Embedding Partial Extended Triple Systems when $\lambda \geq 2$

M. E. Raines and C. A. Rodger
Department of Discrete and Statistical Sciences
120 Math Annex
Auburn University, Alabama
USA 36849-5307

Abstract

In this paper it is shown that any partial extended triple system of order n and index $\lambda \geq 2$ can be embedded in an extended triple system of order v and index λ for all even $v \geq 4n + 6$. This extends results known when $\lambda = 1$.

1 Introduction

Let λK_n^+ be the complete multigraph on n vertices with edge set consisting of λ edges joining each pair of vertices and λ loops incident with each vertex. Define an *extended triple* to be a loop, a loop with an edge attached (known as a *lollipop*), or a copy of K_3 (known as a *triple*). We denote a loop incident with vertex a by $\{a,a,a\}$, a lollipop by $\{a,a,b\}$, $a \neq b$, when the loop of the lollipop is incident with vertex a, and a triple by $\{a,b,c\}$, where a, b, and c are distinct. A (partial) extended triple system of order n and index λ, (P)ETS(n,λ), is an ordered pair (V,B), where B is a set of extended triples defined on the vertex set V which partitions (a subset of) the edges of λK_n^+. If $\lambda = 1$, it has been shown (see, for example, [7]) that a (P)ETS(n) is equivalent to a (partial) totally symmetric quasigroup.

D. M. Johnson and N. S. Mendelsohn [9] first investigated extended triple systems and gave necessary conditions for their existence. Subsequently, these conditions were shown to be sufficient by F. E. Bennett and N. S. Mendelsohn [2].

A PETS$(n,\lambda)(V,B)$ is said to be embedded in an ETS$(v,\lambda)(V',B')$ if $V \subseteq V'$ and $B \subseteq B'$. Cruse and Lindner [4] obtained an embedding of a partial totally symmetric quasigroup of order n in a complete totally symmetric quasigroup of order v for any $v \equiv 0 \pmod 6$, $v \geq 6n$. D. G. Hoffman and C. A. Rodger [7] showed that an ETS$(n,\ 1)$ can be embedded in an ETS$(v,\ 1)$, where $v > n$, if and only if $v \geq 2n$, v is even if n is, and $(n,v) \neq (6k+5, 12k+12)$. Subsequently, M. E. Raines and C. A. Rodger [13] showed that any PETS$(n,\ 1)$ can be embedded in an ETS$(v,\ 1)$, for all $v \geq 4n+6, v \equiv 2 \pmod 4$ and showed that this bound on v can be lowered to $4n+2$ in many cases. Recently, M. E. Raines [12] showed that any PETS$(n,\ 1)$ can be embedded in an ETS$(v,\ 1)$, for all even $v \geq 4n+4$.

All of these embeddings follow upon several landmark results in this area where partial Steiner triple systems were considered. Treash [16] obtained a finite, yet very large, embedding for partial Steiner triple systems. Lindner [10] greatly reduced the size of the containing triple system to $v = 6n+3$. The best result to date is due to Andersen, Hilton, and Mendelsohn [1], and it provides an embedding for admissable $v \geq 4n+1$. Rodger and Stubbs [15] considered partial triple systems of index $\lambda \geq 2$ and found that a partial triple system of order n and index λ ($PTS(n,\lambda)$) can be embedded in a triple system of any odd λ-admissable order greater than $4n$. Subsequently, Hilton and Rodger [6] showed that if 4 divides λ, then any $PTS(n,\lambda)$ can be embedded in a $TS(v,\lambda)$ whenever v is λ-admissable and $v \geq 2n+1$, the best possible lower bound on v. Recently, Johansson [8] gave an embedding of a $PTS(n,\lambda)$ in a $TS(v,\lambda)$ where λ is even, whenever v is λ-admissable and $v \geq 2n+1$.

The focus of this paper is to obtain small embeddings of partial extended triple systems of index $\lambda \geq 2$ (see Corollary 4.4).

Theorem 1.1 *Any partial extended triple system of order n and index* $\lambda \geq 2$ *can be embedded in an extended triple system of order v and index* λ *for all even* $v \geq 4n+6$.

For terms and notation not defined here, we refer the reader to [3].

2 Preliminary Results

We begin with several results that will be used in the proofs of the main results.

Lemma 2.1 **([17])** *If a simple graph G on n vertices contains no* K_3*, then* $\epsilon(G) \leq \lfloor n^2/4 \rfloor$.

The next lemma follows from the previous one.

Lemma 2.2 *If a multigraph G of multiplicity at most* λ *on n vertices contains no* K_3*, then* $\epsilon(G) \leq \lambda \lfloor n^2/4 \rfloor$.

A *near 1-factor* of a graph G is a set of mutually nonadjacent edges in G which saturates all but one vertex of G. We have the following well known result.

Lemma 2.3 *If n is even (odd), then the edges of* K_n *can be partitioned into (near) 1-factors.*

Let Γ be any edge-coloring of a graph G. Let C_α, $\alpha \in \Gamma$ denote the set of edges colored α in this edge-coloring of G; C_α is called a *color class.* Let $C_\alpha(v)$ denote the set of edges incident with v that are colored α. The edge-coloring is said to be *equalized* if $||C_\alpha| - |C_\beta|| \leq 1$, for all $\alpha, \beta \in \Gamma$. The edge-coloring is said to be *equitable* if $||C_\alpha(v)| - |C_\beta(v)|| \leq 1$, for all $\alpha, \beta \in \Gamma$, and all $v \in V(G)$.

The next lemma will be used extensively.

Lemma 2.4 **([11] [18])** *A graph which has a proper n-edge-coloring has an equalized proper n-edge-coloring.*

Lemma 2.5 **([18] [19])** *Any bipartite graph can be given an equitable edge-coloring with n colors, for any* $n \geq 1$.

A *(partial) symmetric quasi-latin square L* of order r and multiplicity λ on n symbols (P)SQ(n, r, λ) is an r x r array such that

(i) for each i, j $(1 \le i, j \le r, i \ne j)$ if a symbol occurs x times in cell (i, j) then it also occurs x times in cell (j, i),

(ii) for each i, j $(1 \le i, j \le r, i \ne j)$, cell (i, j) contains at most λ symbols,

(iii) each symbol occurs (at most) exactly λ times in each row and (at most) exactly λ times in each column, and

(iv) each row contains at most $\lambda(r-1)$ symbols and each column contains at most $\lambda(r-1)$ symbols.

Let $N_L(i)$ denote the number of times symbol i occurs in some partial symmetric quasi-latin square L. Symmetric quasi-latin squares have been used in the embeddings of (extended) triple systems [1] [12] [13]. The following result was proved for the case $\lambda = 1$ in [13]. At first one thinks of proving Theorem 2.6 by splitting L into λ partial symmetric quasi-latin squares of multiplicity 1 and applying the result in [13]. However, this approach is often not possible.

Theorem 2.6 *Let $n \ge 1$ and $r \equiv t (mod\ 2)$. Let L be a partial symmetric quasi-latin square of order r and multiplicity λ on the symbols $1, \ldots, t$ (where $t = 2n$ or $2n+1$), in which row i contains $\lambda(r-2)$ symbols, for $1 \le i \le r$. Then L can be embedded in the top left corner of a symmetric quasi-latin square, L' of order $t+2$ and multiplicity λ on the symbols $1, \ldots, t$ in which the diagonal cells (i, i), for $r+1 \le i \le t+2$, and the near-diagonal cells $(r+2i-1, r+2i)$ and $(r+2i, r+2i-1)$, for $1 \le i \le (t-r+2)/2$, are empty, without adding any symbols to the cells in L if and only if*

(a) $N_L(i) \equiv \lambda(t+2)$ (mod 2), for $1 \le i \le t$, and

(b) $N_L(i) \ge \lambda(2r-t-2)$, for $1 \le i \le t$.

Proof: Necessity: For $1 \le i \le t$, symbol i must be placed $\lambda(t+2)$ times in L'. Since L' is symmetric, symbol i must occur an even number of times in L' outside L, so (a) is necessary. Let $N_A(i)$ and $N_B(i)$ denote the number of times symbol i occurs in A and B as shown in Figure 1. Then $N_L(i) = N_{L'}(i) - N_A(i) - N_B(i) \ge \lambda(t+2) - 2\lambda(t+2-r) = \lambda(2r-t-2)$, so (b) is necessary.

Sufficiency: Let $r \le s \le t$ with $s \equiv t$ (mod 2), and proceed by induction on s. Assume that s rows and columns have been completed so that each row contains $\lambda(s-2)$ symbols, thus forming L^*. Suppose also that, for $1 \le i \le t$, $N_{L^*}(i) \equiv \lambda(t+2)$ (mod 2), $N_{L^*}(i) \ge \lambda(2s-t-2)$, and that the appropriate diagonal and near-diagonal cells are empty.

First suppose $s < t$. Form an $(s+2) \times (s+2)$ array L_1 from L^* as follows. Let B_1 be a bipartite graph with bipartition $(X = \{1, \ldots, t\}, Y =$

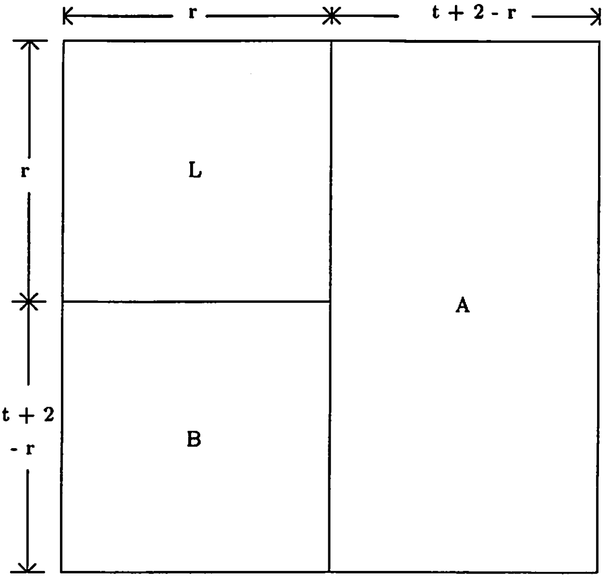

Figure 1:

$\{\rho_1, \ldots, \rho_s\}$) in which vertex i is joined to vertex ρ_j by x edges if and only if symbol i occurs $\lambda - x$ times in row j of L^*. We have that $d_{B_1}(\rho_j) = \lambda(t-(s-2))$ and $d_{B_1}(i) \leq \lambda s - \lambda(2s-t-2) = \lambda(t-(s-2))$, so $\Delta(B_1) = \lambda(t-(s-2))$. Give B_1 an equitable $((t-(s-2))/2)$-edge-coloring, where $C_1, \ldots, C_{(t-(s-2))/2}$ represent the color classes. Let B_1^* be the bipartite subgraph of B_1 induced by the edges in C_1. Now $d_{B_1^*}(\rho_j) = 2\lambda$ for $1 \leq j \leq s$, and $d_{B_1^*}(i) \leq \lceil d_{B_1}(i)/((t-(s-2))/2) \rceil \leq \lceil \lambda(t-(s-2))/((t-(s-2))/2) \rceil = 2\lambda$ for $1 \leq i \leq t$. Give B_1^* an equitable 2-edge-coloring using the colors α_1 and α_2. For every edge $\{i, \rho_j\}$ colored α_k, place symbol i in cells $(s+k, j)$ and $(j, s+k)$. Then each vertex in B_1^* is incident with at most λ edges of each of the colors α_1 and α_2. So, for $1 \leq j \leq s$ we have that row j of L_1 contains $\lambda(s-2) + 2\lambda = \lambda s$ symbols, and each symbol appears at most λ times in each of the added rows and columns. In addition, cells $(s+1, s+1)$, $(s+1, s+2)$, $(s+2, s+1)$, and $(s+2, s+2)$ remain empty since there are no vertices ρ_{s+1} and ρ_{s+2} in B_1. Each symbol occurs at most λ times in each row (column) of L_1 since the number of edges joining vertices i and ρ_j is λ minus the number of occurrences of symbol i in row j of L^*.

We have that $N_{L_1}(i) \equiv \lambda(t+2) \pmod 2$ since each symbol was added an even number of times in forming L_1 from L^*, so (a) is satisfied by L^*. To satisfy (b), we must have that $N_{L_1}(i) \geq \lambda(2(s+2) - t - 2) =$

$\lambda(2s-t+2)$. If $N_{L^*}(i) = \lambda(2s-t-2)+2y$, for some y, $0 \le y \le 2\lambda - 1$, then $d_{B_1}(i) = \lambda(t-(s-2)) - 2y$. Therefore, vertex i is incident with at least $\lfloor d_{B_1}(i)/((t-(s-2))/2)\rfloor = \lfloor 2(\lambda(t-(s-2))-2y)/(t-(s-2))\rfloor = 2\lambda - \lceil 4y/(t-(s-2))\rceil \ge 2\lambda - y$ (since in this case $t-(s-2) \ge 4$) edges colored 1 in B_1. Therefore, symbol i must have been placed at least $2(2\lambda - y)$ times in forming L_1 from L^*, so (b) is also satisfied by L^*.

Now suppose $s = t$, and let L_2 be a partial symmetric quasi-latin square of order t and multiplicity λ on the symbols $1,\ldots,t$. Form a bipartite graph B_2 in the same fashion that B_1 was formed. Since each row of L_2 contains $\lambda(t-2)$ symbols, $d_{B_2}(\rho_j) = 2\lambda$. Since $N_{L_2}(i) \ge \lambda(t-2)$ for $1 \le i \le t$, and since each row contains exactly $\lambda(t-2)$ symbols, it must be that $N_{L_2}(i) = \lambda(t-2)$. Therefore, $d_{B_2}(i) = 2\lambda$ for $1 \le i \le t$. So B_2 is a 2λ-regular graph which can be given an equitable 2-edge-coloring. Both colors, say α_1 and α_2, occur λ times at each vertex of B_2. For each edge $\{i, \rho_j\}$ colored α_k, for $1 \le k \le 2$, place symbol i in cells $(t+k, j)$ and $(j, t+k)$. Clearly, the desired symmetric quasi-latin square L' is obtained. □

We will also need the following companion result to Theorem 2.6. Conditions (a) and (b) in Theorem 2.7 are necessary, but it turns out that $\sum_{i=1}^{t} \max\{0, \lambda(2r-t-2) - N_L(i)\} \le \lambda(r-1)^2$ is a third necessary condition. However, condition (c) below (which implies this third necessary condition) will suffice for our purposes.

Theorem 2.7 *Let $1 \le r \le \lfloor (t+2)/2 \rfloor$, and let $r \equiv (t+1)$ (mod 2). Let L be a partial symmetric quasi-latin square of order r and multiplicity λ on the symbols $1,\ldots,t$ in which, for $1 \le i \le r-1$, row i contains $\lambda(r-2)$ symbols and in which row r contains $\lambda(r-1)$ symbols. Then L can be embedded in the top left corner of a symmetric quasi-latin square, L', of order $t+2$ and multiplicity λ on the symbols $1,\ldots,t$ in which the diagonal cells (i,i), for $r+1 \le i \le t+2$, and the near-diagonal cells $(r+2i, r+2i+1)$ and $(r+2i+1, r+2i)$, for $0 \le i \le (t-r+1)/2$ are empty, without adding any symbols to the cells in L if*

(a) $N_L(i) \equiv \lambda(t+2)$ (mod 2), for $1 \le i \le t$,

(b) $N_L(i) \ge \lambda(2r-t-2)$, for $1 \le i \le t$, and

(c) if $r = (t+2)/2$, then there are no more than $r-1$ symbols satisfying $N_L(i) < \lambda(2(r+1)-t-2)$.

Proof: We consider three cases in turn. In each case we embed L in an $(r+1)\times(r+1)$ array L^* which satisfies the conditions of Theorem 2.6 which shows that L^* can be embedded in L' as required.

Firstly, suppose $1 \le r < \lfloor (t+2)/2 \rfloor$ with $r \equiv (t+1) \pmod 2$. Let L be a PSQLS(r, t, λ) in which, for $1 \le i \le r-1$, row i contains $\lambda(r-2)$ symbols and in which row r contains $\lambda(r-1)$ symbols. Form an $(r+1) \times (r+1)$ array, L^*, as follows. For $1 \le j \le r-1$, greedily fill each of the cells $(j, r+1)$ and $(r+1, j)$ with λ symbols so that symmetry is preserved and so that no row of L^* will contain any symbol more than λ times. Now for $1 \le i \le r-1$, row i of L^* contains $\lambda(r-1)$ symbols, and rows r and $r+1$ of L^* each contain $\lambda(r-1)$ symbols, so L^* is a PSQLS$(r+1, t, \lambda)$. Clearly condition (a) is satisfied for L^*, and condition (b) is satisfied since we only need $N_{L^*}(i) \ge 0$ in any case. Therefore, by Theorem 2.6, L^* can be embedded in a SQLS$(t+2, t, \lambda)$ in which the diagonal cells (i, i), for $r+1 \le i \le t+2$, and the near-diagonal cells $(r+2i, r+2i+1)$ and $(r+2i+1, r+2i)$, for $0 \le i \le (t-r+1)/2$, are empty. Hence, the theorem is true when $r < \lfloor (t+2)/2 \rfloor$.

Secondly, suppose $r = (t+2)/2 \ge 3$, so by (c) the number of symbols for which $N_L(i) < \lambda(2(r+1) - t - 2) = 2\lambda$ is no more than $r-1$. Let $M = \{1, \ldots, k\}$ be the set of $k \le r-1$ symbols which occur less than 2λ times in L. We have that for $1 \le i \le r$, row i contains $\lambda(r-2)$ symbols, and row r contains $\lambda(r-1)$ symbols. Our goal is to form a PSQLS$(r+1, t, \lambda)$, L^*, in which for $1 \le i \le t$, $N_{L^*}(i) \equiv \lambda(t+2) \pmod 2$ and $N_{L^*}(i) \ge \lambda(2(r+1) - t - 2) = 2\lambda$.

We form L^* in the following manner. Let B be a bipartite graph with bipartition $(X = \{\rho_1, \ldots, \rho_{r-1}\}, Y = \{1, \ldots, k\})$, where $\rho_1, \ldots, \rho_{r-1}$ represent the rows $1, \ldots, r-1$ of L and $1, \ldots, k$ represent the elements of M. Join vertices $\rho_j \in X$ and $i \in Y$ with $\lambda - x$ edges if and only if symbol i occurs x times in row j of L. We have that $d_B(\rho_j) \le \lambda k \le \lambda(r-1)$. Also, for $1 \le i \le k$, $\lambda(r-3) < d_B(i) \le \lambda(r-1)$, since symbol i occurs less than 2λ times altogether in L. Give B an equitable $(r-1)$-edge-coloring with color classes $C_1, \ldots, C_{r-1}$. For $1 \le j \le r-1$ and for every edge $\{\rho_j, i\} \in C_1$, place symbol i in cells $(j, r+1)$ and $(r+1, j)$ of L^*. Note that cells $(r, r+1)$, $(r+1, r)$, and $(r+1, r+1)$ of L^* contain no symbols.

In any cell $(\alpha, r+1)$ that contains $\lambda - y$ symbols, greedily place y more symbols and place the same y symbols in cell $(r+1, \alpha)$. (For, row α contains exactly $\lambda(r-1) - y$ symbols, and column $r+1$ contains at most $\lambda(r-1) - y$ symbols. Therefore, there are at least $\lambda(2r-2) - (2\lambda(r-1) - 2y) = 2y$ symbols available to be placed in cell $(\alpha, r+1)$. However, we only need to place y more symbols there, so we can fill the cell). This completes the formation of L^*. Suppose symbol $i \in M$ occurs $2x$ times in L. Then $d_B(i) \ge \lambda(r-1) - 2x$, so vertex i is incident with at least $\lfloor (\lambda(r-1) - 2x)/(r-1) \rfloor \ge \lambda - \lceil 2x/(r-1) \rceil \ge \lambda - x$ (since $r \ge 3$) edges in C_1. Hence, symbol i occurs at least $N_L(i) + 2(\lambda - x) = 2\lambda$ times in L^*. Therefore, for $1 \le i \le k$ (and also for $1 \le i \le t$), $N_{L^*}(i) \ge \lambda(2(r+1) - t - 2)$. Also, vertex ρ_j is incident with at most λ edges in C_1, so for $1 \le j \le r-1$, cells

$(j, r+1)$ and $(r+1, j)$ contain at most λ symbols.

Finally suppose $r = (t+1)/2$. Then condition (a) tells us that $\lambda(t+2)$ is odd, so both λ and $(t+2)$ are odd. We have that L contains $(r-1)\lambda(r-2)+\lambda(r-1) = \lambda(r-1)^2$ symbols. Suppose a symbol occurs at most $2\lambda - 1$ times is L. Then the remaining $t-1 = 2r-2$ symbols occur at least an average of $(\lambda(r-1)^2 - 2\lambda+1)/2r-2 = (\lambda(r^2-2r-1)+2(r-1))/2(r-1) \geq 2+\lambda(r-2)/2 \geq \lambda$ times in L when $r \geq 4$ (when $r = 3$, we can apply Theorem 2.6 since $r \equiv (t+2) \pmod 2$). Hence, there is at least one symbol, say symbol t, which occurs at least λ times in L. We form an $(r+1) \times (r+1)$ array L^* as follows. Suppose symbol i occurs $\lambda - x_i$ times in L; so by (a) x_i is even since λ is odd. Then it occurs at most $\lambda - x_i$ times in any row j of L, where $1 \leq j \leq r-1$. For $1 \leq i \leq 2r-2$, place symbol$i$ x_i times in cell $(\lceil i/2 \rceil, r+1)$ and in cell $(r+1, \lceil i/2 \rceil)$. Since $x_i < \lambda$, we have that each of these cells contains at most λ symbols. In addition, symbol i will occur at least $\lambda - x_i + 2(x_i/2) = \lambda$ times in L^*. Therefore, every symbol appears at least λ times in L^*. Lastly, as in the previous case, greedily place symbols so that each cell $(\alpha, r+1)$ and $(r+1, \alpha)$, for $1 \leq \alpha \leq r-1$, contains exactly λ symbols.

In any case, we have that for $1 \leq i \leq t$, $N_{L^*}(i) \equiv \lambda(t+2) \pmod 2$ since $N_L(i) \equiv \lambda(t+2) \pmod 2$ and since symbols are placed symmetrically when forming L^* from L. In addition, we showed earlier in each case that $N_{L^*}(i) \geq \lambda(2(r+1)-t-2)$ for $1 \leq i \leq t$. Furthermore, for $1 \leq i \leq r-1$, row i of L^* contains $\lambda + \lambda(r-2) = \lambda(r-1)$ symbols and rows r and $r+1$ contain $\lambda(r-1)$ symbols. So L^* is a PSQLS$(r+1, t, \lambda)$ satisfying the conditions of Theorem 2.6. Therefore, by Theorem 2.6, L^* can be embedded in a SQLS $(t+2, t, \lambda)$, L', which contains holes of size 2 down the main diagonal of L' outside L^*, and the proof is complete. □

A *partial triple system* of order n and index λ (PTS(n, λ)) is an ordered pair (S, T), where T is a set of edge-disjoint copies of K_3, or *triples*, that together form a subgraph $G(S)$ of λK_n with vertex set S. We define the *leave* of (S, T) to be the complement of $G(S)$ in λK_n. Let $\mu(n, \lambda)$ denote the maximum possible number of triples in a PTS(n, λ).

Lemma 2.8 ([5])

$$\mu(n,\lambda) = \begin{cases} \lfloor \frac{n}{3} \lfloor \frac{\lambda(n-1)}{2} \rfloor \rfloor - 1 & \text{for } n \equiv 2 \pmod 6 \text{ and } \lambda \equiv 4 \pmod 6 \text{ or} \\ & \text{for } n \equiv 5 \pmod 6 \text{ and } \lambda \equiv 1 \text{ or } 4 \pmod 6 \\ \lfloor \frac{n}{3} \lfloor \frac{\lambda(n-1)}{2} \rfloor \rfloor & \text{otherwise} \end{cases}$$

For a PTS(n, λ) on the vertex set $\{1, \ldots, n\}$, let $r(i)$ denote the number of triples which contain symbol i. If $|r(i) - r(j)| \leq 1$, for $1 \leq i < j \leq n$, the PTS(n, λ) is said to be *equitable*.

Lemma 2.9 ([15]) *Let v, λ, and n be non-negative integers such that $1 \le v \le \mu(n,\lambda)$. Then there is an equitable PTS(n,λ) with v triples.*

Lemma 2.10 ([14]) *There exists an equitable $PTS(n,1)(S,T)$ with $t(n)$ triples such that the leave contains a 1-factor if n is even and a near 1-factor if n is odd if and only if $t(n) \le T(n)$, where*

$$T(n) = \begin{cases} \mu(n) & =n(n-2)/6 & \text{if } n \equiv 0 \pmod 6 \\ \mu(n) - \lfloor n/3 \rfloor & =(n-1)(n-2)/6 & \text{if } n \equiv 1 \pmod 6 \\ \mu(n) & =n(n-2)/6 & \text{if } n \equiv 2 \pmod 6 \\ \mu(n) - n/3 & =n(n-3)/6 & \text{if } n \equiv 3 \pmod 6 \\ \mu(n) - 1 & =(n-4)(n+2)/6 & \text{if } n \equiv 4 \pmod 6 \\ \mu(n) - (n-5)/3 & =(n-1)(n-2)/6 & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

A $PTS(n,\lambda)$ is said to be *regular* if the leave of the PTS is regular. The following lemma follows from Lemma 2.9 and Lemma 2.10.

Lemma 2.11 *Let $n \ge 3$. There exists a regular $PTS(n,1)$ (S,T) containing $R(n)$ triples such that the leave of (S,T) contains a (near) 1-factor if n is even (odd), where*

$$R(n) = \begin{cases} n(n-2)/6 & \text{if } n \equiv 0,2 \pmod 6 \\ n(n-7)/6 & \text{if } n \equiv 1 \pmod 6 \\ n(n-3)/6 & \text{if } n \equiv 3 \pmod 6 \\ n(n-4)/6 & \text{if } n \equiv 4 \pmod 6 \\ n(n-5)/6 & \text{if } n \equiv 5 \pmod 6\ . \end{cases}$$

A *2-factor* of a graph G is a 2-regular spanning subgraph of G. A *near 2-factor* of G is a 2-regular spanning subgraph of $G \setminus \{v\}$, where $v \in V(G)$.

The proofs of the next two lemmas are so similar that we group the proofs into one. Throughout we assume that a doubled edge is a cycle of length 2.

Lemma 2.12 *Let $\lambda = 2$ and $n \ge 3$. Let $t(n,2) \le T(n,2)$, where $T(n,2) = T(n) + R(n)$ if $n \notin \{4,5,7\}$, $T(4,2) = 2$, $T(5,2) = 4$, and $T(7,2) = 10$. Then there exists an equitable $PTS(n,2)(S,T)$ with $t(n,2)$ triples such that: if n is even then the leave contains a 2-factor in which each cycle has even length; and if n is odd then the leave contains a near 2-factor in which each cycle has even length.*

Lemma 2.13 *Let $\lambda = 2$ and $n \ge 3$. Let $t^*(n,2) \le T^*(n,2)$, where $T^*(n,2) = T(n,2)$ if $n \equiv 1,3,4$ or $5 \pmod 6$ and $n \notin \{4,5,7\}$, $T^*(n,2) = T(n,2) - 1$ if $n \equiv 0$ or $2 \pmod 6$, $T^*(4,2) = T^*(5,2) = 3$, and $T^*(7,2) = 9$. Then there exists an equitable $PTS(n,2)$ (S,T) with $t^*(n,2)$ triples such that if n is (even) odd, then the leave contains a (near) 2-factor consisting of all even cycles except for exactly one 3-cycle.*

Proof: Clearly Lemma 2.12 follows from Lemma 2.10 if $t(n,2) \leq T(n)$, and Lemma 2.13 follows from Lemma 2.10 if $t^*(n,2) \leq T(n)$, so we can assume that $T(n) < t(n,2) \leq T(n,2)$, and $T(n) < t^*(n,2) \leq T^*(n,2)$.

Suppose $n \neq 4$, 5 or 7. If $t(n,2) > T(n)$ and if n is (odd) even, then by Lemma 2.11 let (S,T_1) be a regular $PSTS(n)$ with $R(n)$ triples that has a (near) 1-factor, F_1, in the leave, and by Lemma 2.10 let (S,T_2) be an equitable $PSTS(n)$ with $t(n,2) - R(n) \leq T(n,2) - R(n) = T(n)$ triples that has a (near) 1-factor, F_2, in the leave. Clearly, $F_1 \cup F_2$ is a (near) 2-factor in which each cycle has even length.

If $t^*(n,2) > T(n)$, define (S,T_1) and F_1 as above, and if n is (odd) even then let (S,T_2) be an equitable $PSTS(n)$ with $t^*(n,2) - R(n)$ triples that has a (near) 1- factor, F_2, in the leave. Without loss of generality, if n is (odd) even we can assume that F_1 and F_2 differ in exactly (one) two edges. Unless $n \equiv 0$ or 2 (mod 6) and (S,T_2) is maximal (so $|T_1| + |T_2| = T(n,2)$), the leave of $(S, T_1 \cup T_2)$ contains some edge $\{a,b\} \notin F_1 \cup F_2$. If n is odd, then we can assume that $\{a,x\} \in F_1$ and $\{b,x\} \in F_2$ are the edges in which F_1 and F_2 differ. If n is even, let $\{a,y\},\{b,x\} \in F_1$ and $\{a,x\},\{b,y\} \in F_2$. Clearly $\{a,b,x\}$ is a 3-cycle in the leave of $(S, T_1 \cup T_2)$, and the remaining edges of $F_1 \cup F_2$ form even cycles, so the result follows.

Now suppose $n = 4$. Let $T = \{\{1,2,3\},\{1,3,4\},\{2,3,4\}\}$. Define an equitable $PTS(4)$ (S,T') with $k \leq 3$ triples such that T' consists of the first k triples listed in T. If $|T'| \leq 2$, then the leave contains a 2-factor, and if $|T'| = 3$, then the leave contains a 3-cycle.

Next suppose $n = 5$, and let $T = \{\{1,2,3\},\{1,4,5\},\{2,3,4\},\{3,4,5\}\}$. Define an equitable $PTS(5)$ (S,T') with $k \leq 4$ triples such that T' consists of the first k triples listed in T. If $|T'| \leq 4$, then the leave contains a 4-cycle, and if $|T'| \leq 3$, then the leave contains a 3-cycle and a 2-cycle.

Finally suppose $n = 7$, and let $T = \{\{1,3,4\},\{2,5,6\},\{4,6,7\},\{1,5,7\},$ $\{2,3,4\},\{1,6,7\},\{2,3,5\},\{1,3,5\},\{2,4,6\},\{4,5,7\}\}$. Define an equitable $PTS(7)$ (S,T') with $k \leq 10$ triples such that T' consists of the first k triples listed in T. If $|T'| \leq 10$, then the leave contains a near 2-factor consisting of even cycles, and if $|T'| \leq 9$, then the leave contains a 2-factor with exactly one 3-cycle and two 2-cycles. □

3 Embedding a PETS(n,λ) in an ETS$(4n+2,\lambda)$

Given any PETS$(n,\lambda)(V,B)$, define the *deficiency graph*, $G(B)$, to be the graph on the vertex set V whose edge set consists of the edges of λK_n^+ not found in any extended triple in B. Let $p(G(B))$ denote the number of vertices of odd degree in $G(B)$ and let

$$P(G(B)) = \begin{cases} p(G(B)) & \text{if } \epsilon(G(B)) + p(G(B)) \equiv 0 \pmod 3, \\ p(G(B)) + 2 & \text{if } \epsilon(G(B)) + p(G(B)) \equiv 1 \pmod 3, \\ p(G(B)) + 4 & \text{if } \epsilon(G(B)) + p(G(B)) \equiv 2 \pmod 3. \end{cases}$$

We say that (V, B) is *maximal* if $G(B)$ contains no extended triples (and hence no loops).

The embedding process takes two steps. Lemma 3.1 embeds the *PETS* (n, λ) in a *PETS* whose deficiency graph meets the conditions of Proposition 3.2. Applying Proposition 3.2 then completes the embedding.

Lemma 3.1 *Let (V, B) be a maximal PETS(n, λ) with $n \geq 3$ and $\lambda \geq 2$. Then (V, B) can be embedded in a PETS$(2n, \lambda)(V^*, B^*)$ satisfying:*

(i) $\Delta(G(B^*)) < \lambda n$,

(ii) $P(G(B^*)) \leq n$,

(iii a) *If* $\lambda = 2$, *then* $\epsilon(G(B^*)) + P(G(B^*)) \leq 3T(n + 1, 2)$

(iii b) *If* $\lambda > 2$ *then*

$$\epsilon(G(B^*)) + P(G(B^*)) \leq \begin{cases} 3\mu(n + 1, \lambda - \lceil \lambda/n \rceil) & \text{if } n \text{ is odd} \\ 3\mu(n + 1, \lambda - \lceil \lambda/(n - 1) \rceil) & \text{if } n \text{ is even,} \end{cases}$$

(iv) and $G(B^*)$ contains at least 2 vertices of degree at most $\lambda n - 2$.

Proof: Let $V = \{1, \ldots, n\}$ and $V^* = \{1, \ldots, 2n\}$.

Case 1: n is odd, λ is odd.

If $p(G(B)) \neq 0$, we can assume without loss of generality that $p(G(B)) \geq 2$ and that vertices $n - 1$ and n have odd degree.

Define B^* as follows.

(1) $B \subseteq B^*$.

(2a) Suppose $n = 3$ and $p(G(B)) = 0$. If $\epsilon(G(B)) + p(G(B)) \equiv 1 \pmod 3$, let B^* contain λ copies of the lollipops $\{4, 4, 1\}$ and $\{5, 5, 2\}$, $\lambda - 2$ copies of the lollipop $\{6, 6, 3\}$, and the remaining loops at vertex 6. If $\epsilon(G(B)) + p(G(B)) \equiv 2 \pmod 3$, let B^* contain λ copies of the lollipop $\{4, 4, 1\}$, $\lambda - 2$ copies of the lollipops $\{5, 5, 2\}$ and $\{6, 6, 3\}$, and the remaining loops at vertices 5 and 6.

(2b) If $p(G(B)) = 0$ and $n \geq 5$ or if $\epsilon(G(B)) + p(G(B)) \equiv 0 \pmod 3$, then B^* contains λ copies of the lollipops $\{n + i, n + i, i\}$, for $1 \leq i \leq n$.

(2c) If $p(G(B)) \neq 0$ and $\epsilon(G(B))+p(G(B)) \equiv 1 \pmod 3$, then B^* contains λ copies of the lollipops $\{n+i, n+i, i\}$, for $1 \leq i \leq n-2$, $\lambda-1$ copies of the lollipops $\{2n-1, 2n-1, n-1\}$ and $\{2n, 2n, n\}$, and the remaining loops at vertices $2n-1$ and $2n$.

(2d) If $p(G(B)) \neq 0$ and $\epsilon(G(B))+p(G(B)) \equiv 2 \pmod 3$, then B^* contains λ copies of the lollipops $\{n+i, n+i, i\}$ for $1 \leq i \leq n-1$, $\lambda-1$ copies of the lollipop $\{2n, 2n, n\}$, and the loop $\{2n, 2n, 2n\}$.

(3) Using Lemma 2.3, partition the edges of K_n defined on the vertex set $\{n+1, \ldots, 2n\}$ into the near 1-factors $F_1, \ldots, F_n$ with the property that F_v does not saturate vertex $n+v$. For each edge $\{a, b\} \in F_v$, for $1 \leq v \leq n$, let B^* contain λ copies of the triple $\{v, a, b\}$.

We first investigate condition (i). We have that $\Delta(G(B)) \leq \lambda(n-1)$, and in any case of the above construction, we add at most two to the degree of any vertex in $\{1, \ldots, n\}$ when forming $G(B^*)$. In addition, $d_{G(B^*)}(j) \leq 2$, for $n+1 \leq j \leq 2n$. Therefore, $\Delta(G(B^*)) \leq \Delta(G(B)) + 2 \leq \lambda(n-1)+2 < \lambda n$, so (i) is satisfied.

Next we consider condition (ii). Note that in all cases $d_{G(B^*)}(i) = 0$, for $n+1 \leq i \leq 2n$, except that $d_{G(B^*)}(2n-1) = 1$ in case (2c), and $d_{G(B^*)}(2n) = 2, 1$, or 1 in cases (2a), (2c), and (2d), respectively. However, if $d_{G(B^*)}(i) = 1$ (so $i \in \{2n-1, 2n\}$), then $d_{G(B)}(i-n)$ is odd, so $d_{G(B^*)}(i-n)$ is even. So in every case, for $n+1 \leq i \leq 2n$, either vertex i or vertex $i-n$ has even degree, so $p(G(B^*)) \leq n$.

In case (2a), we have $\epsilon(G(B^*)) + p(G(B^*)) \equiv 0 \pmod 3$; therefore, $P(G(B^*)) = p(G(B^*)) = 0 < n$. In case (2b): if $p(G(B)) = 0$, then $P(G(B^*)) \leq 4 \leq n$ if $n \geq 5$; and, if $n = 3$ and $p(G(B)) = 0$ then $\epsilon(G(B)) \equiv 0 \pmod 3$, so $P(G(B^*)) = 0 < n$. It is easily verified in all remaining cases that $\epsilon(G(B^*)) + p(G(B^*)) \equiv 0 \pmod 3$, and since $p(G(B^*)) \leq n$, $P(G(B^*)) \leq n$, so (ii) is satisfied.

Condition (iii a) does not apply since λ is odd, so we now investigate condition (iii b). In (2a), $n = 3$ and $p(G(B)) = 0$. In addition $\epsilon(G(B^*)) + p(G(B^*)) \equiv 0 \pmod 3$, so $\epsilon(G(B^*)) + P(G(B^*)) \leq \lambda\lfloor n^2/4 \rfloor + 4 = 2\lambda + 4$. We see that $3\mu(n+1, \lambda - \lceil \lambda/n \rceil) = 3\mu(4, \lambda - \lceil \lambda/3 \rceil) \geq 4\lambda - 6 \geq \epsilon(G(B^*)) + P(G(B^*))$ when $\lambda \geq 5$. If $\lambda = 3$, then we have $3\mu(4,2) = 12 > \epsilon(G(B^*)) + P(G(B^*))$, so (iii b) is satisfied for case (2a) and for all odd $\lambda \geq 3$.

Consider case (2b). We have that $\epsilon(G(B^*)) + P(G(B^*)) \leq \lambda\lfloor n^2/4 \rfloor + n$. We can calculate that $3\mu(n+1, \lambda - \lceil \lambda/n \rceil) \geq (\lambda(n^2-1) - n^2 - 10)/2$ and so $\epsilon(G(B^*)) + P(G(B^*)) \leq 3\mu(n+1, \lambda - \lceil \lambda/n \rceil)$ for all odd $\lambda \geq 3$ and all odd $n \geq 3$ except possibly for the following special cases: $n = 3$ and $\lambda \leq 7$; $n = 5$ and $\lambda = 3$; and $n = 7$ and $\lambda = 3$. However, direct calculations show that $3\mu(n+1, \lambda - \lceil \lambda/n \rceil) \geq \epsilon(G(B^*)) + P(G(B^*))$ in all of these cases as well, so (iii b) is satisfied in case (2b).

Next consider case (2c). We have that $\epsilon(G(B^*))+P(G(B^*)) \leq \lambda\lfloor n^2/4\rfloor + n+2 \leq (\lambda(n^2-1)-n^2-10)/2 \leq 3\mu(n+1,\lambda-\lceil\lambda/n\rceil)$ when $\lambda \geq 3$ and $n \geq 3$, except possibly for the following special cases: $n=3$ and $\lambda \leq 7$; $n=5$ and $\lambda=3$; and, $n=7$ and $\lambda=3$. However, direct calculations show that $3\mu(n+1,\lambda-\lceil\lambda/n\rceil) \geq \epsilon(G(B^*))+P(G(B^*))$ in all of these cases as well, so (iii b) is satisfied in case (2c).

Finally, we consider case (2d). We have that $\epsilon(G(B^*))+P(G(B^*)) \leq \lambda\lfloor n^2/4\rfloor + n+1 \leq (\lambda(n^2-1)-n^2-10)/2 \leq 3\mu(n+1,\lambda-\lceil\lambda/n\rceil)$ for all odd $n \geq 3$ and all odd $\lambda \geq 3$ except possibly for the following special cases: $n=3$ and $\lambda \leq 7$; $n=5$ and $\lambda=3$; and, $n=7$ and $\lambda=3$. However, direct calculations show that $3\mu(n+1,\lambda-\lceil\lambda/n\rceil) \geq \epsilon(G(B^*))+P(G(B^*))$ in all of these cases as well, so (iii b) is satisfied for (2d) and thus for Case 1.

Clearly, the construction gives a $PETS(2n,\lambda)$ satisfying (iv).

Case 2: n is odd, λ is even.

Again, if $p(G(B)) \neq 0$, we can assume without loss of generality that $p(G(B)) \geq 2$ and that vertices $n-1$ and n have odd degree in $G(B)$.

Define B^* as follows.

(1) $B \subseteq B^*$.

(2a) Suppose $n=3$ and $p(G(B))=0$. If $\epsilon(G(B))+p(G(B)) \equiv 1 \pmod 3$, let B^* contain λ copies of the lollipops $\{4,4,1\}$ and $\{5,5,2\}$, $\lambda-2$ copies of the lollipop $\{6,6,3\}$, and the remaining loops at vertex 6. If $\epsilon(G(B))+p(G(B)) \equiv 2 \pmod 3$, let B^* contain λ copies of the lollipop $\{4,4,1\}$, $\lambda-2$ copies of the lollipops $\{5,5,2\}$ and $\{6,6,3\}$, and the remaining loops at vertices 5 and 6.

(2b) If $p(G(B))=0$ and $n \geq 5$ or if $\epsilon(G(B))+p(G(B)) \equiv 0 \pmod 3$, then B^* contains λ copies of the lollipops $\{n+i,n+i,i\}$, for $1 \leq i \leq n$.

(2c) If $p(G(B)) \neq 0$ and if $\epsilon(G(B))+p(G(B)) \equiv 1 \pmod 3$, then B^* contains λ copies of the lollipops $\{n+i,n+i,i\}$, for $1 \leq i \leq n-1$, $\lambda-2$ copies of the lollipop $\{2n,2n,n\}$, and the two remaining loops at vertex $2n$.

(2d) If $p(G(B)) \neq 0$ and if $\epsilon(G(B))+p(G(B)) \equiv 2 \pmod 3$, then B^* contains λ copies of the lollipops $\{n+i,n+i,i\}$, for $1 \leq i \leq n-1$, $\lambda-1$ copies of the lollipop $\{2n,2n,n\}$, and the loop $\{2n,2n,2n\}$.

(3) Using Lemma 2.3, partition the edges of K_n defined on the vertex set $\{n+1,\ldots,2n\}$ into the near 1-factors $F_1,\ldots F_n$ with the property that F_v does not saturate vertex $n+v$. For each edge $\{a,b\} \in F_v$, for $1 \leq v \leq n$, let B^* contain λ copies of the triple $\{v,a,b\}$.

We consider condition (i). We have that $\Delta(G(B)) \leq \lambda(n-1)$ and $d_{G(B^*)}(j) = d_{G(B)}(j)$, for $1 \leq j \leq n-2$. Furthermore, we have that

$d_{G(B^*)}(j) \leq 2$, for $n+1 \leq j \leq 2n$. In addition, in (2a) we can have without loss of generality that $d_{G(B)}(i) < 2\lambda$, so $d_{G(B^*)}(i) < 2\lambda + 2 \leq \lambda n$, where $i \in \{2,3\}$. In cases (2b) and (2d), we have also that $d_{G(B^*)}(n) \leq d_{G(B)}(n) + 1 \leq \Delta(G(B)) + 1 < \lambda n$. Finally, in case (2c) we have that $d_{G(B^*)}(n) = d_{G(B)}(n) + 2$. However, since $d_{G(B)}(n)$ is odd without loss of generality, $d_{G(B)}(n) < \lambda(n-1)$ so $d_{G(B^*)}(n) < \lambda n$, so (i) is satisfied in all cases.

We now consider (ii). Note that in all cases $d_{G(B^*)}(i) = 0$ for all $n+1 \leq i \leq 2n$, except that $d_{G(B^*)}(2n) = 2$, 2, or 1 in cases (2a), (2c), and (2d), respectively. However, if $d_{G(B^*)}(2n) = 1$ then $d_{G(B)}(n)$ is odd, so $d_{G(B^*)}(n)$ is even. Therefore, in every case, for $n+1 \leq i \leq 2n$, either vertex i or vertex $i-n$ has even degree, so $p(G(B^*)) \leq n$.

In case (2a), we have $\epsilon(G(B^*)) + p(G(B^*)) \equiv 0 \pmod 3$; therefore, $P(G(B^*)) = p(G(B^*)) = 0 < n$. In case (2b): if $p(G(B)) = 0$, then $P(G(B^*)) \leq 4 \leq n$ if $n \geq 5$; and if, $n = 3$ and $p(G(B)) = 0$ then $\epsilon(G(B)) \equiv 0 \pmod 3$, so $P(G(B^*)) = 0 < n$. It is easily verified in all remaining cases that $\epsilon(G(B^*)) + p(G(B^*)) \equiv 0 \pmod 3$, and since $p(G(B^*)) \leq n$, $P(G(B^*)) \leq n$, so (ii) is satisfied.

Consider condition (iii a). We can calculate that $\epsilon(G(B^*))+p(G(B^*)) \leq 2\lfloor n^2/4 \rfloor + n + 2$ and show that if $n \neq 3, 4$, or 6 then $\epsilon(G(B^*))+p(G(B^*)) \leq 3T(n+1,2)$. Since we are assuming that n is odd, we need only consider the case when $n = 3$. We first notice that $\epsilon(G(B^*)) + P(G(B^*)) \leq 2\lfloor n^2/4 \rfloor + 4 \leq 8$; however, since $\epsilon(G(B^*)) + P(G(B^*)) \equiv 0 \pmod 3$, we have that $\epsilon(G(B^*)) + P(G(B^*)) \leq 6 = T(4,2)$. Hence, $\epsilon(G(B^*)) + P(G(B^*)) \leq T(n+1,2)$.

Next we consider condition (iii b). We have that $3\mu(n+1, \lambda - \lceil \lambda/n \rceil) \geq (\lambda(n^2-1) - n^2 - 10)/2$. In (2a), $n = 3$ and $p(G(B^*)) = 0$. Since $\epsilon(G(B^*)) + p(G(B^*)) \equiv 0 \pmod 3$, it follows that $\epsilon(G(B^*)) + P(G(B^*)) \leq \lambda\lfloor n^2/4 \rfloor + 4 = 2\lambda + 4$. Here, we consider $3\mu(4, \lambda - \lceil \lambda/3 \rceil) \geq 4\lambda - 6$, which is greater than $\epsilon(G(B^*)) + P(G(B^*))$ when $\lambda \geq 5$. When $\lambda = 4$, $3\mu(4,2) = 12 \geq \epsilon(G(B^*)) + P(G(B^*))$, so (iii b) is satisfied for case (2a) for all even $\lambda \geq 4$.

We now consider (2b). In this case, $\epsilon(G(B^*))+P(G(B^*)) \leq \lambda\lfloor n^2/4 \rfloor + n$. We can calculate that $3\mu(n+1, \lambda - \lceil \lambda/n \rceil) \geq \epsilon(G(B^*)) + P(G(B^*))$ for all even $\lambda \geq 4$ and all odd $n \geq 3$ except possibly for the special cases when $n = 3$ and $\lambda = 4$ or 6. If $n = 3$ and $\lambda = 4$, then $3\mu(4,2) = 12 > 11 \geq \epsilon(G(B^*)) + P(G(B^*))$; if $n = 3$ and $\lambda = 6$, then $3\mu(4,4) = 24 > 15 \geq \epsilon(G(B^*)) + P(G(B^*))$, so (iii b) is satisfied for (2b).

Next we consider (2c). In this case, $\epsilon(G(B^*)) + P(G(B^*)) \leq \lambda\lfloor n^2/4 \rfloor + n + 2 \leq (\lambda(n^2-1) - n^2 - 10)/2 \leq 3\mu(n+1, \lambda - \lceil \lambda/n \rceil)$ when $\lambda \geq 3$ and $n \geq 3$ except possibly for the special cases when $n = 3$ and $\lambda = 4, 6$, or 8 and when $n = 5$ and $\lambda = 4$. If $n = 3$ and $\lambda = 4$, then $\epsilon(G(B^*)) + P(G(B^*)) \leq 12$ (the calculation actually gives an upper bound of 13, but $\epsilon(G(B^*))+P(G(B^*)) \equiv 0 \pmod 3$). Now $3\mu(4,2) = 12 \geq \epsilon(G(B^*))+P(G(B^*))$. If $n = 3$ and $\lambda = 6$, then $\epsilon(G(B^*)) + P(G(B^*)) \leq 17 < 24 = 3\mu(4,4)$. If $n = 3$ and $\lambda = 8$, then $\epsilon(G(B^*)) + P(G(B^*)) \leq 21 < 27 = 3\mu(4,5)$. If $n = 5$ and $\lambda = 4$, then $\epsilon(G(B^*)) + P(G(B^*)) \leq 31 < 42 = 3\mu(6,3)$, so (iii b) is satisfied for all odd $n \geq 3$ and all even $\lambda \geq 4$ in (2c).

We finally consider case (2d). We have that $\epsilon(G(B^*)) + P(G(B^*)) \leq \lambda\lfloor n^2/4 \rfloor + n + 1 \leq 3\mu(n+1, \lambda - \lceil \lambda/n \rceil)$ when $n \geq 3$ and $\lambda \geq 3$ except possibly for the following special cases: $n = 3$ and $\lambda = 4$ or 6; and $n = 5$ and $\lambda = 4$. If $n = 3$ and $\lambda = 4$, then $\epsilon(G(B^*)) + P(G(B^*)) \leq 12 = 3\mu(4,2)$, and if $n = 3$ and $\lambda = 6$, then $\epsilon(G(B^*)) + P(G(B^*)) \leq 16 < 24 = 3\mu(4,4)$. If $n = 5$ and $\lambda = 4$, then $\epsilon(G(B^*)) + P(G(B^*)) \leq 31 < 42 = 3\mu(6,3)$, so (iii b) is satisfied for all odd $n \geq 3$ and all even $\lambda \geq 4$ in (2d) and in all of the above cases.

Clearly, the construction gives a $PETS(2n, \lambda)$ satisfying (iv).

Case 3: n is even, λ is odd.

Define B^* as follows.

(1) $B \subseteq B^*$.

(2a) If $p(G(B)) = 0$ or if $\epsilon(G(B)) + p(G(B)) \equiv 0 \pmod 3$, then B^* contains λ copies of the lollipops $\{n+i, n+i, n\}$, for $1 \leq i \leq n$.

(2b) If $p(G(B)) \neq 0$ and if $\epsilon(G(B)) + p(G(B)) \equiv 1 \pmod 3$, then B^* contains λ copies of the lollipops $\{n+i, n+i, n\}$, for $1 \leq i \leq n-1$, $\lambda - 2$ copies of the lollipop $\{2n, 2n, n\}$, and the remaining loops at vertex $2n$.

(2c) If $p(G(B)) \neq 0$ and if $\epsilon(G(B)) + p(G(B)) \equiv 2 \pmod 3$, then B^* contains λ copies of the lollipops $\{n+i, n+i, n\}$ for $1 \leq i \leq n-1$, $\lambda - 1$ copies of the lollipop $\{2n, 2n, n\}$, and the remaining loop at vertex $2n$.

(3) Using Lemma 2.3, partition the edges of K_n defined on the vertex set $\{n+1, \ldots, 2n\}$ into the 1-factors $F_1, \ldots, F_{n-1}$. For each edge $\{a, b\} \in F_v$, for $1 \leq v \leq n-1$, let B^* contain λ copies of the triple $\{v, a, b\}$.

We first consider condition (i). We have that $\Delta(G(B)) \leq \lambda(n-1)$, $d_{G(B^*)}(j) = d_{G(B)}(j) \leq \lambda(n-1)$, for $1 \leq j \leq n-1$, and $d_{G(B^*)}(n) \leq d_{G(B)}(n) + 2 < \lambda n$. Clearly, $d_{G(B^*)}(j) \leq 2$, for $n+1 \leq j \leq 2n$, so (i) is satisfied.

We now consider condition (ii). Clearly, $p(G(B)) \leq n$. In any case of the above construction, $d_{G(B^*)}(j) = d_{G(B)}(j)$, for $1 \leq j \leq n-1$, and $d_{G(B^*)}(k) = 0$, for $n+1 \leq k \leq 2n-1$. In cases (2a) and (2b), if $d_{G(B)}(n)$ is even (odd), then $d_{G(B^*)}(n)$ is even (odd), and $d_{G(B^*)}(2n)$ is even. In case (2c), we can assume without loss of generality that $d_{G(B)}(n)$

is odd, so $d_{G(B^*)}(n)$ is even; in addition, $d_{G(B^*)}(2n) = 1$. Therefore, in any case, $p(G(B)) = p(G(B^*))$. In case (2a), if $p(G(B)) = 0$, then clearly $P(G(B^*)) \leq n$, and if $\epsilon(G(B)) + p(G(B)) \equiv 0 \pmod 3$, then $\epsilon(G(B^*)) + p(G(B^*)) \equiv 0 \pmod 3$ since $\epsilon(G(B^*)) = \epsilon(G(B))$ and $p(G(B^*)) = p(G(B))$. Therefore, $P(G(B^*)) = p(G(B^*)) = p(G(B)) \leq n$. In cases (2b) and (2c) we clearly have that $\epsilon(G(B^*)) + p(G(B^*)) \equiv 0 \pmod 3$, so $P(G(B^*)) = p(G(B^*)) = p(G(B)) \leq n$, so (ii) is satisfied in all cases.

Next we observe condition (iii b), as condition (iii a) does not apply. We have that $\epsilon(G(B^*)) \leq \lambda\lfloor n^2/4\rfloor + 2$, and $P(G(B^*)) \leq n$, so $\epsilon(G(B^*)) + P(G(B^*)) \leq \lambda\lfloor n^2/4\rfloor + n + 2$. We can calculate that $3\mu(n+1, \lambda - \lceil\lambda/(n-1)\rceil) \geq (\lambda n^2 - \lambda n - 2\lambda - n^2 - 10)/2$. We have also that $\lambda\lfloor n^2/4\rfloor + n + 2 \leq (\lambda n^2 - \lambda n - 2\lambda - n^2 - 10)/2$ when $n \geq 4$ and $\lambda \geq 3$ except possibly for the following special cases: $n = 4$ and $\lambda < 17$; $n = 6$ and $\lambda < 7$; and, $n = 8$, 10, or 12 and $\lambda = 3$. By direct calculations of $\lambda\lfloor n^2/4\rfloor + n + 2$ and $3\mu(n+1, \lambda - \lceil\lambda/(n-1)\rceil)$, we have $3\mu(n+1, \lambda - \lceil\lambda/(n-1)\rceil) \geq \lambda\lfloor n^2/4\rfloor + n + 2$ for all of the above special cases, so (iii b) is satisfied for all cases considered.

Clearly, the construction gives at least two vertices of degree at most $\lambda n - 2$, so (i) - (iv) are satisfied.

Case 4: n is even, λ is even.

Define B^* as follows.

(1) $B \subseteq B^*$.

(2a) If $p(G(B)) = 0$ or if $\epsilon(G(B)) + p(G(B)) \equiv 0 \pmod 3$, then B^* contains λ copies of the lollipops $\{n+i, n+i, n\}$, for $1 \leq i \leq n$.

(2b) If $p(G(B)) \neq 0$ and if $\epsilon(G(B)) + p(G(B)) \equiv 1 \pmod 3$, then B^* contains λ copies of the lollipops $\{n+i, n+i, n\}$, for $1 \leq i \leq n-1$, $\lambda - 2$ copies of the lollipop $\{2n, 2n, n\}$, and the remaining loops at vertex $2n$.

(2c) If $p(G(B)) \neq 0$ and if $\epsilon(G(B)) + p(G(B)) \equiv 2 \pmod 3$, then B^* contains λ copies of the lollipops $\{n+i, n+i, n\}$, for $1 \leq i \leq n-1$, $\lambda - 1$ copies of the lollipop $\{2n, 2n, n\}$, and the remaining loop at vertex $2n$.

(3) Using Lemma 2.3, partition K_n defined on the vertex set $\{n+1, \ldots, 2n\}$ into the 1-factors $F_1, \ldots, F_{n-1}$. For each edge $\{a, b\} \in F_v$, for $1 \leq v \leq n-1$, let B^* contain λ copies of $\{v, a, b\}$.

We consider condition (i). We have that $\Delta(G(B)) \leq \lambda(n-1)$ and $d_{G(B^*)}(j) = d_{G(B)}(j)$, for $1 \leq j \leq n-1$. Furthermore, we have that $d_{G(B^*)}(j) = 0$, for $n+1 \leq j \leq 2n-1$. In cases (2a) and (2c), we have also that $d_{G(B^*)}(n) \leq d_{G(B)}(n) + 1 \leq \Delta(G(B)) + 1 < \lambda n$. Finally, in case (2b) we have that $d_{G(B^*)}(n) = d_{G(B)}(n) + 2$. However, since $d_{G(B)}(n)$ is odd without loss of generality, $d_{G(B)}(n) < \lambda(n-1)$ so $d_{G(B^*)}(n) < \lambda n$, so (i) is satisfied in all cases.

Next we investigate condition (ii). In all cases, $d_{G(B^*)}(i) = 0$ for $n+1 \leq i \leq 2n$, except that $d_{G(B^*)}(2n) = 2$ or 1 in cases (2b) and (2c), respectively. However, if $d_{G(B^*)}(2n) = 1$ then $d_{G(B)}(n)$ is odd, so $d_{G(B^*)}(n)$ is even. Therefore, in every case, for $n+1 \leq i \leq 2n$, either vertex i or vertex $i-n$ has even degree, so $p(G(B^*)) \leq n$. Furthermore, in every case it is easily verified that $p(G(B^*)) = p(G(B))$ and $\epsilon(G(B^*)) + p(G(B^*)) \equiv 0 \pmod 3$. Therefore, $P(G(B^*)) = p(G(B^*)) \leq n$, so (ii) is satisfied.

We now investigate condition (iii a). We can calculate that $\epsilon(G(B^*)) + P(G(B^*)) \leq 2\lfloor n^2/4 \rfloor + n + 2$ and show that if $n \neq 3, 4,$ or 6 then $\epsilon(G(B^*)) + P(G(B^*)) \leq 3T(n+1, 2)$. Since we are assuming that n is even, we need only consider the cases when $n = 4$ and $n = 6$. We first notice that although $2\lfloor n^2/4 \rfloor + n + 2 = 14$ when $n = 4$, $\epsilon(G(B^*)) + P(G(B^*)) \leq 12$, since $\epsilon(G(B^*)) + P(G(B^*)) \equiv 0 \pmod 3$. Since we defined $T(5,2) = 4$, we have that $3T(5,2) = 12$, so $\epsilon(G(B^*)) + P(G(B^*)) \leq T(n+1, 2)$. In addition, $2\lfloor n^2/4 \rfloor + n + 2 = 26 < 30 = 3T(7,2)$, so $\epsilon(G(B^*)) + P(G(B^*)) \leq T(n+1, 2)$.

Next we consider condition (iii b). We have in all cases that $\epsilon(G(B^*)) \leq \lambda\lfloor n^2/4 \rfloor + 2$ and $P(G(B^*)) \leq n$, so $\epsilon(G(B^*)) + P(G(B^*)) \leq \lambda\lfloor n^2/4 \rfloor + n + 2$. Again we can calculate that $3\mu(n+1, \lambda - \lceil \lambda/(n-1) \rceil) \geq (\lambda n^2 - \lambda n - 2\lambda - n^2 - 10)/2$, so we have that $\lambda\lfloor n^2/4 \rfloor + n + 2 \leq (\lambda n^2 - \lambda n - 2\lambda - n^2 - 10)/2$ when $n \geq 4$ and $\lambda \geq 3$, except possibly for the following cases: $n = 4$ and $\lambda < 19$; $n = 6$ and $\lambda < 7$; and $n = 8$ and $\lambda = 4$. By direct calculations of $\lambda\lfloor n^2/4 \rfloor + n + 2$ and $3\mu(n+1, \lambda - \lceil \lambda/(n-1) \rceil)$, we have that $3\mu(n+1, \lambda - \lceil \lambda/(n-1) \rceil) \geq \lambda\lfloor n^2/4 \rfloor + n + 2$ unless $n = 4$ and $\lambda = 4$. If $n = 4$ and $\lambda = 4$, we have that $\lambda\lfloor n^2/4 \rfloor + n + 2 = 22$ and $3\mu(n+1, \lambda - \lceil \lambda/(n-1) \rceil) = 3\mu(5,2) = 18$. However, if $\epsilon(G(B)) = \lambda\lfloor n^2/4 \rfloor \leq 14$, we will satisfy condition (iii b) since $\epsilon(G(B^*)) + P(G(B^*)) \equiv 0 \pmod 3$ and since in this case $\epsilon(G(B^*)) + P(G(B^*)) \leq \lambda\lfloor n^2/4 \rfloor + n + 2 \leq 20$ (in fact $\epsilon(G(B^*)) + P(G(B^*)) \leq 18 = 3\mu(5,2)$). If $\epsilon(G(B)) = 15$, we can assume without loss of generality that $G(B)$ contains 2 vertices of odd degree, so $\epsilon(G(B)) + p(G(B)) = 17 \equiv 2 \pmod 3$. Hence, we are in case (2c) since $p(G(B)) \neq 0$, and this means that $\epsilon(G(B^*)) + P(G(B^*)) = 18 = 3\mu(5,2)$. If $\epsilon(G(B)) = 16$, we can assume that all four vertices of $G(B)$ have even degree, so we are in case (2a). Now $\epsilon(G(B^*)) + p(G(B^*)) = 16$, so $P(G(B^*)) = 2$; therefore, $\epsilon(G(B^*)) + P(G(B^*)) = 18 = 3\mu(5,2)$, so (iii b) is satisfied when $n = 4$ and $\lambda = 4$, and thus for all cases considered.

Clearly, the construction satisfies condition (iv), so the proof is complete. □

Proposition 3.2 *Let $n \geq 3$ and $\lambda \geq 2$. Any $PETS(2n, \lambda)(V^*, B^*)$ satisfying:*

(i) $\Delta(G(B^*)) < \lambda n$,

(ii) $P(G(B^*)) \leq n$,

(iii a) *If* $\lambda = 2$, *then* $\epsilon(G(B^*)) + P(G(B^*)) \leq 3T(n+1,2)$

(iii b) *If* $\lambda > 2$ *then*

$$\epsilon(G(B^*)) + P(G(B^*)) \leq \begin{cases} 3\mu(n+1, \lambda - \lceil \lambda/n \rceil) & \text{if } n \text{ is odd} \\ 3\mu(n+1, \lambda - \lceil \lambda/(n-1) \rceil) & \text{if } n \text{ is even} \end{cases}$$

(iv) *and* $G(B^*)$ *contains at least 2 vertices of degree at most* $\lambda n - 2$

can be embedded in an $ETS(4n+2, \lambda)(\hat{V}, \hat{B})$.

Proof: Clearly we can assume that (V^*, B^*) is maximal (if necessary, by adding loops and triples, but not lollipops). There are five types of extended triples that will be added to embed $(V^* = \{1, \ldots, 2n\}, B^*)$ in $(\hat{V} = \{1, \ldots, 4n+2\}, \hat{B})$:

(a) lollipops $\{a, a, b\}$, where $a \geq 2n+1, b \leq 2n$;

(b) lollipops and loops on vertices in $\{2n+1, \ldots, 4n+2\}$;

(c) triples $\{a, b, c\}$, where $a, b \leq 2n$, and $c \geq 2n+1$;

(d) triples $\{a, b, c\}$, for $2n+1 \leq a, b, c \leq 4n+2$; and

(e) triples $\{a, b, c\}$, for $a \leq 2n$ and $b, c \geq 2n+1$.

We start by letting $B^* \subseteq \hat{B}$ and then consider each type of extended triple to be placed in $\hat{B}$ in turn.

Type a: Since $|\hat{V}|$ is even, each vertex in $\hat{V}$ must occur together in $\hat{B}$ in an odd number of lollipops if λ is odd and an even number of lollipops if λ is even. We use lollipops to adjust the $p = p(G(B^*))$ vertices in V^* which occur in an even number of lollipops if λ is odd and an odd number of lollipops if λ is even (these are the vertices in $G(B^*)$ which have odd degree). We can assume $\{1, \ldots, p\} \subseteq \{1, \ldots, n\}$ is this set of vertices, and let $\{\{i, 2n+i, 2n+i\} | 1 \leq i \leq p\} \subseteq \hat{B}$.

We must also have that the number of edges remaining to be placed in triples after the Type b extended triples are defined is divisible by 3, so we may need to add up to four further lollipops as follows. Let $\phi \in \mathbb{Z}_3$ with $\phi \equiv \epsilon(G(B^*)) + p(G(B^*)) \pmod 3$. By condition (iv), there are at least ϕ vertices of degree at most $\lambda n - 2$ in $G(B^*)$, which we can name v_i, for $1 \leq i \leq \phi$. If $\phi \geq 1$, we let $\{\{v_i, 2n+p+2i-1, 2n+p+2i-1\}, \{v_i, 2n+p+2i, 2n+p+2i\} | 1 \leq i \leq \phi\} \subseteq \hat{B}$. We have therefore defined exactly $p + 2\phi = P(G(B^*)) \leq n$ Type a extended triples.

Type b: For each $v \in \hat{V} \setminus V^* = \{2n+1, \ldots, 4n+2\}$, in order that v occurs in λ lollipops, we must have that v occurs in λ Type a or Type b extended triples. Since v occurs in exactly one Type a lollipop for $2n+1 \leq v \leq 2n+p+2\phi$, or no Type a lollipops for $v \geq 2n+p+2\phi+1$, we require that v must occur in either $\lambda-1$ or λ Type b extended triples, respectively.

First suppose $\lambda = 2$. By Lemma 2.12 there exists an equitable $PTS(n+1,2)(S = \{2n+1,\ldots,3n+1\},T)$ containing $(\epsilon(G(B^*)) + P(G(B^*)))/3$ triples such that the leave of (S,T) contains a 2-factor consisting of even cycles if $n+1$ is even and a near 2-factor consisting of even cycles if $n+1$ is odd. First suppose $n+1$ is even. Give the 2-factor an orientation forming directed cycles; we can name the vertices in S so that each directed cycle consists of arcs of the form $(x, n(x))$ where $n(x) = x+1$ for each vertex in the cycle except for the largest vertex. For every $a \in \{2n+1,\ldots,3n+1\}\setminus\{2n+1, 2n+3, 2n+5,\ldots,2n+p+2\phi-1\}$, include the lollipop $\{a,a,n(a)\}$. Because we have specified that the 2-factor consists of even cycles, each vertex in $\{2n+1,\ldots,3n+1\}$ is contained in the required number of lollipops. Next, let $\hat{B}$ contain two lollipops of the form $\{3n+2i, 3n+2i, 3n+2i+1\}$, for $1 \leq i \leq (n+1)/2$. Finally, let $\hat{B}$ contain any remaining loops which are not in lollipops.

Next let $n+1$ be odd. Let the near 2-factor saturate all vertices of S except $3n+1$. Form lollipops as before and add them to $\hat{B}$ so that vertices in the set $\{2n+1,\ldots,2n+p+2\phi\}$ occur in exactly one Type b lollipop and all remaining vertices in $S\setminus\{3n+1\}$ are contained in two Type b lollipops. Again, this is possible since the near 2-factor consists of even cycles. Next, let $\hat{B}$ contain two lollipops of the form $\{3n+2i-1, 3n+2i-1, 3n+2i\}$, for $1 \leq i \leq (n+2)/2$. Finally, let $\hat{B}$ contain any remaining loops that are not in lollipops.

Now suppose $\lambda \geq 3$ and n is even. Consider K_n defined on the vertex set $\{2n+1,\ldots,3n\}$. By Lemma 2.3, we can partition the edges of K_n into the 1-factors $F_1,\ldots,F_{n-1}$ such that F_{n-1} contains the edges $\{2n+1,2n+2\},\{2n+3,2n+4\},\ldots,\{3n-1,3n\}$. For $1 \leq v \leq n-1$, orient the edges of F_v to form F'_v, and for each arc $(a,b) \in F'_v$, let $\hat{B}$ contain $\lceil\lambda/(n-1)\rceil - 1$ copies of the lollipop $\{a,a,b\}$. Now every vertex in the set $\{2n+1,\ldots,3n\}$ is contained in $(n-1)(\lceil\lambda/(n-1)\rceil-1)$ Type b lollipops. Additionally, for $1 \leq v \leq (\lambda-1)-(n-1)(\lceil\lambda/(n-1)\rceil-1)$, and for every arc $(a,b) \in F'_v$, let $\hat{B}$ contain one more copy of $\{a,a,b\}$; so now every vertex in $\{2n+1,\ldots,3n\}$ is contained in $\lambda-1$ Type b lollipops. Finally, for $1 \leq i \leq (n-p-2\phi)/2$ and for every arc $(2n+p+2\phi+2i-1, 2n+p+2\phi+2i) \in F'_{n-1}$, let $\{2n+p+2\phi+2i-1, 2n+p+2\phi+2i-1, 2n+p+2\phi+2i\} \in \hat{B}$. Now every vertex in $\{2n+1,\ldots,3n\}$ is contained in the required number of Type b lollipops. Place in $\hat{B}$ all remaining loops which are incident with vertices in $\{2n+1,\ldots,3n\}$. Finally, let $\hat{B}$ contain λ copies of the lollipops $\{3n+2i-1, 3n+2i-1, 3n+2i\}$ and λ copies of the loops at each vertex $3n+2i$, for $1 \leq i \leq (n+2)/2$.

Next suppose $\lambda \geq 3$ and n is odd. Again consider K_{n+1} on the vertex set $\{2n+1, \ldots, 3n+1\}$. Using Lemma 2.3, partition the edges of K_{n+1} into the 1-factors $F_1, \ldots, F_n$ such that F_n contains the edges $\{2n+1, 2n+2\}, \{2n+3, 2n+4\}, \ldots, \{3n, 3n+1\}$. Orient the edges of F_v to form F'_v, for $1 \leq v \leq n$, and for each arc $(a, b) \in F'_v$, let $\hat{B}$ contain $\lceil \lambda/n \rceil - 1$ copies of $\{a, a, b\}$. Now every vertex in the set $\{2n+1, \ldots, 3n+1\}$ is in $n(\lceil \lambda/n \rceil - 1)$ Type b lollipops. Again, for $1 \leq v \leq (\lambda-1) - n(\lceil \lambda/n \rceil - 1)$ and for each arc $(a, b) \in F'_v$, let $\hat{B}$ contain one more copy of $\{a, a, b\}$. Now every vertex in $\{2n+1, \ldots, 3n+1\}$ is contained in $\lambda-1$ Type b lollipops. Finally, for $1 \leq i \leq (n-p-2\phi+1)/2$ and for every arc $(2n+p+2\phi+2i-1, 2n+p+2\phi+2i) \in F'_n$, let $\{2n+p+2\phi+2i-1, 2n+p+2\phi+2i-1, 2n+p+2\phi+2i\} \in \hat{B}$. Therefore, we have that every vertex in $\{2n+1, \ldots, 3n+1\}$ is contained in the required number of Type b lollipops. Now we place in $\hat{B}$ all remaining loops which are incident with vertices in $\{2n+1, \ldots, 3n+1\}$. Furthermore, we let $\hat{B}$ contain λ copies of the lollipops $\{3n+2i, 3n+2i, 3n+2i+1\}$ and λ loops at each vertex of the form $3n+2i+1$, for $1 \leq i \leq (n+1)/2$.

Type c: We form a graph H which consists of the edges of $G(B^*)$ along with the edges (but not the loops) $\{x_i, 2n+\ell\}$, where $i \in \{1, \ldots, p+\phi\}$ and $\ell \in \{1, \ldots, p+2\phi\}$, which occur in Type a lollipops, so $\epsilon(H) = \epsilon(G(B^*)) + P(G(B^*))$. Our goal is to give H an equalized edge-coloring with $n+1$ colors, say $2n+1, \ldots, 3n+1$, where: $|C_{2n+1}| \geq \cdots \geq |C_{3n+1}|$; $|C_i(v)| \leq \lambda$ for $2n+1 \leq i \leq 3n+1$; and for all $v \in V(H)$, and the Type a lollipop edges all receive different colors, say $\{x_i, 2n+\ell\}$ is colored with $2n+\ell$ for $1 \leq \ell \leq p+2\phi$. We do this by forming a graph H' from H by contracting vertices $2n+1, \ldots, 2n+p+2\phi$ into a single vertex γ. We have that $d_{H'}(i) \leq \Delta(G(B^*)) + 1 \leq \lambda n$ for $1 \leq i \leq 2n$, by (i) and (iv), and $d_{H'}(\gamma) \leq p(G(B^*)) + 2\phi = P(G(B^*)) \leq n$ by (ii), so H' can be given a proper equalized $\lambda(n+1)$-edge-coloring by Lemma 2.4. Name the color classes $C_{i,j}$, where $1 \leq i \leq \lambda$ and $2n+1 \leq j \leq 3n+1$, and name the colors so that: $|C_{i,j}| \geq |C_{k,\ell}|$ if and only if (i,j) is lexicographically less than (k, ℓ); and if e_1 and e_2 are any two edges incident with γ in color classes $C_{i,j}$ and $C_{k,\ell}$, respectively, then $j \neq \ell$. Then letting $C_{n+j} = \cup_{i=1}^{\lambda} C_{i,j}$ for $2n+1 \leq j \leq 3n+1$ be the color classes of an equalized edge-coloring of H' produces the desired $(n+1)$-edge-coloring of H. For each edge $\{i, j\}$ in $G(B^*)$ colored k, let $\{i, j, k\} \in \hat{B}$.

Type d: Again consider the edge-coloring of H just obtained and let δ_x denote the number of edges of H colored x. First assume $\lambda = 2$. Let $(S' = \{2n+1, \ldots, 3n+1\}, T')$ be an equitable $PSTS(n+1, 2)$, and let $|T'| = (\epsilon(G(B^*)) + P(G(B^*)))/3$; by the definition of $P(G(B^*))$, this number is an integer. By Lemma 2.12, such a $PSTS$ exists, since by condition (iii a), $(\epsilon(G(B^*)) + P(G(B^*)))/3 \leq T(n+1, 2)$. Name the symbols so that $\delta_x = r(x)$, the number of triples in T' which contain symbol x, for $2n+1 \leq x \leq 3n+1$. We have that no edge of the form $\{a, b\} \subseteq S$ will occur in too many extended triples up to this point since the leave of our $PSTS$ contains a 2-factor from which the Type b lollipops were obtained. Let $T' \subseteq \hat{B}$.

Next assume $\lambda \geq 3$. Let $(S = \{2n+1, \ldots, 3n+1\}, T')$ be an equitable $PSTS(n+1, \lambda - \lceil \lambda/n \rceil)$ if n is odd, and let (S, T') be a $PSTS(n+1, \lambda - \lceil \lambda/(n-1) \rceil)$ if n is even. Furthermore, let $|T'| = (\epsilon(G(B^*)) + P(G(B^*)))/3$; again, by the definition of $P(G(B^*))$, this number is an integer. Name the symbols so that $\delta_x = r(x)$, the number of triples in T' which contain symbol x, for $2n+1 \leq x \leq 3n+1$. By Lemma 2.9, such a $PSTS$ exists, since by condition (iii b)

$$\epsilon(G(B^*)) + P(G(B^*)) \leq \begin{cases} 3\mu(n+1, \lambda - \lceil \lambda/n \rceil) & \text{if } n \text{ is odd} \\ 3\mu(n+1, \lambda - \lceil \lambda/(n-1) \rceil) & \text{if } n \text{ is even.} \end{cases}$$

In addition, since any edge of the form $\{a, b\} \subseteq S$, is in at most $\lceil \lambda/n \rceil$ Type b lollipops if n is odd and at most $\lceil \lambda/(n-1) \rceil$ Type b lollipops if n is even, no such edge will be in too many extended triples up to this point. Let $T' \subseteq \hat{B}$.

Type e: We use Theorem 2.6 and Theorem 2.7 to place the remaining edges in triples. We first form a partial array $L(\infty)$ of order $n+1$ and multiplicity λ on the symbols in $\{\infty\} \cup \{1, \ldots, 2n\}$ as follows:

(1) place symbol $j \leq 2n$ in cell (i, i) if an edge colored $2n+i$ is incident with vertex j in H,

(2) for $1 \leq i < j \leq n+1$, if $\{i+2n, j+2n\}$ is an edge of k_1 triples in T' and a lollipop edge of k_2 Type b lollipops, then fill cells (i, j) and (j, i) with symbol ∞ $k_1 + k_2$ times, and

(3) for $1 \leq i < j \leq n+1$, fill cells (i, j) and (j, i) greedily with $\lambda - (k_1 + k_2)$ symbols in $\{1, \ldots, 2n\}$, preserving symmetry and preserving the property that each symbol occurs at most λ times in each row and at most λ times in each column, and if $n+1$ is odd then at least n symbols occur at least 2λ times in $L(\infty)$.

To see that property (3) can be achieved, we present the following argument.

If $n+1$ is even, this can be done greedily, so suppose $n+1$ is odd. In this case, except when $n = 4$, we show that property (3) can be achieved by ensuring that each symbol in $B = \{n+1, \ldots, 2n\}$ can be placed 2λ times in $L(\infty)$, and in fact this can be done greedily; the case $n = 4$ is relegated to the appendix. For each off-diagonal cell c in $L(\infty)$ let the number of *spaces* in c be the difference between λ and the number of copies of ∞ in c. There are two things to show.

First, we address the greedy issue. Since each symbol in B occurs at most once in each diagonal cell and in an even number of diagonal cells, and since we have the symmetry property, if symbol $j \in B$ occurs less than 2λ times, then it appears less than λ times in every row and in every column; so it can be placed in any off-diagonal cell containing an unfilled space.

Second, we must show that the number of spaces is at least $2n\lambda$. Each of rows $1, \ldots, n$ in L are to contain $\lambda(n-1)$ symbols, and row $n+1$ is to contain λn symbols, so the total number of symbols to occur in L is $\lambda(n-1)n + \lambda n = \lambda n^2$. Of these, $2\epsilon(G(B^*)) + P(G(B^*))$ occur on the diagonal, so the total number of spaces is $\lambda n^2 - 2\epsilon(G(B^*)) + P(G(B^*)) \leq \lambda n^2/2 + n + 4$. For $n \geq 6$, $n^2 - 5n - 4 \geq 0$, so since $\lambda \geq 2$ we have $n^2 - 4n - ((2n+8)/\lambda) \geq 0$, so $\lambda n^2 - (2\epsilon(G(B^*)) + P(G(B^*))) \geq 2n\lambda$ as required. Therefore, property (3) can be achieved.

Let L be formed from $L(\infty)$ by deleting all the ∞ symbols. Then L is a symmetric quasi-latin square of order $n+1$ and multiplicity λ on the symbols in $\{1, \ldots, 2n\}$. In order to apply Theorem 2.6 and Theorem 2.7 (with $r = n+1$ and $t = 2n$), we need to check that L satisfies the appropriate conditions. We first show that every symbol occurs an even number of times on the main diagonal of L, and thus, an even number of times in L, thus satisfying condition (a) of Theorems 2.6 and 2.7. We consider the cases where $d_{G(B^*)}(x)$ is odd and even in turn.

Suppose $d_{G(B^*)}(x) = 2m+1$. Since x has odd degree in $G(B^*)$, the graph H contains a lollipop edge of Type a incident with x colored with some color α. Furthermore, x is contained in $2m+1$ triples of the form $\{x, y, \beta\}$, where β is the color of an edge $\{x, y\}$ in H. Hence, symbol x is placed once in cell $(\alpha - 2n, \alpha - 2n)$ and $2m+1$ times in cells of the form $(\beta - 2n, \beta - 2n)$, and so it is placed $2m+2$ times on the main diagonal of L.

Now suppose $d_{G(B^*)}(x) = 2m$. In this case, vertex x has even degree in $G(B^*)$, so there are either 0 or 2 Type a lollipop edges in H which contain x. If these lollipops exist, the edges were given distinct colors, say α_1 and α_2. Furthermore, x occurs in $2m$ or $2m-2$ triples of the form $\{x, y, \beta\}$ (depending upon whether x occurs in 0 or 2 lollipops, respectively), where β is the color of an edge $\{x, y\}$ in H. Hence, x is placed once in each of the distinct cells $(\alpha_1 - 2n, \alpha_1 - 2n)$ and $(\alpha_2 - 2n, \alpha_2 - 2n)$ and $2m-2$ times in cells of the form $(\beta - 2n, \beta - 2n)$ if such lollipops exist, and x is placed $2m$ times in cells of the form $(\beta - 2n, \beta - 2n)$, otherwise. Therefore, x is placed $2m$ times on the main diagonal of L and, thus, an even number of times in L.

We now check that L contains the appropriate number of symbols in each row as required by Theorems 2.6 and 2.7, considering the cases $n+1$ is even or odd in turn.

Suppose $n+1$ is even. Since $r(2n+i)$ is the number of triples in T' containing symbol $2n+i$, from (1) the number of symbols in cell (i, i) is $2r(2n+i) - 1$ if symbol $2n+i$ occurs in a Type a lollipop and $2r(2n+i)$, otherwise. From (2), the number of times symbol ∞ occurs in row i of $L(\infty)$ is $2r(2n+i)+(\lambda-1)$ if vertex i is contained in a Type a lollipop and $2r(2n+i)+\lambda$, otherwise. Therefore, also from (1), (2), and (3), we get that $\lambda n + 2r(2n+i) - 1 - (2r(2n+i) + (\lambda - 1)) = \lambda n - \lambda = \lambda(n-1)$ symbols are placed in row i of L if symbol $2n+i$ is contained in a Type a lollipop; otherwise, $\lambda n + 2r(2n+i) - (2r(2n+i) + \lambda) = \lambda n - \lambda = \lambda(n-1)$ symbols are placed in row i of L, so each row of L contains $\lambda(n-1)$ symbols if $n+1$ is even.

Suppose $n+1$ is odd. The same argument as the case where $n+1$ is even holds for the first n rows of L, but in this case we must have that row $n+1$ of L contains λn symbols. However, since $n+1$ is odd, vertex $3n+1$ occurs in λ Type b lollipops with vertex $3n+2$, and these lollipops contribute no ∞ symbols to row $n+1$ of $L(\infty)$. However, as before, row $n+1$ of $L(\infty)$ contains symbol ∞ $2r(3n+1)$ times (since vertex $3n+1$ is contained in λ Type b lollipops) and cell $(n+1, n+1)$ of L contains $2r(3n+1)$ symbols. Therefore, row $n+1$ of L contains $\lambda n - 2r(3n+1) + 2r(3n+1) = \lambda n$ symbols.

We now have the following ((5) follows from the third condition imposed on $L(\infty)$ when forming the Type e triples):

(1) $N_L(i)$ is even, for $1 \le i \le 2n$;

(2) L is symmetric;

(3) $N_L(i) \ge \lambda(2(n+1) - 2n - 2) = 0$, for $1 \le i \le 2n$;

(4a) each row of L contains $\lambda(n-1)$ symbols from $\{1, \ldots, 2n\}$ if $n+1$ is even, or

(4b) row j $(1 \le j \le n)$ of L contains $\lambda(n-1)$ symbols, and row $n+1$ contains λn symbols if $n+1$ is odd; and

(5) if $n+1$ is odd then at most n symbols occur less than 2λ times in L.

Therefore, by Theorems 2.6 and 2.7, L can be embedded in the top left corner of a symmetric quasi-latin square L' of order $2n+2$ on the symbols $1, \ldots, 2n$ such that cells (i,i), for $n+2 \le i \le 2n+2$, and cells $(n-x+2i, n-x+2i+1)$ and $(n-x+2i+1, n-x+2i)$, for $1 \le i \le (n+1+x)/2$ are empty, where $x = 0$ or 1 if $n+1$ is even or odd, respectively. We form triples using L': if symbol w occurs in cells (y,z) and (z,y), $y \ne z$, then let $\{y+2n, z+2n, w\} \in \hat{B}$.

This completes the definition of $\hat{B}$, so now it remains to show that $(\hat{V}, \hat{B})$ is an $ETS(4n+2, \lambda)$; that is, we must show that every pair of points of $\hat{V}$ occurs in exactly λ extended triples.

Consider edges of the form $\{x, y\}$, where $x, y \leq 2n$. Some (maybe all) of these edges already occurred in extended triples in B^*. Suppose $\{x, y\}$ occurred in α extended triples in B^*. Then $\{x, y\}$ occurred in $G(B^*)$ as an edge $\lambda - \alpha$ times, and therefore was included in $\lambda - \alpha$ Type c triples. All loops incident with vertices x and y already occurred in B^* since (V^*, B^*) was taken to be maximal. Therefore, every edge of the form $\{x, y\}$ occurs in exactly λ extended triples in $\hat{B}$.

We now consider edges of the form $\{x, y\}$, where $x \leq 2n$ and $y \geq 2n+1$. Since every symbol $1, \ldots, 2n$ occurs exactly λ times in each row of L', $\{x, y\}$ occurs in extended triples of Type a or c if x is in a diagonal cell of L' and of Type e if x is in an off-diagonal cell of L'. In any event $\{x, y\}$ occurs in λ extended triples of $\hat{B}$.

We finally consider edges of the form $\{x, y\}$, where $x, y \geq 2n + 1$. Each cell $(x - 2n, y - 2n)$, $x \neq y$, in L' contains β symbols from $1, \ldots, 2n$, where $0 \leq \beta \leq \lambda$. For every symbol in $1, \ldots, 2n$ found in cell $(x - 2n, y - 2n)$, $\{x, y\}$ is in a Type e triple; the remaining $\lambda - \beta$ edges joining x to y are contained in either Type b lollipops or Type d triples (corresponding to the λ Type b lollipops if (x, y) is a near-diagonal cell of L' outside L, and to the $\lambda - \beta$ copies of ∞ in cell (x, y) of $L(\infty)$ otherwise). In any event, $\{x, y\}$ is contained in exactly λ extended triples, and all loops incident with vertices in $2n + 1, \ldots, 4n + 2$ are contained in λ extended triples of $\hat{B}$.

Therefore, we have that every edge of the form $\{x, y\}$, for $1 \leq x, y \leq 4n + 2$, is in exactly λ extended triples of $\hat{B}$, and the proof is complete. □

We now have the following theorem.

Theorem 3.3 *Any partial extended triple system (V, B) of order n and index $\lambda \geq 2$ can be embedded in an extended triple system $(\hat{V}, \hat{B})$ of order v and index λ for all $v \geq 4n + 2$, $v \equiv 2$ (mod 4).*

Proof: First suppose $n \in \{1, 2\}$. Clearly, any $PETS(1, \lambda)$ can be embedded in an $ETS(v, \lambda)$ for all $v \geq 4n + 2, v \equiv 2 \pmod 4$ since this corresponds to the existence of such $ETS(v, \lambda)$s. If $n = 2$, consider a $PETS(2, \lambda)$ with $\epsilon(G(B)) = \lambda - x$. We can think of this $PETS(2, \lambda)$ as $(\lambda - x)$ $PETS(2, 1)$s and x $ETS(2, 1)$s. Clearly, by [13] each of the $PETS(2, 1)$s and $ETS(2, 1)$s can be embedded in an $ETS(v, 1)$ for all $v \geq 4n + 2, v \equiv 2 \pmod 4$. If we put these λ $ETS(v, 1)$s together, we have the desired $ETS(v, \lambda)$.

Now suppose $n \geq 3, k \geq 0$, and $v = 4(n + k) + 2$. Embed (V, B) in a maximal $PETS(n + k, \lambda)(V_1, B_1)$. By Lemma 3.1, (V_1, B_1) can be embedded in a $PETS(2(n + k), \lambda)$ satisfying conditions (i) – (iv), which by Proposition 3.2 can be embedded in an $ETS(4(n + k) + 2, \lambda)$, and the proof is complete. □

4 Embedding a PETS(n, λ) in an ETS$(4n + 8, \lambda)$

We follow a similar approach when embedding a $PETS(n, \lambda)$ in an $ETS(4n+8, \lambda)$. In most cases we can embed our $PETS(n, \lambda)$ in an $ETS(4n + 4, \lambda)$.

Let $w(G(B))$ denote the number of vertices of even degree in $G(B)$ and let

$$W(G(B)) = \begin{cases} w(G(B)) & \text{if } \epsilon(G(B)) + w(G(B)) \equiv 0 \pmod 3, \\ w(G(B)) + 2 & \text{if } \epsilon(G(B)) + w(G(B)) \equiv 1 \pmod 3, \\ w(G(B)) + 4 & \text{if } \epsilon(G(B)) + w(G(B)) \equiv 2 \pmod 3. \end{cases}$$

As before, the embedding process takes two steps. Lemma 4.1 embeds the $PETS(n, \lambda)$ in a $PETS$ whose deficiency graph meets the conditions of Proposition 4.2. Applying Proposition 4.2 completes the embedding.

Lemma 4.1 *Let (V, B) be a maximal $PETS(n, \lambda)$ with $n \geq 3$ and $\lambda \geq 2$. Let $u = 2n + 1$ or $2n + 3$ if n is odd, and $u = 2n + 3$ or $2n + 5$ if n is even. Then (V, B) can be embedded in a $PETS(u, \lambda)$ (V^*, B^*) satisfying:*

(i)

$$\Delta(G(B^*)) < \begin{cases} \lambda n & \text{, if } n \text{ is odd} \\ \lambda(n+1) & \text{, if } n \text{ is even} \end{cases}$$

(ii a) If λ is even, then

$$P(G(B^*)) \leq \begin{cases} n & \text{, if } n \text{ is odd} \\ n+1 & \text{, if } n \text{ is even} \end{cases}$$

(ii b) If λ is odd, then

$$W(G(B^*)) \leq \begin{cases} n & \text{, if } n \text{ is odd} \\ n+1 & \text{, if } n \text{ is even} \end{cases}$$

(iii a) If $\lambda = 2$, then $\epsilon(G(B^*)) + P(G(B^*)) \leq 3T^*((u+1)/2, 2)$

(iii b) If $\lambda > 2$ and even, then

$$\epsilon(G(B^*)) + P(G(B^*)) \leq \begin{cases} 3\mu((u+1)/2, \lambda - \lceil \lambda/((u-1)/2) \rceil), \\ \quad \text{if } (u+1)/2 \text{ is odd} \\[1em] 3\mu((u+1)/2, \lambda - \lceil \lambda/((u-3)/2) \rceil), \\ \quad \text{if } (u+1)/2 \text{ is even,} \end{cases}$$

(iii c) If $\lambda > 2$ and odd, then

$$\epsilon(G(B^*)) + W(G(B^*)) \leq \begin{cases} 3\mu((u+1)/2, \lambda - \lceil \lambda/((u-1)/2) \rceil), \\ \quad \textit{if } (u+1)/2 \textit{ is odd} \\ \\ 3\mu((u+1)/2, \lambda - \lceil \lambda/((u-3)/2) \rceil), \\ \quad \textit{if } (u+1)/2 \textit{ is even,} \end{cases}$$

(iv) $G(B^*)$ contains at least 2 vertices of degree at most $\lambda n - 2$.

Proof:

Case 1: $u = 2n+1$, n is odd, λ is odd.

Since n is odd we have $w(G(B)) \neq 0$, and we can assume without loss of generality that vertex n has even degree.

Define B^* as follows.

(1) $B \subseteq B^*$.

(2) Using Lemma 2.3, partition the edges of K_{n+1} on the vertex set $\{n+1, \ldots, 2n+1\}$ into the 1-factors $F_1, \ldots, F_n$, where F_n consists of the edges $\{n+i, 2n+2-i\}$, for $1 \leq i \leq (n+1)/2$. For each edge $\{a, b\} \in F_v$, for $1 \leq v \leq n-1$, let B^* contain λ copies of the triple $\{v, a, b\}$, and for each edge $\{a, b\} \in F_n$, let B^* contain $\lambda - 1$ copies of the triple $\{n, a, b\}$.

(3a) If $\epsilon(G(B)) + w(G(B)) + (n+1)/2 \equiv 0 \pmod 3$, let B^* contain the lollipops $\{n+i, n+i, n\}$, for $1 \leq i \leq n+1$, and the remaining loops at vertices $n+1, \ldots, 2n+1$.

(3b) If $\epsilon(G(B)) + w(G(B)) + (n+1)/2 \equiv 1 \pmod 3$, let B^* contain the lollipops $\{n+i, n+i, n\}$, for $3 \leq i \leq n-1$, the lollipops $\{n+1, n+1, 2n+1\}$ and $\{n+2, n+2, 2n\}$, and the remaining loops at vertices $n+1, \ldots, 2n+1$.

(3c) If $\epsilon(G(B)) + w(G(B)) + (n+1)/2 \equiv 2 \pmod 3$, let B^* contain the lollipops $\{n+i, n+i, n\}$, for $2 \leq i \leq n$, the lollipop $\{n+1, n+1, 2n+1\}$, and the remaining loops at vertices $n+1, \ldots, 2n+1$.

We first consider condition (i). Clearly, $\Delta(G(B)) \leq \lambda(n-1)$. We must have at least one vertex of degree at most $\lambda(n-1) - 2$ in $G(B)$, for if all vertices in $G(B)$ had degree $\lambda(n-1)-1$ or greater, then by Lemma 2.2, $G(B)$ would contain at least one triple. Without loss of generality we can assume that $d_{G(B)}(n) \leq \lambda(n-1) - 2$. We have that $d_{G(B^*)}(i) = d_{G(B)}(i)$ for $1 \leq i \leq n-1$, $d_{G(B^*)}(i) = 1$ for $n+1 \leq i \leq 2n+1$, and $d_{G(B^*)}(n) = d_{G(B)}(n)$, $d_{G(B)}(n) + 2$, and $d_{G(B)}(n) + 4$ in cases (3a), (3c), and (3b), respectively (so $w(G(B^*)) = w(G(B)))$. Hence, $\Delta(G(B^*)) \leq \lambda(n-1)+2 < \lambda n$, so (i) is satisfied.

We next consider (ii b). We have observed that $w(G(B)) \leq n$ and $w(G(B^*)) = w(G(B))$. In addition: $\epsilon(G(B^*)) = \epsilon(G(B)) + (n+1)/2$ in (3a); $\epsilon(G(B^*)) = \epsilon(G(B))+(n+1)/2+2$ in (3b); and $\epsilon(G(B^*)) = \epsilon(G(B))+ (n+1)/2+1$ in (3c). Therefore, we have that $\epsilon(G(B^*)) + w(G(B^*)) \equiv 0 \pmod 3$ in all cases, so $W(G(B^*)) \leq n$, satisfying (ii b).

Now we investigate condition (iii c). Since (V, B) is maximal, by Lemma 2.2, $\epsilon(G(B^*)) \leq \lambda n^2/4$, so $\epsilon(G(B^*)) \leq \lambda n^2/4 + 2 + (n+1)/2$. Therefore, $\epsilon(G(B^*)) + W(G(B^*)) \leq (\lambda n^2 + 6n + 10)/4 \leq (\lambda n^2 - n^2 - \lambda - 10)/2 \leq 3\mu(n+1, \lambda - \lceil\lambda/n\rceil)$ when $n \geq 3$ and $\lambda \geq 3$, except possibly for the following special cases: $n = 3$ and $\lambda \leq 9$; $n = 5$ and $\lambda = 3$; $n = 7$ and $\lambda = 3$; and $n = 9$ and $\lambda = 3$. However, by direct calculations of $3\mu(n+1, \lambda - \lceil\lambda/n\rceil)$ for these values, we find that $\epsilon(G(B^*)) + W(G(B^*)) \leq 3\mu(n+1, \lambda - \lceil\lambda/n\rceil)$ in all of these cases, so (iii c) is satisfied.

Condition (iv) is satisfied since $d_{G(B^*)}(i) = 1$ for $n+1 \leq i \leq 2n+1$.

Case 2: $u = 2n+1$, n is odd, λ is even.

Define B^* as follows.

(1) $B \subseteq B^*$.

(2) Using Lemma 2.3, partition the the edges of K_{n+1} on the vertex set $\{n+1, \ldots, 2n+1\}$ into the 1-factors $F_1, \ldots, F_n$, where F_n consists of the edges $\{n+i, 2n+2-i\}$, for $1 \leq i \leq (n+1)/2$. For each edge $\{a, b\} \in F_v$, for $1 \leq v \leq n-1$, let B^* contain λ copies of the triple $\{v, a, b\}$.

(3a) If $\epsilon(G(B)) + p(G(B)) \equiv 0 \pmod 3$, let B^* contain λ copies of the triple $\{a, b, n\}$, for each edge $\{a, b\} \in F_n$, and the remaining loops at vertices in the set $\{n+1, \ldots, 2n+1\}$.

(3b) If $\epsilon(G(B))+p(G(B)) \equiv 1 \pmod 3$, let B^* contain λ copies of the triple $\{a, b, n\}$, for each edge $\{a, b\} \in F_n \setminus \{\{n+1, 2n+1\}\}$, $\lambda - 2$ copies of the triple $\{n, n+1, 2n+1\}$, 2 copies of the lollipops $\{n+1, n+1, n\}$ and $\{2n+1, 2n+1, n\}$, and the remaining loops at vertices in the set $\{n+1, \ldots, 2n+1\}$.

(3c) If $\epsilon(G(B))+p(G(B)) \equiv 2 \pmod 3$, let B^* contain λ copies of the triple $\{a, b, n\}$, for each edge $\{a, b\} \in F_n \setminus \{\{n+1, 2n+1\} \cup \{n+2, 2n\}\}$, $\lambda - 2$ copies of the triples $\{n, n+1, 2n+1\}$ and $\{n, n+2, 2n\}$, 2 copies of the lollipops $\{n+i, n+i, n\}$, where $i \in \{1, 2, n, n+1\}$, and the remaining loops at vertices in the set $\{n+1, \ldots, 2n+1\}$.

We consider condition (i) and note that $\Delta(G(B)) \leq \lambda(n-1)$. Furthermore, for every vertex $i \in \{1, \ldots, n\}$, $d_{G(B^*)}(i) = d_{G(B)}(i)$. In addition,

$d_{G(B^*)}(j) = 0$, for all $j \in \{n+1, \ldots, 2n+1\}$ except for case (3b) in which $d_{G(B^*)}(n+1) = d_{G(B^*)}(2n+1) = 2$, and case (3c) in which $d_{G(B^*)}(j) = 2$, for all $j \in \{n+1, n+2, 2n, 2n+1\}$. Hence, $\Delta(G(B^*)) < \lambda n$.

Next we consider condition (ii a). We have that $p(G(B^*)) = p(G(B)) \leq n$. In addition, $\epsilon(G(B^*)) = \epsilon(G(B)) + 0$, $\epsilon(G(B)) + 2$, and $\epsilon(G(B)) + 4$ in cases (3a), (3b), and (3c), respectively. Therefore, $\epsilon(G(B^*)) + p(G(B^*)) \equiv 0 \pmod 3$, so $P(G(B^*)) = p(G(B^*)) \leq n$.

We now investigate (iii a). It is easily verified that $3T^*(n+1, 2) \geq 2\lfloor n^2/4 \rfloor + n + 4 \geq \epsilon(G(B^*)) + P(G(B^*))$ for all odd $n \geq 5$. If $n = 3$, then $2\lfloor n^2/4 \rfloor + n + 4 = 11$; however, $\epsilon(G(B^*)) + P(G(B^*)) \equiv 0 \pmod 3$, so $\epsilon(G(B^*)) + P(G(B^*)) \leq 9 = 3T^*(4, 2)$, so (iii a) is satisfied for all $n \geq 3$.

Next we consider (iii b). Using Lemma 2.2 we have that $\epsilon(G(B^*)) + P(G(B^*)) \leq \lambda\lfloor n^2/4 \rfloor + n + 4 \leq (\lambda n^2 - n^2 - \lambda - 10)/2 \leq 3\mu(n+1, \lambda - \lceil \lambda/n \rceil)$ when $n \geq 3$ and $\lambda \geq 4$, except possibly for the following special cases: $n = 3$ and $\lambda \leq 8$; and $n = 5$ and $\lambda = 4$. Direct calculations show that $3\mu(n+1, \lambda - \lceil \lambda/n \rceil) \geq \epsilon(G(B^*)) + P(G(B^*))$ except possibly when $n = 3$ and $\lambda = 4$. Suppose $n = 3$ and $\lambda = 4$. We have that $3\mu(4, 2) = 12$ and $\epsilon(G(B^*)) + P(G(B^*)) \leq 15$, so since $\epsilon(G(B^*)) + P(G(B^*)) \equiv 0 \pmod 3$, we need only consider the case when $\epsilon(G(B)) = \lambda\lfloor n^2/4 \rfloor = 8$. But in this case we will have that $P(G(B^*)) = 0$, so $\epsilon(G(B^*)) + P(G(B^*)) \leq 12 = 3\mu(4, 2)$. Therefore, (iii b) is satisfied for all even $\lambda \geq 4$ and all odd $n \geq 3$.

Clearly, the construction gives a $PETS(2n+1, \lambda)$ satisfying (iv).

Case 3: $u = 2n + 3$, n is odd, λ is odd.

Define B^* as follows.

(1) $B \subseteq B^*$.

(2) Using Lemma 2.3, partition the the edges of K_{n+3} on the vertex set $\{n+1, \ldots, 2n+3\}$ into the 1-factors $F_1, \ldots, F_{n+2}$. For each edge $\{a, b\} \in F_v$, for $1 \leq v \leq n$, let B^* contain λ copies of the triple $\{v, a, b\}$.

(3a) Suppose $\epsilon(G(B)) + w(G(B)) + (n+3)/2 \equiv 0 \pmod 3$. For each edge $\{a, b\} \in F_{n+1}$, let B^* contain $(\lambda+1)/2$ copies of the lollipop $\{a, a, b\}$ and $(\lambda-1)/2$ copies of the lollipop $\{b, b, a\}$. For each edge $\{a, b\} \in F_{n+2}$, let B^* contain $(\lambda-1)/2$ copies of the lollipops $\{a, a, b\}$ and $\{b, b, a\}$, and the remaining loops at vertices in the set $\{n+1, \ldots, 2n+3\}$.

(3b) Suppose $\epsilon(G(B)) + w(G(B)) + (n+3)/2 \equiv 1 \pmod 3$. Let $\{x_1, 2n+3\} \in F_{n+1}$. For each edge $\{a, b\} \in F_{n+1} \setminus \{\{x_1, 2n+3\}\}$, let B^* contain $(\lambda+1)/2$ copies of the lollipop $\{a, a, b\}$ and $(\lambda-1)/2$ copies of the lollipop $\{b, b, a\}$. Furthermore, let B^* contain $(\lambda-1)/2$ copies of the lollipop $\{x_i, x_i, 2n+3\}$ and $(\lambda-3)/2$ copies of the lollipop

$\{2n+3, 2n+3, x_i\}$. In addition, for each edge $\{a,b\} \in F_{n+2}$, let B^* contain $(\lambda-1)/2$ copies of the lollipops $\{a,a,b\}$ and $\{b,b,a\}$, and the remaining loops at vertices in the set $\{n+1,\ldots,2n+3\}$.

(3c) Suppose $\epsilon(G(B))+w(G(B))+(n+3)/2 \equiv 2 \pmod 3$. Let $\{x_i, 2n+3\} \in F_{n+i}$, for $1 \leq i \leq 2$. For each edge $\{a,b\} \in F_{n+1} \setminus \{\{x_1, 2n+3\}\}$, let B^* contain $(\lambda+1)/2$ copies of the lollipop $\{a,a,b\}$ and $(\lambda-1)/2$ copies of the lollipop $\{b,b,a\}$. Furthermore, let B^* contain $(\lambda-1)/2$ copies of the lollipop $\{x_1, x_1, 2n+3\}$ and $(\lambda-3)/2$ copies of the lollipop $\{2n+3, 2n+3, x_1\}$. In addition, for each edge $\{a,b\} \in F_{n+2} \setminus \{\{x_2, 2n+3\}\}$, let B^* contain $(\lambda-1)$ copies of the lollipops $\{a,a,b\}$ and $\{b,b,a\}$. Furthermore, let B^* contain $(\lambda-1)/2$ copies of the lollipop $\{x_2, x_2, 2n+3\}$, $(\lambda-3)/2$ copies of the lollipop $\{2n+3, 2n+3, x_2\}$, and the remaining loops at vertices in the set $\{n+1,\ldots,2n+3\}$.

We consider condition (i). Clearly, $\Delta(G(B)) \leq \lambda(n-1)$. Furthermore, $d_{G(B^*)}(i) = d_{G(B)}(i)$, for $1 \leq i \leq n$, and $d_{G(B^*)}(j) = 1, 3$, or 5 (in fact, only vertex $2n+3$ could possibly have degree 5), for $n+1 \leq j \leq 2n+3$ (so $w(G(B^*)) = w(G(B))$). Therefore, $\Delta(G(B^*)) \leq \lambda n$.

Next we consider (ii b). Since $\epsilon(G(B^*)) = \epsilon(G(B)) + (n+3)/2$ in case (2a), $\epsilon(G(B))+(n+3)/2+2$ in case (2b), and $\epsilon(G(B))+(n+3)/2+4$ in case (2c), and since $w(G(B^*)) = w(G(B)) \leq n$, we have $\epsilon(G(B^*))+w(G(B^*)) \equiv 0 \pmod 3$, so $W(G(B^*)) = w(G(B^*)) \leq n$. Therefore, (ii b) is satisfied.

Now we investigate (iii c). We have $\epsilon(G(B^*))+W(G(B^*)) \leq \lambda\lfloor n^2/4\rfloor + (n+3)/2+n+4 \leq (\lambda n^2+\lambda n+2\lambda-n^2-4n-15)/2 \leq 3\mu(n+2, \lambda-\lceil\lambda/n\rceil)$ when $\lambda \geq 3$ and $n \geq 3$ except possibly for the following special cases: $n \leq 11$ and $\lambda = 3$; and $n = 3$ and $\lambda = 5$. However, direct calculations show that $\epsilon(G(B^*)) + W(G(B^*)) \leq 3\mu(n+2, \lambda - \lceil\lambda/n\rceil)$ for these cases as well, so (iii c) is satisfied.

Obviously, condition (iv) is satisfied.

Case 4: $u = 2n+3$, n is odd, λ is even.

Define B^* as follows.

(1) $B \subseteq B^*$.

(2) Using Lemma 2.3, partition the the edges of K_{n+3} on the vertex set $\{n+1,\ldots,2n+3\}$ into the 1-factors $F_1,\ldots,F_{n+2}$. For each edge $\{a,b\} \in F_v$, for $1 \leq v \leq n$, let B^* contain λ copies of the triple $\{v,a,b\}$.

(3a) Suppose $\epsilon(G(B)) + p(G(B)) \equiv 0 \pmod 3$. For each edge $\{a,b\} \in F_i$, $i \in \{n+1, n+2\}$, let B^* contain $\lambda/2$ copies of the lollipop $\{a,a,b\}$ and $\lambda/2$ copies of the lollipop $\{b,b,a\}$.

(3b) Suppose $\epsilon(G(B)) + p(G(B)) \equiv 1 \pmod 3$. Let $\{x_1, 2n+3\} \in F_{n+1}$. For each edge $\{a,b\} \in F_{n+1} \cup F_{n+2} \setminus \{\{x_1, 2n+3\}\}$, let B^* contain $\lambda/2$ copies of the lollipop $\{a,a,b\}$ and $\lambda/2$ copies of the lollipop $\{b,b,a\}$. Furthermore, let B^* contain $(\lambda-2)/2$ copies of the lollipop $\{x_1, x_1, 2n+3\}$, $(\lambda-2)/2$ copies of the lollipop $\{2n+3, 2n+3, x_1\}$, and the remaining loops at vertex $2n+3$.

(3c) Suppose $\epsilon(G(B))+p(G(B)) \equiv 2 \pmod 3$. Let $\{x_i, 2n+3\} \in F_{n+i}$, $i \in \{1,2\}$. For each edge $\{a,b\} \in F_{n+1} \cup F_{n+2} \setminus \{\{x_1, 2n+3\}, \{x_2, 2n+3\}\}$, let B^* contain $\lambda/2$ copies of the lollipop $\{a,a,b\}$ and $\lambda/2$ copies of the lollipop $\{b,b,a\}$. Furthermore, let B^* contain $(\lambda-2)/2$ copies of the lollipops $\{x_i, x_i, 2n+3\}$, $(\lambda-2)/2$ copies of the lollipops $\{2n+3, 2n+3, x_i\}$, for $1 \le i \le 2$, and the remaining loops at vertex $2n+3$.

We consider condition (i) and recall that $\Delta(G(B)) \le \lambda(n-1)$. We have that $d_{G(B^*)}(i) = d_{G(B)}(i)$, for $1 \le i \le n$, and $d_{G(B^*)}(j) = 0$, 2, or 4, for $n+1 \le j \le 2n+3$ (in fact, only vertex $2n+3$ could possibly have degree 4 in $G(B^*)$), so $p(G(B^*)) = p(G(B)) \le n$. In any event, $\Delta(G(B^*)) < \lambda n$.

Consider (ii b). Since $\epsilon(G(B^*)) = \epsilon(G(B))$ in case (2a), $\epsilon(G(B)) + 2$ in case (2b), and $\epsilon(G(B)) + 4$ in case (2c), and since $p(G(B^*)) = p(G(B))$, we have that $\epsilon(G(B^*))+p(G(B^*)) \equiv 0 \pmod 3$, so $P(G(B^*)) = p(G(B^*)) \le n$.

Now we investigate (iii a). It is easily verified that $3T^*(n+2,2) \ge 2\lfloor n^2/4 \rfloor + n + 4 \ge \epsilon(G(B^*)) + P(G(B^*))$ for all odd $n \ge 5$. If $n = 3$, then $2\lfloor n^2/4 \rfloor + n + 4 = 11$; however $\epsilon(G(B^*)) + P(G(B^*)) \equiv 0 \pmod 3$, so $\epsilon(G(B^*)) + P(G(B^*)) \le 9 = 3T^*(5,2)$, so (iii a) is satisfied for all $n \ge 3$.

Next we consider (iii b). We have that $\epsilon(G(B^*))+P(G(B^*)) \le \lambda\lfloor n^2/4 \rfloor + n + 4 \le (\lambda n^2 + \lambda n + 2\lambda - n^2 - 4n - 15)/2 \le 3\mu(n+2, \lambda - \lceil \lambda/n \rceil)$ for all odd $n \ge 3$ and all even $\lambda \ge 4$ except possibly for the case when $n = 3$ and $\lambda = 4$. However, direct calculations show that $\epsilon(G(B^*)) + P(G(B^*)) \le 3\mu(n+2, \lambda - \lceil \lambda/n \rceil)$ in this case as well, so (iii b) is satisfied.

Once again, condition (iv) is satisfied by the construction.

Case 5: n is even.

Suppose (V, B) is a $PETS(n, \lambda)$. Then (V, B) can be embedded in a maximal $PETS(n+1, \lambda)$ (V_1, B_1). Since $n+1$ is odd, we can then apply the constructions to (V_1, B_1) to obtain a $PETS$ (V^*, B^*) in which $|V^*| = 2n+3$ or $2n+5$, where (V^*, B^*) satisfies all of the appropriate conditions, so the proof is complete. □

Proposition 4.2 *Suppose (V^*, B^*) is a maximal $PETS(u, \lambda)$, where $\lambda \ge 2$ and $u = 2n+1$ or $2n+3$ if n is odd and $u = 2n+3$ or $u = 2n+5$ if n is even, and suppose (V^*, B^*) satisfies conditions (i), (ii a), (ii b), (iii a), (iii b), (iii c), and (iv) of Lemma 4.1. Then (V^*, B^*) can be embedded in an $ETS(2u+2, \lambda)$ $(\hat{V}, \hat{B})$.*

Proof: We first note that u is odd in all cases considered. We define five different types of extended triples:

(a) lollipops of the form $\{a, a, b\}$, where $a \geq u+1$ and $b \leq u$;

(b) loops and lollipops on vertices in the set $\{u+1, \ldots, 2u+2\}$;

(c) triples of the form $\{a, b, c\}$, where $a, b \leq u$, and $c \geq u+1$;

(d) triples of the form $\{a, b, c\}$, where $a, b, c \geq u+1$; and

(e) triples of the form $\{a, b, c\}$, where $a \leq u$ and $b, c \geq u+1$.

Type a: First suppose λ is even, and let $\{x_1, \ldots, x_p\}$ be the set of p vertices of odd degree in $G(B^*)$. We have that since u is odd and λ is even, each vertex in $\hat{V}$ must occur in an even number of lollipops in $\hat{B}$. Let $\hat{B}$ contain lollipops of the form $\{u+i, u+i, x_i\}$, for $1 \leq i \leq p$. After the Type b lollipops are defined, the remaining edges are to be placed in triples, so the number of such edges must be divisible by 3. Therefore, we may need to define up to four more lollipops as follows. Let $\phi \in \mathbb{Z}_3$ with $\phi \equiv \epsilon(G(B^*)) + p(G(B^*)) \pmod 3$. By (iv) there are at least ϕ vertices of degree at most $\lambda n - 2$ in $G(B^*)$, and we can name them $p+i$, for $1 \leq i \leq \phi$. If $\phi \geq 1$, let $\{\{p+i, u+p+2i-1, u+p+2i-1\}, \{p+i, u+p+2i, u+p+2i\} | 1 \leq i \leq \phi\} \subseteq \hat{B}$. Now we have defined $p+2\phi = P(G(B^*)) \leq n$ Type a extended triples.

Now suppose λ is odd, and let $\{x_1, \ldots, x_w\}$ be the set of w vertices of even degree in $G(B^*)$. Since u and λ are odd, each vertex in $\hat{V}$ must occur in an odd number of lollipops in $\hat{B}$. Again, after the Type b lollipops are defined, the remaining edges are to be placed in triples, so the number of such edges must be divisible by 3. Therefore, we may need to add up to four more lollipops in much the same manner as before. Again, let $\phi \in \mathbb{Z}_3$ with $\phi \equiv \epsilon(G(B^*)) + w(G(B^*)) \pmod 3$. By (iv) there are at least ϕ vertices of degree at most $\lambda n - 2$ in $G(B^*)$, and we can name them $w+i$, for $1 \leq i \leq \phi$. If $\phi \geq 1$, let $\{\{w+i, u+w+2i-1, u+w+2i-1\}, \{w+i, u+w+2i, u+w+2i\} | 1 \leq i \leq \phi\} \subseteq \hat{B}$. Now we have defined $w + 2\phi = W(G(B^*)) \leq n$ Type a extended triples.

Type b: First suppose $\lambda = 2$ and $u \equiv 1 \pmod 4$. By Lemma 2.13 there exists an equitable $PTS((u+1)/2, 2)$ $(S = \{u+1, \ldots, u+(u+1)/2\}, T)$ containing $(\epsilon(G(B^*)) + P(G(B^*)))/3$ triples such that the leave of (S, T) contains a 2-factor consisting of all even cycles except for exactly one 3-cycle. We can name the vertices in S so that the directed 3-cycle is $(u+1, u+2, u+(u+1)/2)$, and so that otherwise each directed cycle consists of arcs of the form $(x, n(x))$ where $n(x) = x+1$ for each vertex in the cycle except for the largest. For every $a \in \{u+3, \ldots, u+(u-1)/2\} \setminus \{u+3, u+5, u+7, \ldots, u+p+2\phi\}$, let $\hat{B}$ include the lollipop $\{a, a, n(a)\}$. In addition:

if $P(G(B^*)) = 0$ then let $\{u+1, u+1, u+2\}$, $\{u+2, u+2, u+(u+1)/2\}$, and $\{u+(u+1)/2, u+(u+1)/2, u+1\} \in \hat{B}$; and, if $P(G(B^*)) \geq 2$ then let $\hat{B}$ contain $\{u+1, u+1, u+(u+1)/2\}$ and $\{u+2, u+2, u+(u+1)/2\}$. Because of the way the vertices in S were named, each vertex in S is contained in two lollipops (including the Type a lollipops). Next let $\hat{B}$ contain two lollipops of the form $\{u+(u+3)/2+2i, u+(u+3)/2+2i, u+(u+3)/2+2i+1\}$, for $0 \leq i \leq (u-1)/4$, and let $\hat{B}$ contain the remaining loops which are adjacent to vertices in the set $\{u+1, \ldots, 2u+2\}$.

Next suppose $\lambda = 2$ and $u \equiv 3 \pmod 4$. By Lemma 2.13, there exists an equitable $PTS((u+1)/2, 2)$ $(S = \{u+1, \ldots, u+(u+1)/2\}, T)$ containing $(\epsilon(G(B^*)) + P(G(B^*)))/3$ triples such that the leave of (S, T) contains a near 2-factor which saturates all vertices in $S \setminus \{u+(u+1)/2\}$ and consists of all even cycles except for exactly one 3-cycle. We can name the vertices in S so that the directed 3-cycle is $(u+1, u+2, u+(u-1)/2)$ and so that otherwise each directed cycle consists of arcs of the form $(x, n(x))$ where $n(x) = x+1$ for each vertex in the cycle except for the largest. For every $a \in \{u+3, \ldots, u+(u-3)/2\} \setminus \{u+3, u+5, \ldots, u+p+2\phi\}$, let $\hat{B}$ include the lollipop $\{a, a, n(a)\}$. In addition: if $P(G(B^*)) = 0$ then let $\hat{B}$ contain $\{u+1, u+1, u+2\}$, $\{u+2, u+2, u+(u-1)/2\}$, and $\{u+(u-1)/2, u+(u-1)/2, u+1\}$; and, if $P(G(B^*)) \geq 2$ then let $\hat{B}$ contain $\{u+(u-1)/2, u+(u-1)/2, u+1\}$ and $\{u+(u-1)/2, u+(u-1)/2, u+2\}$. Again, because of the way in which we named the vertices in S, each vertex in $S \setminus \{u+(u+1)/2\}$ is contained in two lollipops (including the Type a lollipops). Next let $\hat{B}$ contain two lollipops of the form $\{u+(u+1)/2+2i, u+(u+1)/2+2i, u+(u+1)/2+2i+1\}$, for $0 \leq i \leq (u+1)/4$, and let $\hat{B}$ contain the remaining loops which are incident with vertices in the set $\{u+1, \ldots, 2u+2\}$.

Now suppose $\lambda \geq 4$, λ is even, and $u \equiv 1 \pmod 4$. Let $H_1, \ldots, H_{(u-1)/4}$ be a Hamilton decomposition of $K_{(u+1)/2}$ on the vertex set $\{u+1, \ldots, u+(u+1)/2\}$ where $H_{(u-1)/4} = (u+1, u+2, \ldots, u+(u+1)/2)$. For every edge $\{a, b\} \in H_i$, for $1 \leq i \leq (u-1)/4$, let $\hat{B}$ contain $\lceil \lambda/((u-1)/2) \rceil - 1$ copies of the lollipop $\{a, a, b\}$, so every vertex in $\{u+1, \ldots, u+(u+1)/2\}$ is contained in $((u-1)/2)(\lceil \lambda/((u-1)/2) \rceil - 1)$ Type b lollipops. Next for every edge $\{a, b\} \in H_i$, for $1 \leq i \leq ((\lambda-2) - ((u-1)/2)(\lceil \lambda/((u-1)/2) \rceil - 1))/2$, let $\hat{B}$ contain the lollipop $\{a, a, b\}$. Finally, for every edge $\{a, b\} \in H_{(u-1)/4} \setminus \{\{\{u+2i-1, u+2i\} | 1 \leq i \leq P/2\}\}$, (where $P = P(G(B^*))$) let $\{a, a, b\} \in \hat{B}$. Now every vertex in the set $\{u+1, \ldots, u+P\}$ is in one Type a and $\lambda - 1$ Type b lollipops, and every vertex in the set $\{u+P+1, \ldots, u+(u+1)/2\}$ is in λ Type b lollipops.

Next suppose $\lambda \geq 3$, λ is odd, and $u \equiv 1 \pmod 4$. Again we let $H_1, \ldots, H_{(u-1)/4}$ be a Hamilton decomposition of $K_{(u+1)/2}$ on the vertex set $\{u+1, \ldots, u+(u+1)/2\}$ where $H_{(u-1)/4} = (u+1, u+2, \ldots, u+$

$(u+1)/2)$. For every edge $\{a,b\} \in H_i$, for $1 \le i \le (u-1)/4$, let $\hat{B}$ contain $\lceil \lambda/((u-1)/2) \rceil - 1$ copies of the lollipop $\{a,a,b\}$, so every vertex in $\{u+1,\ldots,u+(u+1)/2\}$ is contained in $((u-1)/2)(\lceil \lambda/((u-1)/2) \rceil - 1)$ Type b lollipops. Next for every edge $\{a,b\} \in H_i$, for $1 \le i \le ((\lambda-1) - ((u-1)/2)(\lceil \lambda/((u-1)/2) \rceil - 1))/2$, let $\hat{B}$ contain the lollipop $\{a,a,b\}$, so every vertex in $\{u+1,\ldots u+(u+1)/2\}$ is contained in $\lambda - 1$ Type b lollipops. Finally, for every edge $\{u+2i-1, u+2i\} \in H_{(u-1)/4}$, for $(W+3)/2 \le i \le (u+1)/4$, (where $W = W(G(B^*))$) let $\hat{B}$ contain the lollipop $\{u+2i-1, u+2i-1, u+2i\}$. Therefore, every vertex in the set $\{u+1,\ldots,u+W\}$ is contained in $\lambda-1$ Type b lollipops and one Type a lollipop, and every vertex in the set $\{u+W+1,\ldots,u+(u+1)/2\}$ is contained in λ Type b lollipops.

Now suppose $\lambda \ge 4$, λ is even, and $u \equiv 3 \pmod 4$. Let $H_1,\ldots,H_{(u-3)/4}$ be a Hamilton decomposition of $K_{(u-1)/2}$ on the vertex set $\{u+1,\ldots,u+(u-1)/2\}$, where $H_{(u-3)/4} = (u+1, u+2, \ldots, u+(u-1)/2)$. For every edge $\{a,b\} \in H_i$, for $1 \le i \le (u-3)/4$, let $\hat{B}$ contain $\lceil \lambda/((u-3)/2) \rceil - 1$ copies of the lollipop $\{a,a,b\}$, so every vertex in $\{u+1,\ldots,u+(u-1)/2\}$ is in $((u-3)/2)(\lceil \lambda/((u-3)/2) \rceil - 1)$ Type b lollipops. For every edge $\{a,b\} \in H_i$, for $1 \le i \le ((\lambda-2) - ((u-3)/2)(\lceil \lambda/((u-3)/2) \rceil - 1))/2$, let $\{a,a,b\} \in \hat{B}$. Finally, for every edge $\{a,b\} \in H_{(u-3)/4} \backslash \{\{u+2i-1, u+2i\} | 1 \le i \le P/2\}$, let $\{a,a,b\} \in \hat{B}$. Now every vertex in the set $\{u+1,\ldots,(u-3)/4\}$ is contained in the appropriate number of Type a and Type b lollipops.

Finally, suppose $\lambda \ge 3$, λ is odd, and $u \equiv 3 \pmod 4$. Again let $H_1,\ldots,H_{(u-3)/4}$ be a Hamilton decomposition of $K_{(u-1)/2}$ on the vertex set $\{u+1,\ldots,u+(u-1)/2\}$, where $H_{(u-3)/4} = (u+1, u+2, \ldots, u+(u-1)/2)$. For every edge $\{a,b\} \in H_i$, for $1 \le i \le (u-3)/4$, let $\hat{B}$ contain $\lceil \lambda/((u-3)/2) \rceil - 1$ copies of the lollipop $\{a,a,b\}$. Next, for every edge $\{a,b\} \in H_i$, for $1 \le i \le ((\lambda-1) - ((u-3)/2)(\lceil \lambda/((u-3)/2) \rceil - 1))/2$, let $\hat{B}$ contain the lollipop $\{a,a,b\}$. Finally, for every edge of the form $\{u+2i, u+2i+1\}$, for $(W+1)/2 \le i \le (u-3)/4$, let $\hat{B}$ contain $\{u+2i, u+2i, u+2i+1\}$. Now every vertex in the set $\{u+1,\ldots,(u-1)/2\}$ is contained in the appropriate number of Type a and Type b lollipops.

In addition: if $u \equiv 1 \pmod 4$, let $\hat{B}$ contain λ copies of the lollipops $\{\{u+2i, u+2i, u+2i+1\} | (u+3)/4 \le i \le (u+1)/2\}$; and if $u \equiv 3 \pmod 4$, let $\hat{B}$ contain λ copies of the lollipops $\{\{u+2i, u+2i, u+2i+1\} | (u+1)/4 \le i \le (u+1)/2\}$. In any event, let $\hat{B}$ contain the remaining loops at vertices in the set $\{u+1,\ldots,2u+2\}$.

Type c: Form a graph H consisting of the edges of $G(B^*)$ along with the Type a lollipop edges (but not the loops) $\{x_i, u+\ell\}$, where $x_i \in \{1,\ldots,u\}$ for $i \in \{1,\ldots,p+\phi\}$ or $i \in \{1,\ldots,w+\phi\}$ and $\ell \in \{1,\ldots,p+2\phi\}$ or $\ell \in \{1,\ldots,w+2\phi\}$ if λ is even or odd, respectively. We want to give H an equalized edge-coloring with $(u+1)/2$ colors, say $u+1,\ldots,u+(u+1)/2$, where: $|C_{u+1}| \geq \cdots \geq |C_{u+(u+1)/2}|$; $|C_i(v)| \leq \lambda$ for $u+1 \leq i \leq u+(u+1)/2$ and for all $v \in V(H)$; and the Type a lollipop edges in H all receive different colors, say $\{x_i, u+\ell\}$ is colored with $u+\ell$ for $1 \leq \ell \leq p+2\phi$ if λ is even and for $1 \leq \ell \leq w+2\phi$ if λ is odd. Do this by forming a graph H' from H by contracting vertices $u+1, \ldots, u+P$ if λ is even and vertices $u+1, \ldots, u+W$ if λ is odd into a single vertex γ. We have $d_{H'}(i) \leq \Delta(G(B^*)) + 1 \leq \lambda n$ if n is odd and $d_{H'}(i) \leq \Delta(G(B^*)) + 1 \leq \lambda(n+1)$ if n is even, for $1 \leq i \leq u$ by (i) and (iv). In addition, $d_{H'}(\gamma) \leq n$ if n is odd and $d_{H'}(\gamma) \leq n+1$ if n is even by (ii a) and (ii b), as $d_{H'}(\gamma) = P(G(B^*))$ if λ is even and $d_{H'}(\gamma) = W(G(B^*))$ if λ is odd. Therefore, H' can be given a proper equalized $\lambda((u+1)/2)$-edge-coloring by Lemma 2.4 (since $(u+1)/2 \geq n+1$ if n is odd and $(u+1)/2 \geq n+2$ if n is even). Name the color classes $C_{i,j}$, where $1 \leq i \leq \lambda$ and $u+1 \leq j \leq u+(u+1)/2$, and name the colors so that: $|C_{i,j}| \geq |C_{k,l}|$ if and only if (i,j) is lexicographically less than (k,l); and if e_1 and e_2 are any two edges incident with γ in color classes $C_{i,j}$ and $C_{k,\ell}$ respectively, then $j \neq \ell$. Then letting $C_{n+j} = \cup_{i=1}^{\lambda} C_{i,j}$ for $u+1 \leq j \leq u+(u+1)/2$ be the color classes of an equalized edge-coloring of H' produces the desired $(n+1)$-edge-coloring of H. For each edge $\{i,j\}$ in $G(B^*)$ colored k, let $\{i,j,k\} \in \hat{B}$.

Type d: Consider the edge-coloring of H just obtained and let δ_x denote the number of edges of H colored x. First assume $\lambda = 2$. Let $(S' = \{u+1, \ldots, u+(u+1)/2\}, T')$ be an equitable $PSTS((u+1)/2, 2)$ and let $|T'| = (\epsilon(G(B^*)) + P(G(B^*)))/3$; by the definition of $P(G(B^*))$, $|T'|$ is an integer. By Lemma 2.13, such a $PSTS$ exists, since by (iii a), $|T'| \leq T^*((u+1)/2, 2)$. Name the symbols so that $\delta_x = r(x)$, the number of triples in T' which contain symbol x, for $u+1 \leq x \leq u+(u+1)/2$. Since the leave of (S', T') contains a (near) 2-factor from which the Type b lollipops were obtained, no edge of the form $\{a,b\}$ will occur in too many extended triples up to this point. Let $T' \subseteq \hat{B}$.

Next assume $\lambda \geq 3$. If $(u+1)/2$ is odd, let $(S = \{u+1, \ldots, u+(u+1)/2\}, T')$ be an equitable $PSTS((u+1)/2, \lambda - \lceil \lambda/((u-1)/2) \rceil)$ and if $(u+1)/2$ is even, let $(S = \{u+1, \ldots, u+(u+1)/2\}, T')$ be an equitable $PSTS((u+1)/2, \lambda - \lceil \lambda/((u-3)/2) \rceil)$. Let

$$|T'| = \begin{cases} (\epsilon(G(B^*)) + P(G(B^*)))/3 & \text{, if } \lambda \text{ is even} \\ (\epsilon(G(B^*)) + W(G(B^*)))/3 & \text{, if } \lambda \text{ is odd.} \end{cases}$$

Such a $PSTS$ exists by Lemma 2.9. Name the symbols so that $\delta_x = r(x)$, the number of triples in T' which contain symbol x, for $u+1 \leq x \leq u+(u+1)/2$. In addition, since any edge of the form $\{a,b\}$, for $u+1 \leq a, b \leq u+(u+1)/2$, is in at most $\lceil \lambda/((u-1)/2) \rceil$ Type b lollipops if $(u+1)/2$ is odd and in at most $\lceil \lambda/((u-3)/2) \rceil$ Type b lollipops if $(u+1)/2$ is even, then no edge of the form $\{a,b\}$, for $u+1 \leq a, b \leq u+(u+1)/2$ will be in too many extended triples in $\hat{B}$ up to this point. Let $T' \subseteq \hat{B}$.

Type e: We use Theorems 2.6 and 2.7 to place the remaining edges in triples. We first form a partial array $L(\infty)$ or order $(u+1)/2$ and multiplicity λ on the symbols $1, \ldots, u$ as follows:

(1) place symbol $j \leq u$ in cell (i, i) if an edge colored $u+i$ is incident with vertex j in H,

(2) for $1 \leq i < j \leq (u+1)/2$, if $\{i+u, j+u\}$ is an edge of k_1 triples in T' and a lollipop edge of k_2 Type b lollipops, then fill cells (i, j) and (j, i) with symbol ∞ $k_1 + k_2$ times, and

(3) for $1 \leq i < j \leq (u+1)/2$, fill cells (i, j) and (j, i) greedily with $\lambda - (k_1 + k_2)$ symbols in $\{1, \ldots, u\}$, preserving symmetry and preserving the property that each symbol occurs at most λ times in each row and at most λ times in each column.

Let L be formed from $L(\infty)$ by deleting all the ∞ symbols. Then L is a symmetric quasi-latin square of order $(u+1)/2$ and multiplicity λ on the symbols in $\{1, \ldots, u\}$. In order to apply Theorems 2.6 and 2.7 (with $r = (u+1)/2$ and $t = u$ in both cases), we need to check that L satisfies the appropriate conditions. We first show that the number of times each symbol occurs on the main diagonal of L is congruent to $\lambda \pmod 2$. We consider the cases where $d_{G(B^*)}(x)$ is odd and even in turn.

Suppose $d_{G(B^*)}(x) = 2m+1$. Since x has odd degree in $G(B^*)$, then H contains a lollipop edge of Type a incident with x colored with some color α only if λ is even. Therefore, symbol x occurs once in a cell of the form $(\alpha - u, \alpha - u)$. In any event, x is contained in $2m+1$ triples of the form $\{x, y, \beta\}$, where β is the color of an edge $\{x, y\}$ in H, so symbol x is placed $2m+1$ times in cells of the form $(\beta - u, \beta - u)$. Therefore, symbol x occurs an even (odd) number of times on the main diagonal of L if λ is even (odd).

Now suppose $d_{G(B^*)} = 2m$. Since x has even degree in $G(B^*)$, H contains a lollipop edge of Type a incident with x colored with some color α only if λ is odd. Therefore, if λ is odd, symbol x occurs in a cell of the form $(\alpha - u, \alpha - u)$. In any event, x is contained in $2m$ triples of the form $\{x, y, \beta\}$ where β is the color of an edge $\{x, y\}$ in H, so symbol x is placed $2m$ times in cells of the form $(\beta - u, \beta - u)$. Therefore, symbol x occurs an even (odd) number of times on the main diagonal of L if λ is even (odd).

In either case, the number of times that symbol x occurs on the main diagonal of L is congruent to $\lambda \pmod 2$.

We now check that L contains the appropriate number of symbols in each row as required by Theorems 2.6 and 2.7, considering the cases $(u+1)/2$ is odd or even in turn.

Suppose $(u+1)/2$ is odd. Then, for $1 \leq i \leq (u+1)/2$, row i of L contains $\lambda((u-3)/2)$ symbols. Since $r(u+i)$ is the number of triples in

T' containing symbol $u+i$, from (1) the number of symbols in cell (i,i) is $2r(u+1)-1$ if symbol i occurs in a Type a lollipop and $2r(u+i)$, otherwise. From (2) and (3), the number of times symbol ∞ occurs in row i of $L(\infty)$ is $2r(u+i)+(\lambda-1)$ if vertex i is contained in a Type a lollipop and $2r(u+i)+\lambda$, otherwise. Hence, from (1), (2), and (3), we get that $\lambda((u-1)/2)+2r(u+i)-1-(2r(u+i)+(\lambda-1)) = \lambda((u-1)/2)-\lambda = \lambda((u-3)/2)$ symbols are placed in row i of L if symbol $u+i$ is contained in a Type a lollipop; otherwise, $\lambda((u-1)/2)+2r(u+i)-(2r(u+i)+\lambda) = \lambda((u-1)/2)-\lambda = \lambda((u-3)/2)$ symbols are placed in row i of L, so each row of L contains $\lambda((u-3)/2)$ symbols if $(u+1)/2$ is odd.

Now suppose $(u+1)/2$ is even. Then $u \equiv 3 \pmod 4$. The same argument holds for the first $(u-1)/2$ rows of L, but in this case we must have that row $(u+1)/2$ of L contains $\lambda((u-1)/2)$ symbols. However, since $u \equiv 3 \pmod 4$, symbol $(u+1)/2$ occurs in λ Type b lollipops with vertex $(u+3)/2$, and these lollipops contribute no ∞ symbols to row $(u+1)/2$ of $L(\infty)$. However, as before, row $(u+1)/2$ of $L(\infty)$ contains symbol ∞ $2r(u+(u+1)/2)$ times and cell $((u+1)/2,(u+1)/2)$ of L contains $2r(u+(u+1)/2)$ symbols. Therefore, row $(u+1)/2$ of L contains $\lambda((u-1)/2)-2r(u+(u+1)/2)+2r(u+(u+1)/2) = \lambda((u-1)/2)$ symbols.

We now have the following:

(1) $N_L(i) \equiv \lambda \pmod 2$, for $1 \le i \le u$;

(2) L is symmetric;

(3) $N_L(i) \ge \lambda(u+1-(u+2)) \ge -\lambda$, for $1 \le i \le u$; and

(4a) each row of L contains $\lambda((u-3)/2)$ symbols from $\{1,\ldots,u\}$ if $(u+1)/2$ is odd, or

(4b) row j $(1 \le j \le (u-1)/2)$ of L contains $\lambda((u-3)/2)$ symbols, and row $(u+1)/2$ contains $\lambda((u-1)/2)$ symbols if $(u+1)/2$ is even.

Therefore, by Theorems 2.6 and 2.7, L can be embedded in the top left corner of a symmetric quasi-latin square L' of order $u+2$ on the symbols $1,\ldots,u$ such that cells (i,i), for $(u+3)/2 \le i \le u+2$ are empty and cells $((u+2(i+x)+1)/2,(u+2(i+x)+3)/2)$ and $((u+2(i+x)+3)/2,(u+2(i+x)+1)/2)$ are empty, where $x \equiv (u+1)/2 \pmod 2$, and $0 \le i \le (u+1)/2$. Form triples using L': if symbol w occurs in cells (y,z) and (z,y), $y \ne z$, then let $\{w,u+y,u+z\} \in \hat{B}$.

This completes the definition of $\hat{B}$, so it now remains to show that $(\hat{V},\hat{B})$ is an $ETS(2u+2,\lambda)$; that is, we must show that every pair of points of $\hat{V}$ occurs in exactly λ extended triples.

Consider edges of the form $\{a,b\}$, $a,b \le u$. Some of these edges already occurred in extended triples in B^*. Suppose $\{a,b\}$ occurred in α extended triples in B^*. Then $\{a,b\}$ occurred as an edge $\lambda - \alpha$ times in $G(B^*)$, and was therefore included in $\lambda - \alpha$ Type c triples. Since (V^*, B^*) was taken to be maximal, all loops incident with vertices a and b already occurred in B^*. Therefore, every edge of the form $\{a,b\}$ occurs in exactly λ extended triples in $\hat{B}$.

Now consider edges of the form $\{a,b\}$, where $a \leq u$ and $b \geq u+1$. Since every symbol $1,\ldots,u$ occurs exactly λ times in each row of L', $\{a,b\}$ occurs in extended triples of Type a or Type c if a is in a diagonal cell of L' and of Type e if a is in an off-diagonal cell of L'. In any event, $\{a,b\}$ occurs in λ extended triples of $\hat{B}$.

Finally, consider edges of the form $\{a,b\}$ where $a,b \geq u+1$. Each cell $(a-u, b-u)$, $a \neq b$, in L' contains β symbols from $1,\ldots,u$, where $0 \leq \beta \leq \lambda$. For every symbol in $1,\ldots,u$ found in cell $(a-u, b-u)$, $\{a,b\}$ is in a Type e triple; the remaining $\lambda - \beta$ edges joining x to y are contained in either Type b lollipops or Type d triples (corresponding to the λ Type b lollipops if (x,y) is a near-diagonal cell of L' outside L, and to the $\lambda - \beta$ copies of ∞ in cell (x,y) of $L(\infty)$ otherwise). In any event, $\{a,b\}$ is contained in exactly λ extended triples, and all loops incident with vertices $u+1,\ldots,2u+2$ are contained in λ extended triples of $\hat{B}$.

Therefore, every edge of the form $\{a,b\}$, for $1 \leq a,b \leq 2u+2$, is in exactly λ extended triples of $\hat{B}$, and the proof is complete. □

This leads to the following theorem.

Theorem 4.3 *Any partial extended triple system (V,B) of order n and index $\lambda \geq 2$ can be embedded in an extended triple system $(\hat{V},\hat{B})$ of order v and index λ for all $v \geq 4n+8$, $v \equiv 0$ (mod 4).*

Proof: First suppose $n \in \{1,2\}$. Clearly, any $PETS(1,\lambda)$ can be embedded in an $ETS(v,\lambda)$ for all $v \geq 4n+4$, $v \equiv 0 \pmod 4$ since this corresponds to the existence of such $ETS(v,\lambda)$s. If $n = 2$, consider a $PETS(2,\lambda)$ with $\epsilon(G(B)) = \lambda - x$. We can think of this $PETS(2,\lambda)$ as $(\lambda - x)$ $PETS(2,1)$s and x $ETS(2,1)$s. By [12] each of the $PETS(2,1)$s and $ETS(2,1)$s can be embedded in an $ETS(v,1)$ for all $v \geq 4n+4$, $v \equiv 0 \pmod 4$. If we put these λ $ETS(v,1)$s together, we have the desired $ETS(v,\lambda)$.

Now suppose $n \geq 3$ and $k \geq 0$. Embed (V,B) in a maximal $PETS(n+k,\lambda)$ (V_1,B_1). If $n+k$ is odd, then by Lemma 4.1 (V_1,B_1) can be embedded in a $PETS(2(n+k)+1,\lambda)$ or a $PETS(2(n+k)+3,\lambda)$ satisfying conditions (i)–(iv). Also, if $n+k$ is even, then by Lemma 4.1 (V_1,B_1) can be embedded in a $PETS(2(n+k)+3,\lambda)$ or a $PETS(2(n+k)+5,\lambda)$ satisfying conditions (i)–(iv). In either case, these $PETS$s can be embedded in an $ETS(4(n+k)+4,\lambda)$ or an $ETS(4(n+k)+8,\lambda)$ if $n+k$ is odd and in an $ETS(4(n+k)+8,\lambda)$ or an $ETS(4(n+k)+12,\lambda)$ if $n+k$ is even. Therefore, we have the desired result and the proof is complete. □

We now have the following corollary.

Corollary 4.4 *Any partial extended triple system of order n and index $\lambda \geq 2$ can be embedded in an extended triple system of order v and index λ for all even $v \geq 4n+6$.*

Proof: This follows directly from Theorem 3.3 and Theorem 4.3. □

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