A family of 4-designs on 38 points

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Abstract

Using a modification of the Kramer-Mesner method, 4-(38,5,λ) designs are constructed with $PSL(2,37)$ as an automorphism group and with λ in the set $\{6,10,12,16\}$. It turns out also that there exists a 4-(38,5,16) design with $PGL(2,37)$ as an automorphism group.

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1 Basic facts related to the constructions

An n-set is a set of cardinality n. Given a group G acting on a ground-set the n-G-orbit is an orbit of n-subsets of the ground-set, which arises by action of G. It will be sometimes convenient to to write " n-orbit of G " or only " n-orbit " or " G-orbit ", instead of " n-G-orbit ".

A *t-(v,k,λ) design* [3] is an incidence structure on the v-ground-set, which consists of some k-subsets (called *blocks*) of the ground-set, without repetitions and which satisfies the property that each t-subset of the ground-set is contained in exactly λ blocks.

1.1 The Kramer-Mesner method

The well-known Kramer-Mesner method [4] for constructing t-(v,k,λ) designs with a prescribed group G of automorphisms, works as follows:

Let λ_{ij} ([3], pp. 185) denote the number of elements of the j-th k-G-orbit, that contain a fixed arbitrary element of the i-th t-G-orbit, $t < k$. This notion is well-defined, since each t-set of a t-G-orbit is contained within the same number of k-sets of a k-G-orbit.

The matrix (λ_{ij}) will be denoted here as $\Lambda(G;t,k)$; the same matrix was denoted as $A(G;H;t,k)$ in [4] and as $A_{t,k}$ in [5]; it can be called the *orbit incidence matrix* for t-G-orbits and k-G-orbits. If $n(G,i)$ denotes the number of i-G-orbits, then the size of $\Lambda(G;t,k)$ is $n(G,t) \times n(G,k)$. The row sums in $\Lambda(G;t,k)$ are uniform and equal to $\lambda_{max} = \binom{v-t}{k-t}$.

The key idea of the method is to find a *proper* subset S (if exists) of the columns of $\Lambda(G;t,k)$ with uniform row sums λ. Blocks of the required design are all k-subsets of the v-ground-set that belong to the k-G-orbits corresponding to columns of S. In other words, a t-(v,k,λ) design with G as a group of automorphisms can be recognized as a *proper* submatrix D of $\Lambda(G;t,k)$ consisting of whole columns and also has uniform row sums λ in all t rows. One can easily conclude by using complementary submatrices that it suffices to search λ for $\lambda \leq \frac{1}{2} \cdot \lambda_{max}$.

In this way, blocks of a t-(v,k,λ) design are obtained as a union of several k-G-orbits.

A modification of the Kramer-Mesner method is applied here to the case $t=4$, $k=5$, $G=PSL(2,37)$; it follows that $v=38$ and $\lambda_{max}=34$. We have applied some other modifications of the Kramer-Mesner method in the papers [1] and [2].

1.2 A construction of $PGL(2,37)$ and $PSL(2,37)$

A computer aided construction begins with considering action of the linear group $GL(2,37)$ upon the 2-dimensional vector space $V(2,37)$ over the field $GF(37)$. This action is implemented as a multiplication of a row vector from $V(2,37)$ with a 2×2 matrix from $GL(2,37)$. Next step is constructing $PGL(2,37)$ by introducing projectivity in this action; this requires replacement of matrices by their representatives of homotethy classes and transition from vectors to their corresponding points on the projective line (ground-set) $\{0,1,...,36\}\cup\{\infty\}$.

Matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ of the group $PGL(2,37)$ can be constructed using the following two loops:

** choosing $a_{11}=1$ and arbitrary $a_{12},a_{21},a_{22}\in\{0,...,36\}$ so that $a_{22}-a_{12}\cdot a_{21}\neq 0$;

** choosing $a_{11}=0$, $a_{12}=1$, and arbitrary $a_{21}\in\{1,...,36\}$ and $a_{22}\in\{0,...,36\}$.

The group $PSL(2,37)$ is constructed from the group $SL(2,37)$ of 2×2 matrices over $GF(37)$ with determinant 1, reducing by the group of homotethies. This is done so that precisely one of any two matrices from $SL(2,37)$ of the form $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and $\begin{pmatrix} 36\cdot b_{11} & 36\cdot b_{12} \\ 36\cdot b_{21} & 36\cdot b_{22} \end{pmatrix}$ – is included into $PSL(2,37)$; denote the included matrix by $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

The matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ of $PSL(2,37)$ can be constructed by using the following two loops: choosing $a_{11}\in\{1,...,18\}$ and $a_{12},a_{21}\in$

$\{0, ..., 36\}$ in all possible ways and determining each time a_{22} as the solution of the equation $a_{11} \cdot a_{22} - a_{12} \cdot a_{21} = 1$ within $GF(37)$; choosing $a_{12} \in \{1, ..., 18\}$ and $a_{22} \in \{0, ..., 36\}$ in all possible ways and each time taking $a_{11} = 0$ and determining a_{21} as the solution of the equation $- a_{12} \cdot a_{21} = 1$ within $GF(37)$.

When applying matrices of $PSL(2, 37)$ or $PGL(2, 37)$, points of the projective line are represented by their homogeneous coordinates as row vectors; that is, $x = (x, 1)$ for $x \in \{0, 1, ..., 36\}$ and $\infty = (1, 0)$.

1.3 Reduced orbits

It is well-known (a consequence of two statements in [3], pp. 171 and 169) that $PSL(2, 37)$ and $PGL(2, 37)$ are respectively a 2-homogeneous and a 3-homogeneous group; any of these two groups will be denoted by G. When constructing k-G-orbits, it suffices to use those k-subsets of the projective line that are supersets of $\{0, \infty\}$ (respectively $\{0, 1, \infty\}$); we call these k-subsets "special". "Special" k-subsets within a k-G-orbit constitute a *reduced* k-G-orbit. An analogous reduction is applied to t-G-orbits.

Reduced k-G-orbits are constructed by applying elements of G to their representative k-subsets; the image k-subsets are recorded iff they are "special". Together with the ordinal numbers of these k-subsets in the lexicographic order, it suffices to keep the ordinal numbers of k-G-orbits containing "special" k-subsets in computer memory.

Reduced t-G-orbits and reduced k-G-orbits are sufficient for construction of the matrix $\Lambda(G; t, k)$, since the set-inclusion preserves the "speciality"; that is, all k-supersets of a "special" t-subset are "special" k-subsets.

2 4-designs arising from $PSL(2, 37)$

Throughout this section, " n-orbits" will be an abbreviation for n-G-orbits, where $G = PSL(2, 37)$.

The main result of this paper reads:

Theorem 1 *There exist 4-(38,5,λ) designs with* $PSL(2, 37)$ *as an automorphism group and with each* λ *in the set* $\{6, 10, 12, 16\}$.

Proof. The proof will be given by exhibiting four 4-$(38, 5, \lambda)$ designs with $PSL(2, 37)$ as a group of automorphisms and with the four values of λ above, accompanied with the necessary data for documenting the constructed designs. These data include:

a) data for identification of 4- and 5-orbits of of $PSL(2, 37)$; (Tables 1 and 2)

b) matrix $\Lambda(PSL(2,37);4,5)$; (Table 3)

c) column combinations (sets of columns) of $\Lambda(PSL(2,37);4,5)$ corresponding to the designs.

It turns out that there are 9 4-orbits and 29 5-orbits. In accordance with discussion held in Subsection 1.3., the representatives of all 4-orbits and 5-orbits may be assumed to be supersets of $\{0,\infty\}$.

In order to enable the identification of 4-orbits and 5-orbits, associated with rows and columns of the matrices, the following data will be given in Tables 1 and 2:

- the ordinal number of an orbit, associated to the corresponding row (column) of the matrix $\Lambda(PSL(2,37);4,5)$.
- the elements of the lexicographically first "special" representative, apart from the compulsory elements 0 and ∞.
- the number of "special" subsets within the orbit.

Example. The denotations | 8 | 2 8 | 54 | in Table 1 and | 18 | 1 3 17 | 180 | in Table 2 mean that the 8th 4-orbit contains the representative $\{0,2,8,\infty\}$ and a total of 54 "special" 4-subsets, while the 18th 5-orbit contains the representative $\{0,1,3,17,\infty\}$ and a total of 180 "special" 6-subsets.

1	1	2	54	2	1	3	108	3	1	4	54
4	1	5	108	5	1	6	108	6	1	8	108
7	1	11	18	8	2	8	54	9	2	17	18

Table 1. Data describing 4-orbits of $PSL(2,37)$

1	1	2	3	180	2	1	2	4	180	3	1	2	5	360
4	1	2	6	360	5	1	2	7	180	6	1	2	8	360
7	1	2	9	360	8	1	2	11	360	9	1	2	14	360
10	1	3	4	180	11	1	3	7	360	12	1	3	9	180
13	1	3	12	360	14	1	3	13	360	15	1	3	14	180
16	1	3	15	180	17	1	3	16	360	18	1	3	17	180
19	1	3	22	180	20	1	3	26	360	21	1	3	29	360
22	1	4	5	180	23	1	4	11	60	24	1	4	17	180
25	1	5	8	180	26	1	5	22	180	27	1	5	24	180
28	1	6	8	180	29	2	8	22	60					

Table 2. Data describing 5-orbits of $PSL(2,37)$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	4	4	4	4	2	4	4	4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	2	0	2	0	0	0	0	2	2	4	2	2	4	2	1	2	2	1	2	2	0	0	0	0	0	0	0	0
3	0	2	4	0	0	0	0	4	0	4	0	0	8	0	0	0	0	0	0	4	0	4	2	2	0	0	0	0	0
4	0	0	4	2	0	0	4	0	2	1	2	1	0	2	0	2	0	0	2	2	2	2	0	0	2	2	2	0	0
5	0	0	0	2	4	2	0	2	2	0	2	0	2	2	0	2	2	0	2	2	2	1	0	2	0	1	0	2	0
6	0	0	2	2	0	2	2	2	2	0	0	0	2	0	3	0	2	3	0	2	2	0	0	2	2	0	2	2	0
7	0	0	0	0	0	0	0	12	0	0	12	0	0	0	0	0	0	0	0	0	0	0	4	0	6	0	0	0	0
8	2	0	0	0	0	4	4	0	0	0	0	4	0	0	0	0	8	0	0	0	4	0	0	0	0	4	0	2	2
9	0	0	0	0	0	12	0	0	0	0	0	0	0	12	0	0	0	0	0	0	0	0	0	0	0	0	6	0	4

Table 3. The 9×29 matrix $\Lambda(PSL(2,37);4,5)$

The existence of a 4-(38,5,λ) design will be proved in each particular case by exhibiting a proper subset P of the column-set (a combination of columns) of matrix $\Lambda(PSL(2,37);4,5)$, satisfying that the sum of elements of any row within the columns of P is equal to λ.

Let C denote the set of ordinal numbers of those columns of $\Lambda(PSL(2,37);4,5)$ that constitute a required proper subset P. Four possible sets C are listed below, for λ within the set $\{6,10,12,16\}$:

$\lambda = 6: \quad C = \{1,5,10,12,24,25,27\}$;
$\lambda = 10: \quad C = \{1,4,5,10,12,20,23,25,27,28,29\}$;
$\lambda = 12: \quad C = \{4,6,7,11,13,20,21\}$;
$\lambda = 16: \quad C = \{3,4,6,9,10,11,12,20,21,23,24,28,29\}$. □

On the other hand, it immediately follows from the last row of $\Lambda(PSL(2,37);4,5)$ that no column combination has uniform row sums λ, with λ odd or with λ in the set $\{2,8,14\}$. A straightforward combinatorial argument related to entries 4 in 7th and 9th row, possible entries 2 in 3rd and 6th row and their consequences – implying that $\lambda = 4$ is also not possible.

Results of a computer search show that there exist 2, 4, 6, 11 column combinations of $\Lambda(PSL(2,37);4,5)$ corresponding to λ equal to 6, 10, 12, 16, respectively.

3 About orbits and 4-design(s) from $PGL(2,37)$ and $PSL(2,37)$

We recall that the group $PSL(2,q)$ is a subgroup of index 2 of $PGL(2,q)$, for each prime power q. This fact, combined with the following lemma, implies a number of consequences listed below.

Lemma 1 *If H is a subgroup of index k of a group G, then a G-orbit includes at most k H-orbits.*

Proof. Let $G = \bigcup_{i=1}^{k} g_i H$ be the partition of G into left cosets modulo H. Then the G-orbit determined by a set X can be represented as $\{X^g | g \in G\} = \bigcup_{i=1}^{k} \{X^{g_i h} | h \in H\}$. The set $S_i = \{X^{g_i h} | h \in H\}$ is an H-orbit determined by X^{g_i}. Therefore, the number of H-orbits within the considered G-orbit cannot be greater than k. It may happen that H-orbits determined by X^{g_i} and X^{g_j} coincide (this happens iff $X^{g_j} \in \{X^{g_i h} | h \in H\}$). □.

Consequence 1. A low homogeneous subgroup H of a highly homogeneous group G may be useful for searching designs. All designs composed of H-orbits are also composed of G-orbits. The subgroup H, although possibly less homogeneous, preserves all λ-values found with G and leaves a possibility for finding some new λ-values. This is exactly what happened with $G = PGL(2, 37)$ and $H = PSL(2, 37)$.

Consequence 2. Each orbit of $PGL(2, q)$ consists of either one or two orbits of $PSL(2, q)$.

Consequence 3. All designs that have $PGL(2, q)$ as a group of automorphisms can be obtained by applying the Kramer-Mesner method to the group $PSL(2, q)$.

Using the fact that $PGL(2, q)$ is 3-homogeneous, one also has

Consequence 4. The application of the Kramer-Mesner method to the group $PSL(2, q)$ produces 3-designs, for each prime power q.

Let $PG(n, i)$ (resp., $PS(n, i)$) denote the i-th n-orbit of $PGL(2, 37)$ (resp., $PSL(2, 37)$). Inclusion relationships among orbits $PG(n, i)$ and $PS(n, j)$ are listed in Table 4:

PG(4, 1) = PS(4, 1); PG(4, 2) = PS(4, 2);
PG(4, 3) = PS(4, 3) + PS(4, 8); PG(4, 4) = PS(4, 4);
PG(4, 5) = PS(4, 5); PG(4, 6) = PS(4, 6);
PG(4, 7) = PS(4, 7) + PS(4, 9); PG(5, 1) = PS(5, 1) + PS(5, 2);
PG(5, 2) = PS(5, 3) + PS(5, 7); PG(5, 3) = PS(5, 4) + PS(5, 9);
PG(5, 4) = PS(5, 5); PG(5, 5) = PS(5, 6) + PS(5, 8);
PG(5, 6) = PS(5,10) + PS(5,12); PG(5, 7) = PS(5,11) + PS(5,14)
PG(5, 8) = PS(5,13) + PS(5,17); PG(5, 9) = PS(5,15) + PS(5,18)
PG(5,10) = PS(5,16) + PS(5,19); PG(5,11) = PS(5,20) + PS(5,21)
PG(5,12) = PS(5,22) + PS(5,26); PG(5,13) = PS(5,23) + PS(5,29)
PG(5,14) = PS(5,24) + PS(5,28); PG(5,15) = PS(5,25) + PS(5,27)

Table 4. Inclusion relationships among 4-orbits and 5-orbits of $PGL(2,37)$ and $PSL(2,37)$

Theorem 2 *There exists a 4-(38,5,16) design with* $PGL(2,37)$ *as an automorphism group.*

Proof. Joining together those orbits of $PSL(2,37)$ that belong to the same orbit of $PGL(2,37)$ (Table 4), one transforms the matrix $\Lambda(PSL(2,37);4,5)$ (Table 3) into the matrix $\Lambda(PGL(2,37);4,5)$ (Table 5):

8	8	8	2	8	0	0	0	0	0	0	0	0	0	0
4	0	4	0	0	4	8	4	4	2	4	0	0	0	0
2	4	0	0	4	4	0	8	0	0	4	4	2	2	0
0	8	4	0	0	2	4	0	0	4	4	4	0	0	4
0	0	4	4	4	0	4	4	0	4	4	2	0	4	0
0	4	4	0	4	0	0	4	6	0	4	0	0	4	4
0	0	0	0	12	0	12	0	0	0	0	0	4	0	6

Table 5. The 7×15 matrix $\Lambda(PGL(2,37);4,5)$ and a corresponding 4-(38, 5, 16) design

The column combination of $\Lambda(PGL(2,37);4,5)$, consisting of the 2nd, 3rd, 7th, 8th, 13th and 14th column (rounded columns in Table 5) is the only one that corresponds to a 4-design. This column combination can be also recognized in Section 2; it is equivalent to one of the mentioned 11 combinations of $\Lambda(PSL(2,37);4,5)$ corresponding to $\lambda = 16$; precisely to the combination containing the 3rd, 4th, 7th, 9th, 11th, 13th, 14th, 17th, 23rd, 24th, 28th and 29th column. □

Remark. Note that the family F of blocks of a 4-$(38,5,16)$ design exhibited in Section 2 cannot be represented as the union of whole orbits of $PGL(2,37)$. For example, the orbit $PS(5,3)$ is included into F, but $PS(5,7)$ is not; consequently, the orbit $PG(5,2)$ is only partly included into F.

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