Block Graphs of Z-transformation Graphs of Perfect Matchings of Plane Elementary Bipartite Graphs

Heping Zhang
Department of Mathematics
Lanzhou University
Lanzhou, Gansu 730000
P. R. China
email: zhanghp@lzu.edu.cn

Fuji Zhang
Department of Mathematics
Xiamen University
Xiamen, Fujian 361005
P. R. China

ABSTRACT. Let G be a connected plane bipartite graph. The Z-transformation graph $Z(G)$ is a graph where the vertices are the perfect matchings of G and where two perfect matchings are joined by an edge provided their symmetric difference is the boundary of an interior face of G. For a plane elementary bipartite graph G it is shown that the block graph of Z-transformation graph $Z(G)$ is a path. As an immediate consequence, we have that $Z(G)$ has at most two vertices of degree one.

Since a hexagonal system with at least one perfect matching is the skeleton of a benzenoid hydrocarbon molecule, graphs of this kind are of chemical significance and their topological properties have been extensively studied[2-4]. In Refs.[7,8] the concept of Z-transformation graphs of hexagonal systems was introduced. By virtue of this concept a complete characterization for the hexagonal systems with forcing edges [5] was given, see[9]. In [10] the present authors extended the Z-transformation graph of hexagonal systems to general plane bipartite graphs in a natural way. For a plane elementary bipartite graph some analogous results were obtained.

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Now let us recall some related concepts and properties. Some terminologies on the connection of graphs are taken from [14]. A connected graph is said to be 2-connected or nonseparable if it has no cut-vertex. A block in a graph G is a maximal 2-connected subgraph of G, that is, a 2-connected subgraph of G that is not contained in any other. Let G be a plane bipartite graph. The boundary of an interior face of G is called a ring if it is a cycle. A perfect matching of G is a set of disjoint edges covering all the vertices of G. A connected bipartite graph is said to be elementary or 1-extendable if each edge of it is contained in a perfect matching of G. It is known that an elementary bipartite graph is 2-connected. For other various properties, the interested reader may be referred to [6,10]. The symmetric difference of two finite sets A and B is defined as $A \oplus B =: (A \cup B) \setminus (A \cap B)$. For any concepts of graphs and directed graphs not defined in this paper see[1].

Definition 1. *Let G be a plane bipartite graph with perfect matchings. Z-transformation graph of G, denoted by $Z(G)$, is defined as a simple graph in which the vertices are the perfect matchings of G and two perfect matchings M_1 and M_2 are joined by an edge provided the symmetric difference $M_1 \oplus M_2$ consists exactly of a ring of G.*

Theorem 2. [10]. *Let G be a plane elementary bipartite graph. Then*

(a) *$Z(G)$ is connected bipartite graph,*

(b) *$Z(G)$ has at most two vertices of degree 1, and*

(c) *$Z(G)$ is either a path or a graph of girth 4 different from cycles.*

For hexagonal systems H and polyomino graphs P (or square systems), the above results hold. Here we would like to emphasize the following results: the connectivity of $Z(H)$ is equal to the minimum degree[8]; the same result holds for $Z(P)$ except for only two graphs [13]. For general plane (elementary) bipartite graphs the situations are more complicated. To find the relations between the blocks and cut-vertices of Z-transformation graphs we will study its block-graph.

Definition 3. *Let G be a graph. Let U and V be the sets of blocks and cut-vertices of G, respectively. The block-graph of G, denoted by $Blk(G)$, is defined as a bipartite graph $G(U, V)$ such that a vertex B of U and a vertex x of V are adjacent if and only if the block B includes the cut-vertex x of G.*

It is well known that the block-graph of a connected graph is a tree[14]. In this paper we obtain the following main result.

Theorem 4. *Let G be a plane elementary bipartite graph. Then the block-graph of Z-transformation graph of G is a path.*

To prove the theorem it needs to orientate every edge of Z-transformation graph $Z(G)$ and obtain a directed graph $\vec{Z}(G)$. First let us introduce some important concepts. Let G be a plane bipartite graph with a perfect matching M. A cycle C of G is called an M-alternating cycle if the edges of C appear alternately in M and $E(G)\backslash M$. An M-alternating path can be defined similarly. For convenience all the vertices of G are always colored properly black and white. In the following we restrict our consideration to plane elementary bipartite graphs G with a perfect matching M.

Definition 5. [11,12]. *An M-alternating cycle C of G is called a proper M-alternating cycle if each edge of C belonging to M goes from white vertex to black vertex by the orientation of C clockwise; otherwise C is known as an improper M-alternating cycle.*

Definition 6. *Let G be a plane bipartite graph. Orientate all the edges of $Z(G)$ and result in a directed Z-transformation graph $\vec{Z}(G)$ according to the following way: the orientaion of an edge M_1M_2 of $Z(G)$ is from M_1 to M_2 if and only if $M_1 \oplus M_2$ is a proper M_1-alternating ring of G.*

Lemma 7. *Assume that M_1 and M_2 are two perfect matchings of G such that $M_1 \oplus M_2$ is a proper M_1-alternating cycle. Then $\vec{Z}(G)$ has a directed path from M_1 to M_2.*

Proof: Let $C = M_1 \oplus M_2$. The proof is by induction on the number N of faces contained in the interior of the cycle C. If $N = 1$, the lemma is true obviously. In what follows assume that $N > 1$. There must exist an edge $e \in E(C)$ in the interior of C such that an end vertex of e lies on the C. Since G is elementary, G has a perfect matching M_3 such that $e \in M_3$. Hence the symmetric difference $M_1 \oplus M_3$ has an $M_1(M_3)$-alternating cycle C^* containing the egde e. Let P be a path on C^* such that two end vertices lie on C and the internal vertices of P lie in the interior of C. Since P is an M_1-,M_2- and M_3-alternating path and two end edges belong to M_3, the two end vertices of P are of different colors. C and P form two new cycles C' and C'' such that C' is an M_1-alternating cycle. Hence C' is a proper M_1-alternating cycle and C'' is a proper $M_1 \oplus C'$-alternating cycle. Since the numbers of faces contained in C' and C'' are all less than N, by introduction hypothesis we have that $\vec{Z}(G)$ has a directed path from M_1 to $M_1 \oplus C'$ and a directed path from $M_1 \oplus C'$ to $M_2 = M_1 \oplus C = M_1 \oplus C' \oplus C''$. The lemma is proved. □

Corollary 8. *In the interior of each proper (improper) M-alternating cycle of G there must exist a proper (improper) M-alternating ring.*

Proof: Assume that C is a proper M-alternating cycle. Denote by $G[C]$ the subgraph of G consisting of C together with its interior. It is easily shown that $G[C]$ is elementary. Let M_0 denote the restriction of M on $G[C]$. By lemma 7 $\vec{Z}(G[C])$ has a directed path from M_0 to $M_0 \oplus C$, namely, it has an arc with the tail M_0. Hence G has a proper M_0-alternating ring in the interior of C. □

Lemma 9. *Let Z be a subgraph of $Z(G)$. Denote by $\vec{Z}$ the orientation of Z such that the orientation of each edge of Z is the same as $\vec{Z}(G)$. Assume that $Z(G)$ has a cut-vertex M. Then*

(1) *Each proper M-alternating ring of G intersects each improper M-alternating ring of G.*

(2) *M belongs exaxtly to two blocks Z_1 and Z_2 of $Z(G)$, and*

(3) *M must be the source (a vertex of in-degree 0) of one and the sink (a vertex of out-degree 0) of the other one in $\vec{Z}_1$ and $\vec{Z}_2$.*

Proof: Let M be a cut-vertex of $Z(G)$. Let $f_i, i = 1, ..., s$, be all proper M-alternating rings of G; $g_j, 1, 2, ..., t$, all improper M-alternating rings of G. The neighbour set of M in $Z(G)$ is defined as $N(M) := \{M \oplus g_j : j = 1, 2, ..., t\} \cup \{M \oplus f_i : i = 1, 2, ..., s\}$. Let $V_1 = \{M \oplus_{i \in S} f_i : S \subseteq \{1, 2, ..., s\}\}$, where $M \oplus_{i \in S} f_i$ denote the symenetric difference of M and all $f_i, i \in S$. Let $V_2 = \{M \oplus_{j \in T} g_j : T \subseteq \{1, 2, ...t\}\}$. It is easy to know that all proper M-alternating rings f_i are disjoint and all improper M-alternating rings g_j are disjoint as well. Hence the induced subgraphs of $Z(G)$ by V_1 and V_2 are s- and t-dimensional hypercubes, respectively. It is obvious that $s \geq 1$ and $t \geq 1$. Hence M belongs exactly to two blocks Z_1 and Z_2 of G containing V_1 and V_2 respectively. So (2) follows.

It is clear that $(M, M \oplus f_i), i = 1, 2, ..., s$, are arcs of $\vec{Z}_1$; $(M \oplus g_j, M), j = 1, 2, ..., t$ are arcs of $\vec{Z}_2$. Thus M is both a source of $\vec{Z}_1$ and a sink of $\vec{Z}_2$, i.e. (3) follows. We now prove that $f_i \cap g_j \neq \emptyset$ for all $i = 1, ..., s$ and $j = 1, ..., t$. Suppose that $f_i \cap g_j = \emptyset$ for a certain pair (i, j) of subscripts. Then the induced subgraph of $Z(G)$ by $\{M, M \oplus g_j, M \oplus f_i, M \oplus g_j \oplus f_i\}$ is a cycle of length 4 containing M, which contradicts that M is a cut-vertex of $Z(G)$. □

Lemma 10. *$\vec{Z}(G)$ has no directed cycles.*

Proof: By contrary. Suppose that $\vec{Z}(G)$ has a directed cycle $M_1 M_2 \cdots M_t M_1$ such that $M_i \oplus M_{i+1} = s_i$ is a proper M_i-alternating ring, $i = 1, 2, ..., t$(the subscripts modulo t). Let f be any face of G. The depth $d(f)$ of f is defined as the length of shortest path of the dual graph G^* between two vertices corresponding to f and the exterior face (infinite face) of G. If f is an interior face of G, put $d(\partial f) := d(f)$, where ∂f denotes the boundary of f. Without loss of generality, assume that $d(s_1) = min\{d(s_i) : i = 1, 2, ..., t\}$. G must have a face f_0 such that the ∂f_0 and s_1 have an edge e in common and $d(f_0) = d(s_1) - 1$. It is obvious that $e \notin E(s_i), i = 2, ..., t$, i.e.

$e \notin M_i \oplus M_{i+1}, i = 2, ..., t$. In the process of $M_2 \to M_3 \to \cdots \to M_t \to M_1$, the matched way of e remains unchanged. Thus $e \notin M_2 \oplus M_1$. On the other hand, $s_1 = M_1 \oplus M_2$ and $e \in M_1 \oplus M_2$, a contradiction. □

Lemma 11. *$\vec{Z}(G)$ has exactly one source and one sink.*

Proof: If the out-degree of each vertex of $\vec{Z}(G)$ exceeds zero, $\vec{Z}(G)$ must have a directed cycle by its finiteness, which contradicts Lemma 10. Let M' be a sink of $\vec{Z}(G)$. Then G has no proper M'-alternating ring. Moreover, G has no proper M'-alternating cycles by Corollary 8. Suppose M'' is a sink of $\vec{Z}(G)$ other than M'. By the same reason as above G has a no proper M''-altenating cycles. But $M' \oplus M''$ must have a proper M'- or M''-alternating cycle, a contradiction. Therefore $\vec{Z}(G)$ has a unique sink. Similarly, it follows that $\vec{Z}(G)$ has a unique source. □

Lemma 12. *If the number of blocks of $Z(G) \geq 2$, then $Z(G)$ has exactly two extremal blocks, which corresponds to vertices of degree 1 in $Blk(Z(G))$.*

Proof: For any block B of $Z(G)$, it is obvious that $\vec{B}$ has no directed cycles. Hence $\vec{B}$ has at least one sink and one source. In particular, assume that B is extremal block of $Z(G)$. Then B contains exactly one cut-vertex of $Z(G)$. By Lemma 9(3) the cut-vertex must be a source or a sink of $\vec{B}$. Hence $\vec{B}$ has at least a sink or a source which is also a sink or a source of $\vec{Z}(G)$. By Lemma 11 it follows that $Z(G)$ has exactly two extremal blocks. □

Proof of Theorem 4: Since $Z(G)$ is connected, the block-graph of $Z(G)$ is a tree. On the other hand, by Lemma 12 it follows that the block-graph of $Z(G)$ has at most two vertices of degree one. Therefore the block-graph of $Z(G)$ is a path (single-vertex graph may be viewed as a degenerated case of path). □

The above results can be used to deduce Part (b) of Theorem 2 straightforwardly. Furthermore, we have

Corollary 13. *Let G be a plane elementary bipartite graph. For every block B of $Z(G)$, $\vec{B}$ has exactly one sink and one source.*

Corollary 14. *Let G be a plane elementary bipartite graph. Let t and s be the sink and source of $\vec{Z}(G)$ respectively, then s and t are not cut-vertices of $Z(G)$ and are contained in the two extremal blocks. For any other vertex w, $\vec{Z}(G)$ has a directed path from w to t and a directed path from s to w.*

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