Infinite designs with more point orbits than block orbits

Bridget S. Webb
Department of Pure Mathematics,
The Open University, Walton Hall,
Milton Keynes, MK7 6AA,
United Kingdom.

Abstract

Block's Lemma states that every automorphism group of a finite 2-(v, k, λ) design acts with at least as many block orbits as point orbits: this is not the case for infinite designs. Evans constructed a block transitive 2-$(v, 4, 14)$ design with two point orbits using ideas from model theory and Camina generalized this method to construct a family of block transitive designs with two point orbits. In this paper we generalize the method further to construct designs with n point orbits and l block orbits with $l < n$, where both k and λ are finite. In particular, we prove that for $k \geq 4$ and $n \leq k/2$, there exists a block transitive 2-(v, k, λ) design, for some finite λ, with n point orbits. We also construct 2-$(v, 4, \lambda)$ designs with automorphism groups acting with n point orbits and l block orbits, $l < n$, for every permissible pair (n, l).

1 Introduction

Let $\mathcal{D}$ be a 2-(v, k, λ) design and let $G \leq \text{Aut } \mathcal{D}$. It is well known that when v is finite, G acts with at least as many orbits on the blocks of $\mathcal{D}$ as on the points. This is known as Block's Lemma [1]. However, infinite designs with automorphism groups acting with more point orbits than block orbits are not difficult to construct when λ is infinite; see Webb [11] for some examples. An example of a block transitive point intransitive design with $\lambda = 1$ and k infinite is attributed to Valette [2]. Another example with an automorphism group acting with two block orbits and three point orbits was constructed by Prazmowski [9] by generalizing a linear space of Strambach [10]. The first example of a design with more point orbits than block orbits and both k and λ finite, is a block transitive design with two point orbits constructed by Evans [7] using a model theoretic construction of Hrushovski [8]. This method was generalized by Camina [6] to show that a block transitive 2-$(v, k, k + 1)$ design with two point orbits exists for all finite $k \geq 6$. Here we generalize this method further, to construct designs with n point orbits and l block orbits. We show that 2-(v,k,λ) designs with automorphism groups acting with more point orbits than block orbits, and finite k and λ, are not uncommon.

It is worth noting here that no design is known with $\lambda = 1$, k finite and an automorphism group acting with more point orbits than block orbits. When $k = 2$ or 3, and $\lambda = 1$ every automorphism group acts with at least as many block orbits as point orbits [4] (in fact, this is also the case for $k = 2$ or 3 and any λ [11]). The 'smallest' possible example of a block transitive point intransitive design with $\lambda = 1$ is therefore a 2-$(v,4,1)$ design with two point orbits. The question of the existence of such a design was posed by Cameron and Praeger [5] in 1993, and despite receiving attention from a number of mathematicians, remains unanswered. A related problem (Doyen, see Cameron [3]) is to construct infinite linear spaces with more point orbits than block orbits with (n,l) not equal to $(2,1)$ and $(3,2)$.

2 Preliminaries

Let $\mathcal{D}$ be a design with both k and λ finite, and let G act on $\mathcal{D}$ with a finite number n of point orbits $\mathcal{P}_1,\ldots,\mathcal{P}_n$. Let C be a set of n distinct colours. Then we colour the points of $\mathcal{D}$ so that each point in $\mathcal{P}_i$ is of colour c_i. This induces a colouring on the blocks of $\mathcal{D}$ which depends only on the colours of the points incident with the block. The point colouring is G-invariant and so the induced colouring of the blocks is also G-invariant.

Now, $\mathcal{D}$ is a design, so there are precisely λ blocks incident with each pair of points. Therefore, for each colour (that does not correspond to a fixed point) there must be some block orbit whose blocks are incident with at least two points of this colour, and likewise for each pair of colours there must be some block orbit whose blocks are incident with at least one point of each of these colours. For fixed k, we get a lower bound for the number of block orbits in terms of n, call this $\mathcal{F}_k(n)$; see [11]. Here we are not concerned with the exact values taken by $\mathcal{F}_k(n)$, but with designs with l block orbits under G where $\mathcal{F}_k(n) \le l < n$.

We begin by generalizing Lemma 1 of [11].

Lemma 1 *Let $\mathcal{D}$ be a 2-(v,k,λ) design with k and λ finite. Let G be an automorphism group of $\mathcal{D}$ acting with a finite number of finite length point orbits. Let $\mathcal{P}_1$ denote an infinitely long point orbit under G. Then there is some block orbit such that each block in this orbit is incident with at least two points of $\mathcal{P}_1$ and no points of finite length orbits.*

Proof: Let p be a point of $\mathcal{P}_1$. Since there are only finitely many points in finite length orbits, each in λ blocks incident with p, only finitely many blocks incident with p contain a point in a finite orbit. However, there are infinitely many points $p' \neq p$ of $\mathcal{P}_1$, and so infinitely many blocks incident with p and another point of $\mathcal{P}_1$. Therefore, there is some block orbit with each block incident with at least two points of $\mathcal{P}_1$ and no points of a finite length orbit. □

In light of this Lemma, since we are concerned with designs with relatively few block orbits ($\mathcal{F}_k(n) \leq l < n$), we can restrict our attention to designs with automorphism groups acting only with infinitely long point orbits.

In this paper we generalize the method of Evans [7] of constructing block transitive 2-(v,k,λ) designs with two point orbits, to construct 2-(v,k,λ) designs with n point orbits and l block orbits ($\mathcal{F}_k(n) \leq l < n$). In particular, we prove:

Theorem 1 *Let $k \geq 4$. Then, for some finite natural number λ, there exists a block transitive 2-(v,k,λ) design with n point orbits, for every $n \leq k/2$.*

The values of l satisfying $\mathcal{F}_4(n) \leq l < n$ can easily be determined. This leads to the following:

Theorem 2 *Let l and n be natural numbers such that $\mathcal{F}_4(n) \leq l < n$. Then, for some finite natural number λ, there exists a 2-$(v,4,\lambda)$ design with n point orbits and l block orbits.*

We make the following conjecture:

Conjecture *Let n, k and l be finite natural numbers such that $\mathcal{F}_k(n) \leq l < n$. Then, for some finite natural number λ, there exists a 2-(v,k,λ) design with an automorphism group acting with n point orbits and l block orbits.*

3 An overview of the method

This method, introduced by Evans [7], for constructing designs is quite involved and makes use of a powerful model theory theorem of Hrushovski [8]. We therefore give a brief overview of the main ideas before going into more detail in the next section.

Let V be a set of *points* and E_V a set of distinct subsets of V of size 3, called *edges*. Then the pair (V, E_V) is a 3-*uniform hypergraph*. (Note that in standard graph theory terms the elements of V are usually called vertices of the hypergraph. Here we call these elements points as we shall be using them to form the points of the design we construct.)

Let C be a set of n distinct colours, where n is finite. Then we call the triple $\mathcal{V} = (V, E_V, C_V)$ a 3-*uniform coloured hypergraph*, where C_V assigns a colour from $\mathcal{C}$ to each point of V. We call $\mathcal{W} = (W, E_W, \mathcal{C}_W)$ a *coloured subgraph* of $\mathcal{V}$, and write $\mathcal{W} \subseteq \mathcal{V}$, if $W \subseteq V$, $E_W \subseteq E_V$ is a set of edges incident only with points of W and $\mathcal{C}_W$ is a restriction of $\mathcal{C}_V$ to $\mathcal{W}$.

We can summarize the method used to construct the designs as follows:

1. We use a model-theoretic result to show that there exists a special countably infinite 3-uniform coloured hypergraph $\mathcal{M}$ with: n colours; a large automorphism group that acts transitively on the points of $\mathcal{M}$ of each colour; and a function μ on certain pairs of finite coloured subgraphs of $\mathcal{M}$. At this point we do not specify the values that this function takes, simply that its value is always at least 2. The automorphism group is 'large' in the sense that any isomorphism between certain finite coloured subgraphs can be extended to an automorphism of the coloured hypergraph $\mathcal{M}$.

2. We then construct a design on $\mathcal{M}$: the points of this design $\mathcal{D}$ are the points of $\mathcal{M}$ and the blocks of $\mathcal{D}$ are chosen to be those coloured subgraphs (of size k) whose isomorphism type is one of a fixed collection (of size l).

3. The automorphism group of $\mathcal{M}$ induces an automorphism group of $\mathcal{D}$; the orbits on the points of $\mathcal{D}$ are simply the sets of points of each colour. Because of the 'largeness' of the automorphism group, if two blocks are isomorphic as coloured subgraphs of $\mathcal{M}$, then they lie in the same orbit under the induced action of the automorphism group; thus there are l block orbits.

4. The final (and more technical) part of the construction is to find possible values for the function $\mathcal{M}$ (in terms of n and l) such that any pair of points of $\mathcal{D}$ is incident with a constant number of blocks of $\mathcal{D}$. This determines the value of λ and proves a result for this n and l.

4 The countably infinite hypergraph $\mathcal{M}$

In this section we give the method, introduced by Evans [7], to construct the 3-uniform coloured hypergraph with the required properties for the construction of the design $\mathcal{D}$. This method uses model theoretic results of Hrushovski [8].

Let $\mathcal{V} = (V, E_V, \mathcal{C}_V)$. If $\mathcal{X} \subseteq \mathcal{V}$ and $\mathcal{W} \subseteq \mathcal{V}$, then we define the union $\mathcal{X} \cup \mathcal{W}$ to be the coloured subgraph of $\mathcal{V}$ given by

$$\mathcal{X} \cup \mathcal{W} = (X \cup W, E_{X \cup W}, \mathcal{C}_{X \cup W}),$$

where $E_{X \cup W}$ comprises *all* the edges of $\mathcal{V}$ incident only with the points of $X \cup W$.

We define

$$d_0(\mathcal{V}) = |V| - |E_V|.$$

Notice that hypergraphs with positive values of d_0 have comparatively few edges.

If $\mathcal{X} \subseteq \mathcal{W} \subseteq \mathcal{V}$, then we define

$$d(\mathcal{X}, \mathcal{W}) = \min\{d_0(\mathcal{Z}) \mid \mathcal{X} \subseteq \mathcal{Z} \subseteq \mathcal{W}\}.$$

We call $\mathcal{X}$ *self-sufficient* in $\mathcal{W}$ if $d(\mathcal{X}, \mathcal{W}) = d_0(\mathcal{X})$.

Now let $\mathcal{X} \subseteq \mathcal{V}$ and $\mathcal{W} \subseteq \mathcal{V}$ be disjoint coloured subgraphs. Then we call $\mathcal{X}$ *simply algebraic* over $\mathcal{W}$ if $\mathcal{W}$ is self-sufficient in $\mathcal{X} \cup \mathcal{W}$ and $d_0(\mathcal{X} \cup \mathcal{W}) = d_0(\mathcal{W})$; that is, if

$$d(\mathcal{W}, \mathcal{X} \cup \mathcal{W}) = d_0(\mathcal{W}) = d_0(\mathcal{X} \cup \mathcal{W}).$$

Further, we call $\mathcal{X}$ *minimally simply algebraic* over $\mathcal{W}$ if there is no proper coloured subgraph $\mathcal{Z} \subset \mathcal{W}$ such that $\mathcal{X}$ is simply algebraic over $\mathcal{Z}$.

For example, consider the coloured subgraphs $\mathcal{X} = (\{x\}, \emptyset, C_X)$, $\mathcal{Y} = (\{y_1, y_2\}, \emptyset, C_Y)$ and $\mathcal{W} = (\{w_1, w_2\}, \emptyset, C_W)$ of $\mathcal{V}$, where x, y_1, y_2, w_1 and w_2 are in V and the edges $\{x, w_1, w_2\}$, $\{y_1, w_1, w_2\}$ and $\{y_2, w_1, w_2\}$ are in E_V. Now,

$$\mathcal{X} \cup \mathcal{W} = (\{x, w_1, w_2\}, \{\{x, w_1, w_2\}\}, C_{X \cup W})$$

and

$$d(\mathcal{W}, \mathcal{X} \cup \mathcal{W}) = d_0(\mathcal{W}) = d_0(\mathcal{X} \cup \mathcal{W}) = 2,$$

so $\mathcal{X}$ is simply algebraic over $\mathcal{W}$. Notice that the only proper coloured subgraphs of $\mathcal{W}$ are $\mathcal{Z}_1 = (\{w_1\}, \emptyset, C_{Z_1})$ and $\mathcal{Z}_2 = (\{w_2\}, \emptyset, C_{Z_2})$. We have

$$d(\mathcal{Z}_1, \mathcal{X} \cup \mathcal{Z}_1) = d(\mathcal{Z}_2, \mathcal{X} \cup \mathcal{Z}_2) = 1 \neq d_0(\mathcal{X} \cup \mathcal{Z}_1) = d_0(\mathcal{X} \cup \mathcal{Z}_2) = 2,$$

so $\mathcal{X}$ is minimally simply algebraic over $\mathcal{W}$. Also,

$$\mathcal{Y} \cup \mathcal{W} = (\{y_1, y_2, w_1, w_2\}, \{\{y_1, w_1, w_2\}, \{y_2, w_1, w_2\}\}, C_{Y \cup W})$$

and

$$d(\mathcal{Y}, \mathcal{Y} \cup \mathcal{W}) = d_0(\mathcal{Y}) = d_0(\mathcal{Y} \cup \mathcal{W}) = 2,$$

so $\mathcal{W}$ is simply algebraic over $\mathcal{Y}$. The only proper coloured subgraphs of $\mathcal{Y}$ are $\mathcal{Z}_3 = (\{y_1\}, \emptyset, C_{Z_3})$ and $\mathcal{Z}_4 = (\{y_2\}, \emptyset, C_{Z_4})$. We have

$$d(\mathcal{Z}_3, \mathcal{W} \cup \mathcal{Z}_3) = d(\mathcal{Z}_4, \mathcal{W} \cup \mathcal{Z}_4) = 1 \neq d_0(\mathcal{W} \cup \mathcal{Z}_3) = d_0(\mathcal{W} \cup \mathcal{Z}_4) = 2,$$

so $\mathcal{W}$ is minimally simply algebraic over $\mathcal{Y}$. The only examples of minimally simply algebraic coloured subgraphs that we shall consider will be isomorphic, as (uncoloured) subgraphs, to one of these two examples.

Let $\mathcal{X}$ and $\mathcal{W}$ be coloured subgraphs of $\mathcal{V}$, such that $\mathcal{X}$ is minimally simply algebraic over $\mathcal{W}$ and $|X| \geq 1$. Then we define an integer valued class function μ, defined on the pairs $(\mathcal{X}, \mathcal{W})$ such that the value of $\mu(\mathcal{X}, \mathcal{W})$ is dependent only on the isomorphism type of the embedding of $\mathcal{X}$ in $\mathcal{X} \cup \mathcal{W}$. We do not at this point assign exact values to this function, but note that it is at least the value of $d_0(\mathcal{W})$.

We are now in a position to quote the definition from [7] of the class of 3-uniform coloured hypergraphs that we shall need.

Definition 1 *For a given class function μ, let C be the class of all finite 3-uniform coloured hypergraphs $\mathcal{V}$ which satisfy the following:*

(a) for all coloured subgraphs $\mathcal{X}$ of $\mathcal{V}$, $d_0(\mathcal{X}) \geq \min\{|X|, 2\}$;

(b) let $\mathcal{W}$ and $\mathcal{X}_1, \ldots, \mathcal{X}_t$ be pairwise disjoint coloured subgraphs of $\mathcal{V}$ such that $\mathcal{X}_i \cup \mathcal{W}$ are pairwise isomorphic under (colour preserving) automorphisms of $\mathcal{V}$ fixing $\mathcal{W}$, and each $\mathcal{X}_i$ is minimally simply algebraic over $\mathcal{W}$. Then $t \leq \mu(\mathcal{X}_i, \mathcal{W})$.

Notice that, by part (a) of Definition 1, if $\mathcal{V} \in C$, then $\mathcal{V}$ has at most $|V| - 2$ edges, and part (b) states that the number of any such subgraphs is finite and bounded above by μ.

The results of Hrushovski [8], as interpreted by Evans [7] prove the existence of the 3-uniform coloured hypergraph with the special properties that we require. Note that we need a hypergraph with n colours, for finite n, so we make the additional modification of taking a language with a single ternary predicate and n unary predicates (see Section 5.3 of Hrushovski's paper [8]).

Theorem 3 *There exists a countably infinite 3-uniform coloured hypergraph $\mathcal{M} = (M, E_M, C_M)$ (dependant on C and hence on μ) with the following properties:*

(i) the class of all (isomorphism classes of) finite coloured subgraphs of $\mathcal{M}$ is C;

(ii) let $\mathcal{V}_1 \in C$ and $\mathcal{V}_2 \in C$ be self-sufficient in $\mathcal{M}$. If there is a (colour preserving) isomorphism from $\mathcal{V}_1$ to $\mathcal{V}_2$, then this isomorphism can be extended to a (colour preserving) automorphism of $\mathcal{M}$.

Note that any subset of size 2 is self-sufficient in $\mathcal{M}$. Hence, the orbits on pairs of points are determined only by the colours of the points.

We also need the following definitions and lemmas from [6].

Definition 2 *Let $\mathcal{W} \subseteq \mathcal{Z} \subseteq \mathcal{V}$ be 3-uniform coloured hypergraphs with the following properties: $\mathcal{Z}$ is a union of subgraphs $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_s$ such that $\mathcal{W} \subseteq \mathcal{Y}_i$ for each i, and each $\mathcal{Y}_i$ is pairwise isomorphic over $\mathcal{W}$; if $i \neq j$, then $\mathcal{Y}_i \cap \mathcal{Y}_j = \mathcal{W}$ and no edge contains points of both $\mathcal{Y}_i \backslash \mathcal{W}$ and $\mathcal{Y}_j \backslash \mathcal{W}$. Then $\mathcal{Z}$ is called the free amalgam of $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_s$ over $\mathcal{W}$.*

Definition 3 *An edge configuration is an almost special case of a free amalgam in which each component $\mathcal{Y}_i$ is just one edge and $|W| = 2$; but the $\mathcal{Y}_i$ need not be isomorphic to each other; that is, their colourings may vary.*

Lemma 2 *Any sufficiently small edge configuration satisfies the conditions of Definition 1. Hence it is in C, and occurs as a subgraph of $\mathcal{M}$; the size restriction is given by part (b) of Definition 1.*

Lemma 3 *Any sufficiently small free amalgam in which $|Y_i| = 4$ and $|W| = 2$ and the only edges within $\mathcal{Y}_i$ are the two 3-subsets containing $\mathcal{W}$, satisfies the conditions of Definition 1. Hence, it is in C and occurs as a subgraph of $\mathcal{M}$; the size restriction is given by part (b) of Definition 1.*

5 Proof of Theorem 1

We prove the Theorem in four parts as outlined in Section 3.

1. We fix $k \geq 4$ and let $|\mathcal{C}| = n$. One of the countably infinite 3-uniform coloured hypergraphs $\mathcal{M} = (M, E_M, C_M)$ whose existence is guaranteed by Theorem 3 will be used to construct $\mathcal{D}$. The automorphism group of $\mathcal{M}$ acts with n point orbits; it acts transitively on the points of each of the n colours. At this stage we do not specify the choice of the function μ; we do this in the final part of the proof, once the necessary properties have been established. Recall that μ always takes values of at least 2.

2. The points of the design $\mathcal{D}$ are the points of the 3-uniform coloured hypergraph $\mathcal{M}$. We fix $s_1 = 2$ and $s_2 \geq 2$, $s_3 \geq 2$, $\ldots$, $s_n \geq 2$ such that $\sum_{i=1}^{n} s_i = k$. This is possible since $n \leq k/2$. The blocks of our structure $\mathcal{D}$ are the subgraphs $\mathcal{X} = (X, E_X, C_X)$:

$$\begin{aligned} X &= \{{}^1x_1, {}^1x_2, {}^2x_1, {}^2x_2, \ldots, {}^2x_{s_2}, {}^3x_1, \ldots, {}^3x_{s_3}, \ldots, {}^nx_1, \ldots, {}^nx_{s_n}\} \\ E_X &= \{\{{}^1x_1, {}^1x_2, y\} : y \in X \backslash \{{}^1x_1, {}^1x_2\}\} \end{aligned}$$

The point ${}^ix_j$ is coloured $c_i \in \mathcal{C}$. Notice that $|X| = k$ and $|E_X| = k - 2 = \sum_{i=2}^{n} s_i$, and so $d_0(\mathcal{X}) = 2$. Such blocks are contained in $\mathcal{M}$ for a suitable function μ by Lemma 2.

3. The automorphism group of $\mathcal{M}$ induces an automorphism group of $\mathcal{D}$. We have already established that there are n point orbits under this action.

Now, $d_0(\mathcal{X}) = 2$, so by part (a) of Definition 1, $\mathcal{X}$ is self-sufficient in $\mathcal{M}$. Hence, by part (ii) of Theorem 3, the automorphism group acts transitively on the blocks of $\mathcal{D}$.

4. Here we show how to choose the function μ so that the structure $\mathcal{D}$ is a design. We do this by showing how the existence and value of λ are related to the value of μ on certain basic configurations. We first relate the values of μ to the counts of these configurations, then we count the number of blocks through a pair of points in terms of these counts. The details of the counts are given in the following table.

We shall show that each of these counts must be equal to the value of $\mu(\mathcal{Z}, \mathcal{W})$; this is the value of μ on the isomorphism class of embeddings. Since the automorphism group of μ is transitive on pairs of points, it follows that each of the counts is independent of the fixed pair of points chosen, and also that the free amalgam of the required type can be found.

We use Lemma 2 or Lemma 3 to prove that the free amalgam of t' copies of $\mathcal{Z} \cup \mathcal{W}$ over $\mathcal{W}$ is in $\mathcal{C}$ for any $t' \leq \mu(\mathcal{Z}, \mathcal{W})$; thus the count is at least $\mu(\mathcal{Z}, \mathcal{W})$. However, by part (b) of Definition 1, the count is at most $\mu(\mathcal{Z}, \mathcal{W})$. Hence, the count is equal to $\mu(\mathcal{Z}, \mathcal{W})$.

Count symbol	What is counted	Configuration (Z, W)
α_i	For a fixed pair p, q coloured c_1, the number of edges $\{{}^ix, p, q\}$ with ${}^ix$ coloured c_i.	$(\{{}^ix\}, \{p, q\})$
β_i	For a fixed pair p, coloured c_1, and ${}^ix$, coloured c_i, the number of edges $\{q, p, {}^ix\}$ with q coloured c_1.	$(\{q\}, \{p, {}^ix\})$
γ_i	For a fixed pair ${}^ix$, ${}^ix'$ coloured c_i, the number of pairs of points p and q coloured c_1, such that the only edges in $\{p, q, {}^ix, {}^ix'\}$ are $\{{}^ix, p, q\}$ and $\{{}^ix', p, q\}$.	$(\{p, q\}, \{{}^ix, {}^ix'\})$
γ_{ij}	For a fixed pair ${}^ix$, coloured c_i, and ${}^jx$, coloured c_j, the number of pairs of points p and q coloured c_1, such that the only edges in $\{p, q, {}^ix, {}^jx\}$ are $\{{}^ix, p, q\}$ and $\{{}^jx, p, q\}$.	$(\{p, q\}, \{{}^ix, {}^jx\})$

In each case $1 < i, j \leq n$.

We now count the number of blocks incident with a given pair of points.

- Two points of colour c_1:
  the number of blocks containing them is

$$\prod_{i=2}^{n} \binom{\alpha_i}{s_i}.$$

- Two points coloured c_j $(j \neq 1)$:
  the number of blocks containing them is

$$\gamma_j \binom{\alpha_j - 2}{s_j - 2} \prod_{\substack{i=2 \\ i \neq j}}^{n} \binom{\alpha_i}{s_i}.$$

- One point coloured c_1 and one coloured c_j $(j \neq 1)$:
  the number of blocks containing them is

$$\beta_j \binom{\alpha_j - 1}{s_j - 1} \prod_{\substack{i=2 \\ i \neq j}}^{n} \binom{\alpha_i}{s_i}.$$

- One point coloured c_j and one coloured c_l $(j, l \neq 1)$:
  the number of blocks containing them is

$$\gamma_{jl} \binom{\alpha_j - 1}{s_j - 1} \binom{\alpha_l - 1}{s_l - 1} \prod_{\substack{i=2 \\ i \neq j \\ i \neq l}}^{n} \binom{\alpha_i}{s_i}.$$

Hence we require values for μ so that

$$\begin{aligned} \lambda &= \prod_{i=2}^{n} \binom{\alpha_i}{s_i} = \gamma_j \binom{\alpha_j - 2}{s_j - 2} \prod_{\substack{i=2 \\ i \neq j}}^{n} \binom{\alpha_i}{s_i} \\ &= \beta_j \binom{\alpha_j - 1}{s_j - 1} \prod_{\substack{i=2 \\ i \neq j}}^{n} \binom{\alpha_i}{s_i} = \gamma_{jl} \binom{\alpha_j - 1}{s_j - 1} \binom{\alpha_l - 1}{s_l - 1} \prod_{\substack{i=2 \\ i \neq j \\ i \neq l}}^{n} \binom{\alpha_i}{s_i}. \end{aligned}$$

This implies that, for each $j = 2, \ldots, n$,

$$\gamma_j = \frac{\alpha_j(\alpha_j - 1)}{s_j(s_j - 1)}, \qquad \beta_j = \frac{\alpha_j}{s_j}$$

and for each pair $j = 2, \ldots, n$ and $l = 2, \ldots, n$ $(j \neq l)$

$$\gamma_{jl} = \frac{\alpha_j \alpha_l}{s_j s_l}.$$

We have $s_i \geq 2$, for all i, so we can set $\alpha_i = s_i^2$, for each i, giving integer solutions of at least 2 for α_i, γ_i, β_i and γ_{ij}, for all i and j. This results in

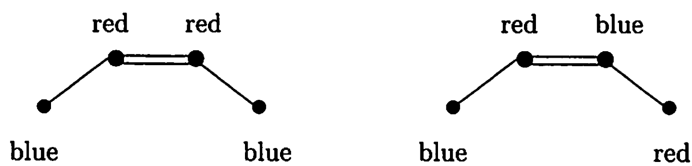
$$\lambda = \prod_{i=2}^{n} \binom{s_i^2}{s_i}.$$

Thus $\mathcal{D}$ is a 2-(v, k, λ) design with one block orbit and n point orbits as required.

In most cases it is possible to set smaller values for the α_i and obtain integer solutions of at least 2 for α_i, γ_i, β_i and γ_{ij}. This results in designs with lower values of λ.

6 Some examples

1. A block transitive 2-$(v, 4, 6)$ design exists with two point orbits. The blocks of this design are the sets of four points, two coloured (say) red and two (say) blue, such that the edges of the underlying 3-uniform hypergraph contain both red points and one of the blue points, as illustrated below. A block of the block transitive 2-$(v, 4, 14)$ design given by Evans [7] with two point orbits, is also illustrated.

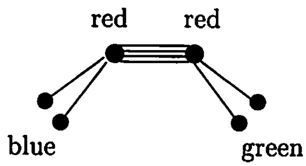


Blocks of the 2-$(v, 4, 6)$ *design,* *the* 2-$(v, 4, 14)$ *design.*

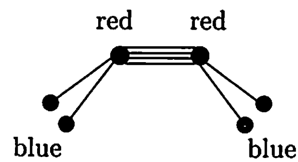
2. A block transitive 2-$(v, 5, 84)$ design exists with two point orbits. In fact, we can set $\alpha_2 = 2s_2 = 6$, so that $\beta_2 = 2$, $\gamma_2 = 5$ and $\lambda = 20$, giving a block transitive 2-$(v, 5, 20)$ design with two point orbits.

3. When $k = 6$, setting $\alpha_i = s_i^2$ gives the following designs: a block transitive 2-$(v, 6, 36)$ design with three point orbits and a block transitive 2-$(v, 6, 1820)$ design with two point orbits. However, by setting $\alpha_2 = 3s_2 = 12$ we obtain a block transitive 2-$(v, 6, 495)$ design with two point orbits. A block from each of these designs is illustrated below, along with a block from the 2-$(v, 6, 7)$ design given by Camina [6]. The blocks of the 2-$(v, 6, 1820)$ and the 2-$(v, 6, 495)$ designs have the same colour type.

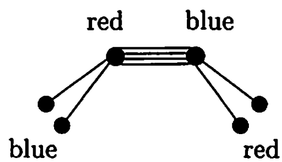
Notice that although the values for λ are large, relatively smaller values of λ arise for designs with as many point orbits as possible. In many cases it is also possible to construct designs with lower values for λ by choosing a different colouring for the coloured subgraph used to define the blocks.



Blocks of the 2-$(v, 6, 36)$ *design,*



the 2-$(v, 6, 495)$ *design,*



the 2-$(v, 6, 7)$ *design.*

7 Proof of Theorem 2

The values of n and l that we need to consider are $n = 2$, $l = 1$; $n = 3$, $l = 2$ and $n = 4$, $l = 3$. We consider each case separately.

$n = 2$, $l = 1$
Theorem 1 proves the existence of a block transitive 2-$(v, 4, 6)$ design with two point orbits, as required.

$n = 3$, $l = 2$
We first consider the case where $n = 3$ and $l = 2$. Again we use the four parts as outlined in Section 3.

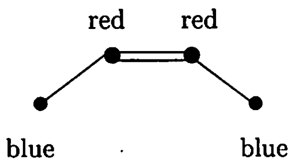
1. We have $k = 4$ and $|\mathcal{C}| = 3$. We use one of the countably infinite 3-uniform coloured hypergraphs $\mathcal{M} = (M, E_M, \mathcal{C}_M)$ whose existence is guaranteed by Theorem 3. The automorphism group of $\mathcal{M}$ acts with 3 point orbits; it acts transitively on the points of each of the 3 colours. At this stage we do not specify the choice of the function μ; we do this in the final part of the proof, once the necessary properties have been established. Recall that μ always takes values of at least 2.

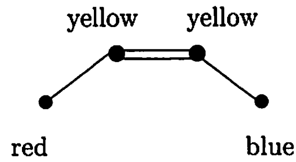
2. The points of the design $\mathcal{D}$ are the points of the 3-uniform coloured hypergraph $\mathcal{M}$. The blocks of our structure $\mathcal{D}$ are the subgraphs $\mathcal{X} = (X, E_X, \mathcal{C}_X)$ and $\mathcal{Y} = (Y, E_Y, \mathcal{C}_Y)$:

$$\begin{array}{ll} X = \{{}^1x_1, {}^1x_2, {}^2x_1, {}^2x_2\} & E_X = \{\{{}^1x_1, {}^1x_2, {}^2x_1\}\{{}^1x_1, {}^1x_2, {}^2x_2\}\} \\ Y = \{{}^3x_1, {}^3x_2, {}^1x, {}^2x\} & E_Y = \{\{{}^3x_1, {}^3x_2, {}^1x\}\{{}^3x_1, {}^3x_2, {}^2x\}\} \end{array}$$

The point ${}^ix_j$ is coloured $c_i \in \mathcal{C}$.

Let c_1 be *red*, c_2 be *blue* and c_3 be *yellow*, then we can illustrate the two types of blocks as follows.





Notice that $|E_X| = |E_Y| = 2$, and so $d_0(\mathcal{X}) = d_0(\mathcal{Y}) = 2$. Such blocks are contained in $\mathcal{M}$ for a suitable function μ by Lemma 2.

3. The automorphism group of $\mathcal{M}$ induces an automorphism group of $\mathcal{D}$: there are 3 point orbits under this action. Now, $d_0(\mathcal{X}) = d_0(\mathcal{Y}) = 2$, so by part (a) of Definition 1, $\mathcal{X}$ and $\mathcal{Y}$ are self-sufficient in $\mathcal{M}$. Hence, by part (ii) of Theorem 3, the automorphism group acts with two orbits on the blocks of $\mathcal{D}$; it acts transitively on the blocks of each colour type.

4. Here we choose the function μ so that the structure $\mathcal{D}$ is a design. The details of the counts that we need are given in the following table.

Count symbol	What is counted	Configuration (Z, W)
α_{12}	For a fixed pair r, r' coloured c_1, the number of edges $\{b, r, r'\}$ with b coloured c_2.	$(\{b\}, \{r, r'\})$
β_{12}	For a fixed pair r, coloured c_1, and b, coloured c_2, the number of edges $\{r', r, b\}$ with r' coloured c_1.	$(\{r'\}, \{r, b\})$
γ_2	For a fixed pair b, b' coloured c_2, the number of pairs of points r and r' coloured c_1, such that the only edges in $\{r, r', b, b'\}$ are $\{b, r, r'\}$ and $\{b', r, r'\}$.	$(\{r, r'\}, \{b, b'\})$
α_{31}	For a fixed pair y, y' coloured c_3, the number of edges $\{r, y, y'\}$ with r coloured c_1.	$(\{r\}, \{y, y'\})$
α_{32}	For a fixed pair y, y' coloured c_3, the number of edges $\{b, y, y'\}$ with b coloured c_2.	$(\{b\}, \{y, y'\})$
β_{31}	For a fixed pair y, coloured c_3, and r, coloured c_1, the number of edges $\{y', y, r\}$ with y' coloured c_3.	$(\{y'\}, \{y, r\})$
β_{32}	For a fixed pair y, coloured c_3, and b, coloured c_2, the number of edges $\{y', y, b\}$ with y' coloured c_3.	$(\{y'\}, \{y, b\})$
γ_{12}	For a fixed pair r, coloured c_1, and b, coloured c_2, the number of pairs of points y and y' coloured c_3, such that the only edges in $\{y, y', r, b\}$ are $\{r, y, y'\}$ and $\{b, y, y'\}$.	$(\{y, y'\}, \{r, b\})$

Each of the counts is independent of the fixed pair of points chosen, and also the free amalgam of the required type can be found. We use Lemma 2 or Lemma 3 to prove that the free amalgam of t' copies of $\mathcal{Z} \cup \mathcal{W}$ over $\mathcal{W}$ is in C for any $t' \leq \mu(\mathcal{Z}, \mathcal{W})$; thus the count is at least $\mu(\mathcal{Z}, \mathcal{W})$. However, by part (b) of Definition 1, the count is at most $\mu(\mathcal{Z}, \mathcal{W})$. Hence, the count is equal to $\mu(\mathcal{Z}, \mathcal{W})$.

We now count the number of blocks incident with a given pair of points. The following table shows the counts of the number of blocks containing a pair of points of the given colours.

	c_1	c_2	c_3
c_1	$\binom{\alpha_{12}}{2}$	$\beta_{12}(\alpha_{12}-1)+\gamma_{12}$	$\beta_{31}\alpha_{32}$
c_2		γ_2	$\beta_{32}\alpha_{31}$
c_3			$\alpha_{31}\alpha_{32}$

Hence we require values for μ so that

$$\begin{aligned}\lambda = \binom{\alpha_{12}}{2} &= \gamma_2 = \alpha_{31}\alpha_{32} = \beta_{12}(\alpha_{12}-1)+\gamma_{12} \\ &= \beta_{31}\alpha_{32} = \beta_{32}\alpha_{31}.\end{aligned}$$

This implies that

$$\beta_{31} = \alpha_{31} \quad \text{and} \quad \alpha_{32} = \beta_{32}.$$

Also,

$$\frac{\alpha_{12}(\alpha_{12}-1)}{2} = \beta_{12}(\alpha_{12}-1)+\gamma_{12}.$$

Since β_{12} and γ_{12} are at least two, we have

$$\alpha_{12}(\alpha_{12}-1) \geq 4\alpha_{12},$$

which implies that $\alpha_{12} \geq 5$.

We set $\alpha_{12} = 5$, so that $\beta_{12} = \gamma_{12} = 2$, then $\gamma_2 = 10$. We set $\alpha_{31} = \beta_{31} = 2$ and $\alpha_{32} = \beta_{32} = 5$ to get a solution to the above equation with $\lambda = 10$.

Thus $\mathcal{D}$ is a 2-$(v, 4, 10)$ design with two block orbits and three point orbits as required.

$n = 4$, $l = 3$

Now we consider the case where $n = 4$ and $l = 3$. Again we use the four parts as outlined in Section 3.

1. We have $k = 4$ and $|C| = 4$. We use one of the countably infinite 3-uniform coloured hypergraphs $\mathcal{M} = (M, E_M, C_M)$ whose existence is guaranteed by Theorem 3. The automorphism group of $\mathcal{M}$ acts with 4 point orbits; it acts transitively on the points of each of the 4 colours. At

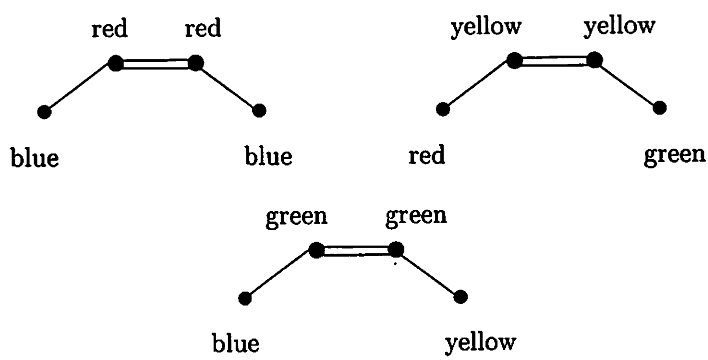
this stage we do not specify the choice of the function μ; we do this in the final part of the proof, once the necessary properties have been established. Recall that μ always takes values of at least 2.

2. The points of the design $\mathcal{D}$ are the points of the 3-uniform coloured hypergraph $\mathcal{M}$. The blocks of our structure $\mathcal{D}$ are the subgraphs $\mathcal{X} = (X, E_X, \mathcal{C}_X)$, $\mathcal{Y} = (Y, E_Y, \mathcal{C}_Y)$ and $\mathcal{U} = (U, E_U, \mathcal{C}_U)$:

$$\begin{array}{ll} X = \{{}^1x_1, {}^1x_2, {}^2x_1, {}^2x_2\} & E_X = \{\{{}^1x_1, {}^1x_2, {}^2x_1\}\{{}^1x_1, {}^1x_2, {}^2x_2\}\} \\ Y = \{{}^3x_1, {}^3x_2, {}^1x, {}^4x\} & E_Y = \{\{{}^3x_1, {}^3x_2, {}^1x\}\{{}^3x_1, {}^3x_2, {}^4x\}\} \\ U = \{{}^4x_1, {}^4x_2, {}^2x, {}^3x\} & E_U = \{\{{}^4x_1, {}^4x_2, {}^2x\}\{{}^4x_1, {}^4x_2, {}^3x\}\} \end{array}$$

The point ${}^ix_j$ is coloured $c_i \in \mathcal{C}$.

Again, let c_1 be *red*, c_2 be *blue* and c_3 be *yellow*, and let c_4 be *green*, then we can illustrate the three types of blocks as follows.



Notice that $|E_X| = |E_Y| = |E_U| = 2$, and so $d_0(\mathcal{X}) = d_0(\mathcal{Y}) = d_0(\mathcal{U}) = 2$. Such blocks are contained in $\mathcal{M}$ for a suitable function μ by Lemma 2.

3. The automorphism group of $\mathcal{M}$ induces an automorphism group of $\mathcal{D}$: there are 4 point orbits under this action. Now, $d_0(\mathcal{X}) = d_0(\mathcal{Y}) = d_0(\mathcal{U}) = 2$, so by part (a) of Definition 1, $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{U}$ are self-sufficient in $\mathcal{M}$. Hence, by part (ii) of Theorem 3, the automorphism group acts with three orbits on the blocks of $\mathcal{D}$; it acts transitively on the blocks of each colour type.

4. Here we choose the function μ so that the structure $\mathcal{D}$ is a design. The details of the counts that we need are given in the following table.

Count symbol	What is counted	Configuration (Z, W)
α_{12}	For a fixed pair r, r' coloured c_1, the number of edges $\{b, r, r'\}$ with b coloured c_2.	$(\{b\}, \{r, r'\})$
β_{12}	For a fixed pair r, coloured c_1, and b, coloured c_2, the number of edges $\{r', r, b\}$ with r' coloured c_1.	$(\{r'\}, \{r, b\})$
γ_2	For a fixed pair b, b' coloured c_2, the number of pairs of points r and r' coloured c_1, such that the only edges in $\{r, r', b, b'\}$ are $\{b, r, r'\}$ and $\{b', r, r'\}$.	$(\{r, r'\}, \{b, b'\})$
α_{31}	For a fixed pair y, y' coloured c_3, the number of edges $\{r, y, y'\}$ with r coloured c_1.	$(\{r\}, \{y, y'\})$
α_{34}	For a fixed pair y, y' coloured c_3, the number of edges $\{g, y, y'\}$ with g coloured c_4.	$(\{g\}, \{y, y'\})$
β_{31}	For a fixed pair y, coloured c_3, and r, coloured c_1, the number of edges $\{y', y, r\}$ with y' coloured c_3.	$(\{y'\}, \{y, r\})$
β_{34}	For a fixed pair y, coloured c_3, and g, coloured c_4, the number of edges $\{y', y, g\}$ with y' coloured c_3.	$(\{y'\}, \{y, g\})$
γ_{14}	For a fixed pair r, coloured c_1, and g, coloured c_4, the number of pairs of points y and y' coloured c_3, such that the only edges in $\{y, y', r, g\}$ are $\{r, y, y'\}$ and $\{g, y, y'\}$.	$(\{y, y'\}, \{r, g\})$
α_{42}	For a fixed pair g, g' coloured c_4, the number of edges $\{b, g, g'\}$ with b coloured c_2.	$(\{b\}, \{g, g'\})$
α_{43}	For a fixed pair g, g' coloured c_4, the number of edges $\{y, g, g'\}$ with y coloured c_3.	$(\{y\}, \{g, g'\})$
β_{42}	For a fixed pair g, coloured c_4, and b, coloured c_2, the number of edges $\{g', g, b\}$ with g' coloured c_4.	$(\{g'\}, \{g, b\})$
β_{43}	For a fixed pair g, coloured c_4, and y, coloured c_3, the number of edges $\{g', g, y\}$ with g' coloured c_4.	$(\{g'\}, \{g, y\})$
γ_{23}	For a fixed pair b, coloured c_2, and y, coloured c_3, the number of pairs of points g and g' coloured c_4, such that the only edges in $\{g, g', b, y\}$ are $\{b, g, g'\}$ and $\{y, g, g'\}$.	$(\{g, g'\}, \{b, y\})$

Each of the counts is independent of the fixed pair of points chosen, and also the free amalgam of the required type can be found. We use Lemma 2 or Lemma 3 to prove that the free amalgam of t' copies of $\mathcal{Z} \cup \mathcal{W}$ over $\mathcal{W}$ is in C for any $t' \leq \mu(\mathcal{Z}, \mathcal{W})$; thus the count is at least $\mu(\mathcal{Z}, \mathcal{W})$. However, by part (b) of Definition 1, the count is at most $\mu(\mathcal{Z}, \mathcal{W})$. Hence, the count is equal to $\mu(\mathcal{Z}, \mathcal{W})$.

The following table shows the counts of the number of blocks containing a pair of points of the given colours.

	c_1	c_2	c_3	c_4
c_1	$\binom{\alpha_{12}}{2}$	$\beta_{12}(\alpha_{12}-1)$	$\beta_{31}\alpha_{34}$	γ_{14}
c_2		γ_2	γ_{23}	$\beta_{42}\alpha_{43}$
c_3			$\alpha_{31}\alpha_{34}$	$\beta_{34}\alpha_{31} + \beta_{43}\alpha_{42}$
c_4				$\alpha_{42}\alpha_{43}$

Hence we require values for μ so that

$$\begin{aligned}\lambda = \binom{\alpha_{12}}{2} &= \gamma_2 = \alpha_{31}\alpha_{34} = \alpha_{42}\alpha_{43} = \beta_{12}(\alpha_{12}-1) = \beta_{31}\alpha_{34} \\ &= \gamma_{14} = \gamma_{23} = \beta_{42}\alpha_{43} = \beta_{34}\alpha_{31} + \beta_{43}\alpha_{42}.\end{aligned}$$

This implies that

$$\beta_{12} = \alpha_{12}/2, \quad \beta_{31} = \alpha_{31} \quad \text{and} \quad \beta_{42} = \alpha_{42}$$

We set $\alpha_{12} = 6$ so that $\beta_{12} = 3$. We also set

$$\alpha_{31} = \beta_{31} = \beta_{42} = \alpha_{42} = \beta_{43} = 3,$$

and

$$\beta_{34} = 2 \quad \text{and} \quad \alpha_{34} = \alpha_{43} = 5.$$

Thus, with

$$\gamma_2 = \gamma_{14} = \gamma_{23} = 15,$$

we have a solution to the above equation with $\lambda = 15$.

Thus $\mathcal{D}$ is a 2-$(v, 4, 15)$ design with three block orbits and four point orbits as required. This completes the proof.

8 Block intransitive designs

The results of this paper and those of Camina [6] show that infinite designs with more point orbits than block orbits are not uncommon. Based on these findings we make the following conjecture:

Conjecture *Let n, k and l be finite natural numbers such that $\mathcal{F}_k(n) \leq l < n$. Then, for some finite natural number λ, there exists a 2-(v, k, λ) design with an automorphism group acting with n point orbits and l block orbits.*

This essentially claims that, whenever it is feasible to find a design with blocks of length k, with n point orbits and l block orbits, then for some finite λ such a 2-(v, k, λ) design exists. The difficulty in proving this arises in choosing the function μ and hence solving the equation for λ. The number of counts required increases as k, n or l are increased. For example, in the proof of Theorem 2, the 2-$(v, 4, 10)$ design with $n = 3$, $l = 2$ requires 8 counts and the 2-$(v, 4, 15)$ design with $n = 4$, $l = 3$ requires 13 counts. The author has constructed a 2-$(v, 6, 4)$ design with $n = 4$ and $l = 2$ in the same way and this requires 18 counts. To try to construct a 2-$(v, 6, \lambda)$ design with $n = 6$, $l = 3$, then either 30 or 32 counts are required, depending on the way in which the blocks are coloured. The complexity of this increases greatly as k, n and l are made general.

Acknowledgement The author would like to thank Chris Rowley for his input into this work, and Dave Evans for his helpful comments.

References

[1] R.E. Block. On the orbits of collineation groups. *Mathematische Zeitschrift* **96** (1967) 33–49.

[2] F. Buekenhout. Remarques sur l'homogénéité des espaces linéaires et des systèmes de blocs. *Mathematische Zeitschrift* **104** (1968) 144–146.

[3] P.J.Cameron. Infinite versions of some topics in finite geometry, in *Geometrical combinatorics* Research Notes in Mathematics Series 114 (ed. F.C.Holroyd and R.J.Wilson) Pitman (1984), 13–20.

[4] P.J. Cameron. An orbit theorem for Steiner triple systems. *Discrete Mathematics* **125** (1994) 97–100.

[5] P.J. Cameron and C.E. Praeger. Proposed problem at the 14th British Combinatorial Conference. 1993.

[6] A.R. Camina. Block transitive point intransitive block designs, in *Combinatorial designs and their applications* Research Notes in Mathematics Series 403 (ed. F.C.Holroyd, K.A.S.Quinn, C.Rowley, B.S.Webb) Chapman and Hall (1999), 71–82.

[7] D.M. Evans. A block transitive 2-design with two point orbits. *Journal of Combinatorial Theory (A)*, to appear.

[8] E. Hrushovski. A new strongly minimal set. *Annals Pure and Applied Logic* **62** (1993) 147–166.

[9] K. Prazmowski. An axiomatic description of the Strambach planes. *Geometrica Dedicata* **32** (1989) 125–156.

[10] K. Strambach. Zur Klassification von Satzmann-Ebenen mit dreidimensionaler Kollineationsgruppe.*Mathematische Annelen* **179** (1968) 15–30.

[11] B.S. Webb. Orbits of infinite block designs. *Discrete Mathematics* **169** (1997) 287–292.