a path of length three $v_i, u, v_j (u \in U, v_i, v_j \in V)$, there is a path from v_i to v_j in G_1. If $v_i, v_j \in U_1$, then both v_i and v_j must be in the same cyclic permutation in τ by the definition of the interlaced graph. This contradicts Lemma 2.1. Hence, without loss of generality, it is sufficient to consider the following two cases arise.

1. Case for $v_i \in U_1$ and $v_j \in V_1$.
   By the definition of G_1, $v_iv_j \in E(G_1)$.

2. Case for $v_i, v_j \in V_1$.
   By the definition of the interlaced graph, there is a vertex $v_k \in U_1$ adjacent to u. Since $v_iv_k, v_kv_j \in E(G_1)$ from the definition of G_1, there is a (v_i, v_j)-path in G_1.

Therefore, if G has a (v_i, v_j)-path, G_1 has also a (v_i, v_j)-path. From the construction method of G_1, if G has no (v_i, v_j)-path, G_1 also has no paths from v_i to v_j. Hence the number of connected components of G is equal to that of G_1. ∎

Remaining object is to construct an interlaced forest from G_1. Let G_1 be a forest with partite sets $U_1 = \cup_{0\leq j\leq\alpha-1}\Phi_j$ and $V_1 = \cup_{\alpha\leq j\leq d_n-1}\Phi_j$, and G_1 satisfies $\sum_{v\in\Phi_j}\deg_{G_1}(v) = d_m$, $(\alpha \leq j \leq d_n - 1)$. Construct G_2 from G_1 as follows.

1. The edge set of G_1 is $e_0, e_1, \dots, e_{\alpha' d_m - 1}$.

2. Subdivide the edges of G_1 and let w_i be the added vertex on the edge e_i.

The resulting graph is G_2.

Next, construct a graph $K(G_1)$ from G_2 as follows.

$$\begin{aligned} V(K(G_1)) &= \{w_0, w_1, \dots, w_{\alpha' d_m - 1}\}, \\ E(K(G_1)) &= \left\{ ww' \,\middle|\, \begin{array}{l} w, w' \in N_{G_2}(\Phi_j),\ \alpha \le j \le d_n - 1 \\ \text{or} \\ w \in N_{G_2}(v),\ w' \in N_{G_2}(v'), \\ v, v' \in \Phi_j,\ v \ne v',\ 0 \le j \le \alpha - 1 \end{array} \right\}. \end{aligned}$$

Consider a vertex coloring of $K(G_1)$ such that adjacent vertices have different colors. For some coloring of $K(G_1)$, the vertices $w_0, w_1, \dots, w_{\alpha' d_m - 1}$ of G_2 are allowed the same coloring as $K(G_1)$. Since $|N_{G_2}(\Phi_j)| = d_m$, $\alpha \le j \le d_n - 1$, the *chromatic number* $\chi(K(G_1))$ satisfies $\chi(K(G_1)) \ge d_m$.

Lemma 2.5 *If vertices $w_0, w_1, \dots, w_{\alpha' d_m - 1}$ of G_2 are colored with the same coloring of $K(G_1)$, then G_2 satisfies the following conditions:*

1. *for $\alpha \le j \le d_n - 1$, vertices in $N_{G_2}(\Phi_j)$ have different colors,*

2. *for $0 \le j \le \alpha - 1$, if $v, v' \in \Phi_j$, $v \ne v'$, then each vertex in $N_{G_2}(v)$ has a different color from any vertex in $N_{G_2}(v')$.*

When $\chi(K(G_1)) = d_m$, let us define a graph G_3 from G_2 as follows.

1. For a given d_m-coloring of $K(G_1)$, give the same color to the vertices $w_0, w_1, \dots, w_{\alpha' d_m - 1}$ of G_2. This d_m-colors are referred to $C = \{c_0, c_1, \dots, c_{d_m - 1}\}$.

2. Repeat the following procedure for all vertex in $U_1 = \cup_{0 \le j \le \alpha - 1} \Phi_j$ and all colors in C:

   (a) for $v \in U_1$, let $S_i(v)$ be the set of vertices in $N_{G_2}(v)$ colored with c_i,

   (b) if $S_i(v) \ne \emptyset$, delete the vertices in $S_i(v)$ and add a vertex $s_i(v)$, and join edges to $s_i(v)$ and vertices adjacent with $S_i(v)$ in G_2.

Lemma 2.6 *If $\chi(K(G_1)) = d_m$, then G_3 is a forest satisfying the following conditions,*

1. *for $\alpha \le j \le d_n - 1$, $|N_{G_3}(\Phi_j)| = d_m$ and vertices in $N_{G_3}(\Phi_j)$ have different colors,*

2. for $0 \le j \le \alpha - 1$, if $v, v' \in \Phi_j$, $v \ne v'$, then each vertex in $N_{G_3}(v)$ is colored with a different color from any vertex in $N_{G_3}(v')$.

3. for $0 \le j \le \alpha - 1$, $|N_{G_3}(\Phi_j)| \le d_m$.

If $\chi(K(G_1)) = d_m$, we can construct an interlaced graph from the above graph G_3.

Theorem 2.7 *If $\chi(K(G_1)) = d_m$, an interlaced forest is constructed from G_3.*

Proof. For $v \in U_1$, let $CS(v)$ be a set of colors of vertices in $N_{G_3}(v)$. By Lemma 2.6, $CS(v) \cap CS(v') = \emptyset$ for $v, v' \in \Phi_j$, $v \ne v'$ and $0 \le j \le \alpha - 1$.

For $0 \le j \le \alpha - 1$, since $|N_{G_3}(\Phi_j)| \le d_m$, the set of colors C can be partitioned such that

$$C = C_j^0 \cap C_j^1 \cap \cdots \cap C_j^{\beta-1}, \quad CS(v_j^i) \subset C_j^i, \quad j = 0, 1, \ldots, \beta - 1.$$

For each i, j $(0 \le i \le d_m - 1, 0 \le j \le \alpha - 1)$, add $|C_j^i - CS(v_j^i)|$ vertices to G_3 and add edges connecting these vertices to v_i^j. Color these vertices with different colors in $C_j^i - CS(v_j^i)$. Then, give a label u_i^j to the vertex in $N_{G_3}(\Phi_j)$ colored by c_i.

The resulting graph G is a forest which divides $K(m, n)$ under σ and τ. Moreover, since

$$N_G(\Phi_j) = \{u_i^j | 0 \le i \le d_m - 1\}, \quad (0 \le j \le \alpha - 1),$$

G is an interlaced forest. ∎

2.3 Existence of interlaced forests

From the results of the previous section, if a bipartite graph G_1 with a bipartition $U_1 \cup V_1$ satisfies

$$\sum_{v \in \Phi_j} \deg_{G_1}(v) = d_m, \quad \alpha \le j \le d_n - 1,$$

and $\chi(K(G_1)) = d_m$, an interlaced forest is constructed from G_1. In this section, we show the existence of such a forest G_1.

Construction Algorithm of G_1

Step 1. Determine integers $a_0, a_1, \ldots, a_{\alpha'\beta}$ and $b_0, b_1, \ldots, b_c$ such that

1. $a_0 = b_0 = 0$,
2. $a_r \ge 2$ $(1 \le r \le \alpha'\beta)$, $b_r \ge 1$ $(1 \le r \le c)$,

3. $\sum_{j=1}^{\beta} a_{i\beta+j} = d_m = \beta + \beta'$, $(0 \le i \le \alpha' - 1)$,

$\sum_{j=1}^{c} b_j = \alpha'\beta.$

Step 2. Determine integers h_{ij}, $(0 \le i \le c,\ 0 \le j \le b_i)$ such that

1. $h_{0j} = h_{i0} = 0$,

2. $h_{ij} = a_k$, where $k = \left(\sum_{0 \le l \le i-1} b_l\right) + j$.

Step 3. Construct graph H_{ij}.
The graph H_{ij}, $1 \le i \le c$, $1 \le j \le b_i$, is a complete bipartite graph $K(1, h_{ij})$ which has vertex sets $U_{ij} = \{v_k\ ;\ k = \alpha\beta + (\sum_{0 \le k \le i-1} b_k) + j - 1\}$ and

$$V_{ij} = \left\{ v_r \;\middle|\; \begin{array}{l} s_i + \sum_{k=0}^{j-1} h_{ik} - j + 1 \le r \le s_i + \sum_{k=0}^{j} h_{ik} - j, \\ \text{where } s_i = \sum_{k=0}^{i-1} \sum_{l=0}^{b_k} h_{kl} - \sum_{k=0}^{i-1} b_k + (i-1) \end{array} \right\}.$$

Step 4. $G_1 = \bigcup_{1 \le i \le c, 1 \le j \le b_i} H_{ij}$.

Lemma 2.8 *The graph G_1 constructed by the above algorithm satisfies $\chi(K(G_1)) = d_m$.*

Proof. Subdivide the edges of G_1, and give a label u_{i+j} for the added vertex on the edge $v_{\alpha\beta+i}v_j$. The resulting graph is G_2.
Let $\Phi_j = \{v_{j\beta}, v_{j\beta+1}, \dots, v_{j\beta+(\beta-1)}\}$, $0 \le j \le d_n - 1$.
Then, $N_{G_2}(\Phi_j) = \{u_{jd_m}, u_{jd_m+1}, \dots, u_{jd_m+(d_m-1)}\}$, for $0 \le j \le \alpha' - 1$.

Let $j = i \bmod d_m$, and give the color c_j to the vertex u_i. Then $N_{G_2}(\Phi_j)$, $0 \le j \le d_n - 1$, contains d_m vertices, which have different colors each other. Further, vertices in $N_{G_2}(\Phi_j)$, $(0 \le j \le \alpha - 1)$ have different colors. Therefore, $\chi(K(G_1)) = d_m$. ∎

From the above theorem, an interlaced forest can be constructed for m, n such that $m + n - c$ divides mn and $\theta = 1$. Therefore, Theorem 1.6 is proved.

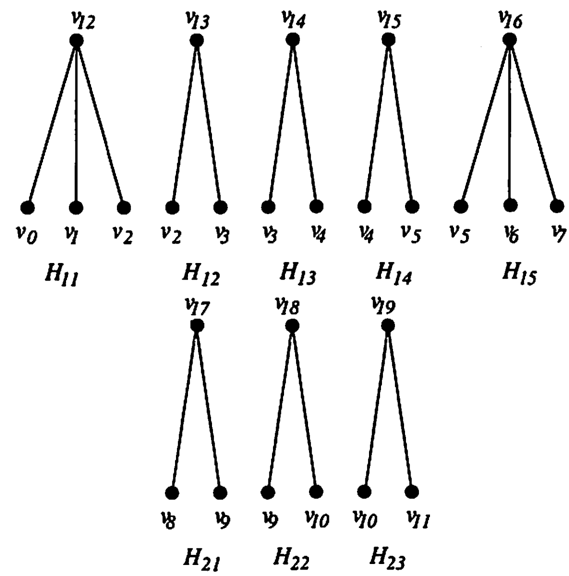
Putting $c = 1$, we obtain the following corollary on the isomorphic factorization of complete bipartite graphs into trees.

Corollary 2.9 ([6]) *A complete bipartite graph $K(m,n)$ is divisible by a tree if and only if $m + n - 1$ divides mn.*

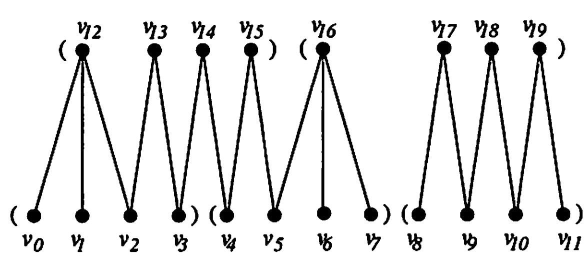
2.4 Example

Based on the algorithm described in Section 2.3, an example of the construction of G in the case of $c = 2$, $[m, n] = [27, 20]$ is shown. In this case, $d_m = 9$, $\alpha = 3$, $\alpha' = 2$, and $d_n = 5$, $\beta = 4$, $\beta' = 5$.

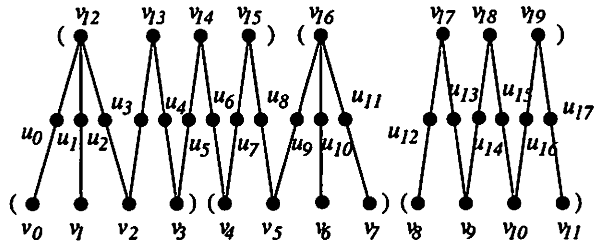
1. H_{ij}



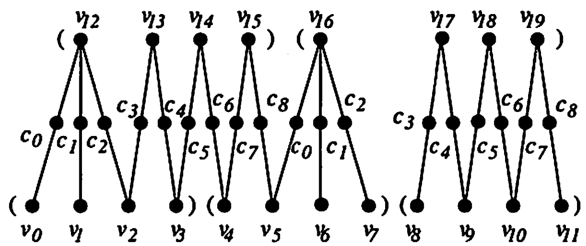
2. G_1



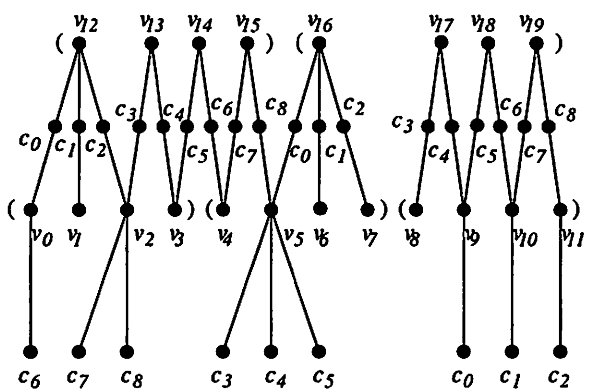
3. G_2



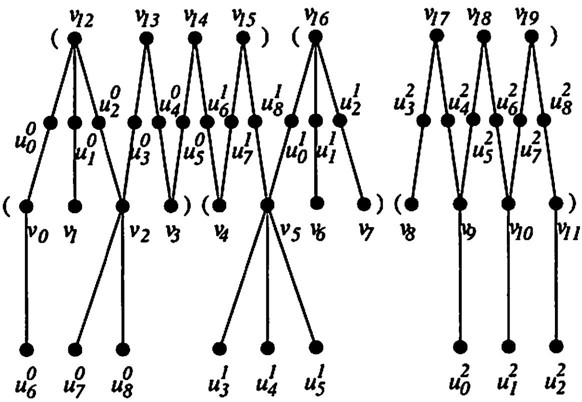
4. Coloring of G_2



5. G_3



6. G



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