Bounds on the Size of Critical Edge-Chromatic Graphs

Dawit Haile
Department of Mathematics
Virginia State University
Petersburg, VA 23806

Abstract

By Vizing's theorem, the chromatic index $\chi'(G)$ of a simple graph G satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$; if $\chi'(G) = \Delta(G)$, then G is class 1, otherwise G is class 2. A graph G is called critical edge-chromatic graph if G is connected, class 2 and $\chi'(H) < \chi'(G)$ for all proper subgraphs H of G. We give new lower bounds for the size of Δ-critical edge-chromatic graphs, for $\Delta \geq 9$.

1 Introduction

All graphs we consider are undirected and have neither loops nor multiple edges. We denote the vertex set of a graph G by $V(G)$ and the edge set by $E(G)$. The order of G is $|V(G)|$ and the size of G is $|E(G)|$. We denote the degree of a vertex v in G by $d_G(v)$ and the maximum degree of G by $\Delta(G)$. A vertex of maximum degree is called a major vertex; otherwise it is a minor vertex. The number of vertices in G of degree k is denoted by $n_k = n_k(G)$. For disjoint subsets A, B of $V(G)$, $E(A, B)$ denotes the set of edges one end of which is in A and the other end of which is in B. We write $e(A, B)$ for $|E(A, B)|$ and $e(v, B)$ for $e(\{v\}, B)$. A planar graph is a graph which can be embedded in the plane in such a way that no two edges intersect geometrically except at a vertex to which they are both incident. If a connected graph G is embedded in the plane in this way, it is called a plane graph. The points of the plane not on G are partitioned into a number of connected regions. The closures of these regions are called the faces of G and the number of such faces is denoted by f. Our notation and terminology generally follows [1].

The chromatic index $\chi'(G)$ of a graph G is the minimum number of colors required to color the edges of G so that no two adjacent edges receive the same color. Vizing [8] showed that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. A graph G is of class 1 provided that $\chi'(G) = \Delta(G)$, and G is of class 2 provided that $\chi'(G) = \Delta(G)+1$. We say G is Δ-critical if and only if $\Delta(G) = \Delta$, G is connected, G is of class 2 and $\chi'(G-e) < \chi'(G)$ for every edge e of G. For a Δ-critical graph G of order n and size m, Vizing [8] conjectured that $m \geq \frac{1}{2}[(\Delta-1)n+3)]$. Jakobsen [6] showed that $m \geq \frac{4}{3}n$ for $\Delta = 3$, thus verifying the conjecture. Fiorini and Wilson [4] showed that

$$m \geq \begin{cases} \frac{5}{3}n & \text{if } \Delta = 4, \\ \frac{9}{5}n & \text{if } \Delta = 5, \\ 2n & \text{if } \Delta = 6, \end{cases}$$

thus verifying the conjecture for $\Delta = 4$. Yap [10] further improved these results by showing

$$m \geq \begin{cases} 2n+1 & \text{if } \Delta = 5, \\ \frac{1}{4}(9n+1) & \text{if } \Delta = 6, \\ \frac{5}{2}n & \text{if } \Delta = 7. \end{cases}$$

Kayathri [7] improved this results of Yap by establishing the nonexistence of a 5-critical graph of size $2n+1$, thus verifying the conjecture. In general, the best bounds are due to Fiorini [3] who showed that

$$m \geq \begin{cases} \frac{1}{4}(\Delta+1)n & \text{if } \Delta \text{ is odd}, \\ \frac{1}{4}(\Delta+2)n & \text{if } \Delta \text{ is even}. \end{cases}$$

Recently, Clark and Haile [2] further improved the bounds of Yap [10] and Fiorini [3] by showing that

$$m \geq f(\Delta)n$$

where

$$f(\Delta) = \begin{cases} \frac{\Delta+1}{3} & \text{for } 6 \leq \Delta \leq 8, \\ \frac{\Delta+4}{4} & \text{for } 9 \leq \Delta \leq 12, \\ \frac{3\Delta+20}{14} & \text{for } 13 \leq \Delta \leq 16, \\ \frac{3\Delta+30}{16} & \text{for } 17 \leq \Delta \leq 21. \end{cases}$$

In this paper we give new lower bounds that will improve all known lower bounds on the size of Δ-critical graphs for $\Delta \geq 9$.

2 Preliminary Results

We give here some further notations and preliminary results we use, some of which are well-known.

Theorem 2.1 [4, 8] *Let G be a Δ-critical graph and let $vw \in E(G)$ where $d_G(v) = k$. Then*
(i) if $k < \Delta$ then w is adjacent to at least $\Delta - k + 1$ major vertices of G.
(ii) if $k = \Delta$ then w is adjacent to at least two major vertices of G. □

Theorem 2.1 is known as the Vizing Adjacency Lemma (VAL).

Theorem 2.2 [4, 11] *A critical graph contains no cut vertex. Moreover, there are no regular Δ-critical graphs for $\Delta \geq 3$.* □

We now fix a Δ-critical graph G with $\Delta \geq 3$. Let

$$A_k = \{v \in V(G) \ : \ d_G(v) = k\} \ \text{ for } 2 \leq k \leq \Delta; \ \ |A_k| = n_k.$$

$$A_{kl} = \{v \in A_k \ : \ e(v, A_\Delta) = l\} \ \text{ for } 2 \leq l \leq k \leq \Delta - 1; \ \ |A_{kl}| = a_{kl}.$$

For $2 \leq k \leq \Delta - 1$, VAL gives $\displaystyle\sum_{l=2}^{k} a_{kl} = n_k$.

Now we may restate VAL(i) in terms of the notation introduced above.

Lemma 2.3 *If $v \in A_k$ with $2 \leq k \leq \Delta - 1$ and $vw \in E(G)$, then w is adjacent to at least $\Delta - k + 1$ major vertices, hence,*

$$w \in A_\Delta \cup \bigcup_{\Delta-1 \geq q > r \geq \Delta-k+1} A_{qr}.$$ □

The following two results are corollaries of Lemma 2.3.

Lemma 2.4 *The sets A_{k2} for $2 \leq k \leq \Delta - 2$ are independent.* □

Lemma 2.5 *The sets A_k for $2 \leq k \leq \lceil \frac{\Delta}{2} \rceil$ are independent.* □

Using Lemma 2.3 and Lamma 2.5, we obtain

Corollary 2.6 *For $0 \leq j \leq \lceil \frac{\Delta}{2} \rceil - 3$, we have*

$$\sum_{k=3}^{\lceil \frac{\Delta}{2} \rceil - j} \sum_{l=2}^{k-1} (k-l) a_{kl} \leq \begin{cases} \displaystyle\sum_{p=\frac{\Delta}{2}+j+2}^{\Delta-1} \ \sum_{q=\frac{\Delta}{2}+j+1}^{p-1} (p-q) a_{pq} & \textit{if } \Delta \textit{ is even;} \\ \displaystyle\sum_{p=\frac{\Delta+1}{2}+j+1}^{\Delta-1} \ \sum_{q=\frac{\Delta+1}{2}+j}^{p-1} (p-q) a_{pq} & \textit{if } \Delta \textit{ is odd.} \end{cases}$$ □

The following result can be read out of the proof of Theorem 13.5 of [4]. We give a proof that is presented in [2]. See [5] for an entirely different proof.

Lemma 2.7

$$n_\Delta \geq \sum_{k=2}^{\Delta-1}\sum_{l=2}^{k}\frac{la_{kl}}{k-1}. \tag{2.1}$$

Proof. For $v \in A_\Delta$, consider $(v_2,\ldots,v_{\Delta-1})$ where $v_k = e(v, A_k)$ for $2 \leq k \leq \Delta-1$. Let $T = \{(v_2,\ldots,v_{\Delta-1}) : v \in A_\Delta\}$; $T^* = T - \{(0,\ldots,0)\}$; $B(i_2,\ldots,i_{\Delta-1}) = \{v \in A_\Delta : (v_2,\ldots,v_{\Delta-1}) = (i_2,\ldots,i_{\Delta-1})\}$ and $b(i_2,\ldots,i_{\Delta-1}) = |B(i_2,\ldots,i_{\Delta-1})| (\neq 0)$ for $(i_2,\ldots,i_{\Delta-1}) \in T$. Observe that $\{B(i_2,\ldots,i_{\Delta-1}) : (i_2,\ldots,i_{\Delta-1}) \in T\}$ partitions A_Δ. For each $2 \leq k \leq \Delta-1$,

$$\sum_{l=2}^{k} la_{kl} = e(A_\Delta, A_k) = \sum_{(i_2,\ldots,i_{\Delta-1})\in T^*} i_k b(i_2,\ldots,i_{\Delta-1})$$

hence,

$$\begin{aligned}\sum_{k=2}^{\Delta-1}\sum_{l=2}^{k}\frac{la_{kl}}{k-1} &= \sum_{k=2}^{\Delta-1}\sum_{(i_2,\ldots,i_{\Delta-1})\in T^*}\frac{i_k b(i_2,\ldots,i_{\Delta-1})}{k-1}\\ &= \sum_{(i_2,\ldots,i_{\Delta-1})\in T^*} b(i_2,\ldots,i_{\Delta-1})\sum_{k=2}^{\Delta-1}\frac{i_k}{k-1}.\end{aligned}$$

Fix $(i_2,\ldots,i_{\Delta-1}) \in T^*$, and let $q = q(i_2,\ldots,i_{\Delta-1}) = \min\{k : i_k \neq 0\}$, so that, $2 \leq q \leq \Delta-1$. Observe that for $v \in B(i_2,\ldots,i_{\Delta-1})$ with $i_q \neq 0$, there exists $vw \in E(G)$ with $w \in A_q$. By VAL(i), v is adjacent to at least $\Delta - q + 1$ major vertices, so, at most $q-1$ minor vertices. Hence, $i_q + \cdots + i_{\Delta-1} = i_2 + \cdots + i_{\Delta-1} \leq q-1$. Consequently,

$$\begin{aligned}\sum_{k=2}^{\Delta-1}\sum_{l=2}^{k}\frac{la_{kl}}{k-1} &\leq \sum_{(i_2,\ldots,i_{\Delta-1})\in T^*} b(i_2,\ldots,i_{\Delta-1}) \sum_{k=q=q(i_2,\ldots,i_{\Delta-1})}^{\Delta-1}\frac{i_k}{q-1}\\ &\leq \sum_{(i_2,\ldots,i_{\Delta-1})\in T^*} b(i_2,\ldots,i_{\Delta-1}) \leq n_\Delta.\end{aligned}$$ □

Using VAL and Lemma 2.4 we obtain the following result which is proved in [2].

Theorem 2.8 *For $\Delta \geq 5$, we have*

$$n_\Delta \geq 2n_2 + 3\sum_{k=3}^{\Delta-2}\frac{n_k}{k-1} + \frac{n_{\Delta-1}}{\Delta-3}.$$ □

3 Main Result

Observe that in the expression $m = \frac{1}{2}\sum_{k=2}^{\Delta} kn_k$, the coefficient of $n_{\Delta-1}$ gives the coefficient of n in Vizing's conjecture. So we may use the coefficient of $n_{\Delta-1}$ in (2.1) to increase the coefficients of some of the smaller minor vertices and obtain the following result.

Theorem 3.9 *For $\Delta \geq 9$, we have*

$$n_\Delta \geq 2n_2 + \sum_{k=3}^{\lfloor\sqrt{\Delta-1}+2\rfloor} \frac{k}{k-1}n_k \quad + \sum_{k=\lfloor\sqrt{\Delta-1}+2\rfloor+1}^{\lceil\frac{\Delta}{2}\rceil} \left[\frac{2}{k-1} + \frac{\Delta-k+1}{\Delta-2}\right] n_k$$
$$+ \sum_{\lceil\frac{\Delta}{2}\rceil+1}^{\Delta-3} \frac{2}{k-1}n_k + \frac{1}{(\Delta-3)}n_{\Delta-2}.$$

Proof. Suppose Δ is even. Using Corollary (2.6), we obtain the following system of $\Delta/2-2$ linear inequalities

$$\begin{aligned}
&a_{32} + a_{43} + 2a_{42} + \cdots + a_{\frac{\Delta}{2}(\frac{\Delta}{2}-1)} + 2a_{\frac{\Delta}{2}(\frac{\Delta}{2}-2)} + \cdots + (\Delta/2-2)a_{\frac{\Delta}{2}2}\\
&\quad\leq a_{(\frac{\Delta}{2}+2)(\frac{\Delta}{2}+1)} + a_{(\frac{\Delta}{2}+3)(\frac{\Delta}{2}+2)} + 2a_{(\frac{\Delta}{2}+3)(\frac{\Delta}{2}+1)} + \cdots + a_{(\Delta-1)(\Delta-2)}\\
&\qquad+2a_{(\Delta-1)(\Delta-3)}+\cdots+(\Delta/2-2)a_{(\Delta-1)(\frac{\Delta}{2}+1)} \qquad (1)\\
&a_{32}+a_{43}+2a_{42}+\cdots+a_{(\frac{\Delta}{2}-1)(\frac{\Delta}{2}-2)}+2a_{(\frac{\Delta}{2}-1)(\frac{\Delta}{2}-3)}+\cdots+(\Delta/2-3)a_{(\frac{\Delta}{2}-1)2}\\
&\quad\leq a_{(\frac{\Delta}{2}+3)(\frac{\Delta}{2}+2)} + a_{(\frac{\Delta}{2}+4)(\frac{\Delta}{2}+3)} + 2a_{(\frac{\Delta}{2}+4)(\frac{\Delta}{2}+2)} + \cdots + a_{(\Delta-1)(\Delta-2)}\\
&\qquad+2a_{(\Delta-1)(\Delta-3)}+\cdots+(\Delta/2-3)a_{(\Delta-1)(\frac{\Delta}{2}+2)} \qquad (2)\\
&\qquad\vdots\\
&a_{32}+a_{43}+2a_{42} \leq a_{(\Delta-2)(\Delta-3)}+a_{(\Delta-1)(\Delta-2)}+2a_{(\Delta-1)(\Delta-3)} \qquad (\tfrac{\Delta}{2}-3)\\
&a_{32} \leq a_{(\Delta-1)(\Delta-2)} \qquad (\tfrac{\Delta}{2}-2)
\end{aligned}$$

Notice that the coefficient of $a_{(\Delta-1)l}$ in (2.1) is $\frac{l}{\Delta-2}$. Now we like to choose suitable multipliers so that in taking a linear combination of the above $\Delta/2-2$ inequalities, the coefficient of $a_{(\Delta-1)l}$, for each $\frac{\Delta}{2}+1 \leq l \leq \Delta-2$, does not exceed $\frac{l}{\Delta-2}$. This can be done by multiplying both sides of inequality (1) with

$$\frac{\Delta/2+1}{(\Delta-2)(\Delta/2-2)} = \frac{\Delta+2}{(\Delta-2)(\Delta+4)}$$

and both sides of inequality (j), for $2 \le j \le \frac{\Delta}{2} - 2$, with

$$\frac{1}{\Delta-2}\left(\frac{\Delta+2j}{\Delta-2j-2} - \frac{\Delta+2j-2}{\Delta-2j}\right)$$

Then taking the sum of the resulting inequalities, we obtain

$$\sum_{k=3}^{\frac{\Delta}{2}}\sum_{l=2}^{k-1}\left[\tfrac{\Delta+2}{(\Delta-2)(\Delta-4)} + \tfrac{1}{\Delta-2}\sum_{j=2}^{\frac{\Delta}{2}+1-k}\left(\tfrac{\Delta+2j}{\Delta-2j-2} - \tfrac{\Delta+2j-2}{\Delta-2j}\right)\right](k-l)a_{kl}$$

$$\le \sum_{p=\frac{\Delta}{2}+2}^{\Delta-1}\sum_{q=\frac{\Delta}{2}+1}^{p-1} \tfrac{q(p-q)}{(\Delta-2)(\Delta-q-1)}a_{pq} \tag{3.2}$$

But for any $r \ge 2$ we have

$$\sum_{j=2}^{r}\left(\frac{\Delta+2j}{\Delta-2j-2} - \frac{\Delta+2j-2}{\Delta-2j}\right) = \frac{\Delta+2r}{\Delta-2r-2} - \frac{\Delta+2}{\Delta-4}. \tag{3.3}$$

Using (3.3) in (3.2), we get

$$\sum_{k=3}^{\frac{\Delta}{2}}\sum_{l=2}^{k-1}\tfrac{(\Delta-k+1)(k-l)}{(k-2)(\Delta-2)}a_{kl} \le \sum_{p=\frac{\Delta}{2}+2}^{\Delta-1}\sum_{q=\frac{\Delta}{2}+1}^{p-1}\tfrac{q(p-q)}{(\Delta-2)(\Delta-q-1)}a_{pq}. \tag{3.4}$$

For $\frac{\Delta}{2}+1 \le q \le p-1 \le \Delta-4$,

$$\begin{aligned}\frac{q}{p-1} - \frac{q(p-q)}{(\Delta-2)(\Delta-q-1)} &= \frac{q(\Delta+p-q-2)(\Delta-p-1)}{(p-1)(\Delta-2)(\Delta-q-1)}\\ &\ge \frac{4q}{(p-1)(\Delta-4)}\\ &\ge \left(2+\frac{12}{\Delta-4}\right)\frac{1}{p-1}.\end{aligned} \tag{3.5}$$

For $\frac{\Delta}{2}+1 \le q \le p-1 = \Delta-3$, we have

$$\frac{q}{p-1} - \frac{q(p-q)}{(\Delta-2)(\Delta-q-1)} \ge \frac{1}{(\Delta-3)}. \tag{3.6}$$

Now using (3.4), (3.5) and (3.6) in (2.1) we obtain

$$\begin{aligned} n_\Delta \ge 2n_2 &+ \sum_{k=3}^{\frac{\Delta}{2}}\sum_{l=2}^{k-1}\left[\frac{l}{k-1} + \frac{(\Delta-k+1)(k-1)}{(k-2)(\Delta-2)}\right]a_{kl} + \sum_{k=3}^{\frac{\Delta}{2}}\frac{k}{k-1}a_{kk}\\ &+ \sum_{k=\frac{\Delta}{2}+1}^{\Delta-3}\frac{2}{k-1}n_k + \frac{1}{(\Delta-3)}n_{\Delta-2}.\end{aligned}$$

If Δ is odd, by a similar argument, we obtain

$$\begin{aligned} n_\Delta \geq 2n_2 &+ \sum_{k=3}^{\frac{\Delta+1}{2}} \sum_{l=2}^{k-1} \left[\frac{l}{k-1} + \frac{(\Delta-k+1)(k-1)}{(k-2)(\Delta-2)} \right] a_{kl} + \sum_{k=3}^{\frac{\Delta+1}{2}} \frac{k}{k-1} a_{kk} \\ &+ \sum_{k=\frac{\Delta+1}{2}+1}^{\Delta-3} \frac{2}{k-1} n_k + \frac{1}{(\Delta-3)} n_{\Delta-2}. \end{aligned}$$

Consequently, for any Δ, we have

$$\begin{aligned} n_\Delta \geq 2n_2 &+ \sum_{k=3}^{\lceil\frac{\Delta}{2}\rceil} \sum_{l=2}^{k-1} \left[\frac{l}{k-1} + \frac{(\Delta-k+1)(k-l)}{(k-2)(\Delta-2)} \right] a_{kl} + \sum_{k=3}^{\lceil\frac{\Delta}{2}\rceil} \frac{k}{k-1} a_{kk} \\ &+ \sum_{k=\lceil\frac{\Delta}{2}\rceil+1}^{\Delta-3} \frac{2}{k-1} n_k + \frac{1}{(\Delta-3)} n_{\Delta-2}. \end{aligned}$$

For a fixed k, $3 \leq k \leq \lceil\frac{\Delta}{2}\rceil$,

$$\begin{aligned} &\min_{2\leq l\leq k-1} \left\{ \frac{l}{k-1} + \frac{(\Delta-k+1)(k-l)}{(k-2)(\Delta-2)} \right\} \\ &= \begin{cases} 1 + \dfrac{\Delta-k+1}{(k-2)(\Delta-2)} & \text{if } 3 \leq k \leq \lfloor\sqrt{\Delta-1}+2\rfloor; \\ \dfrac{2}{k-1} + \dfrac{\Delta-k+1}{\Delta-2} & \text{if } \lceil\sqrt{\Delta-1}+2\rceil \leq k \leq \lceil\frac{\Delta}{2}\rceil. \end{cases} \end{aligned}$$

Now as $1+\frac{\Delta-k+1}{(k-2)(\Delta-2)} \geq \frac{k}{k-1}$ for $3 \leq k \leq \lfloor\sqrt{\Delta-1}+2\rfloor$, and $\frac{2}{k-1}+\frac{\Delta-k+1}{\Delta-2} \leq \frac{k}{k-1}$ for $\lfloor\sqrt{\Delta-1}+2\rfloor+1 \leq k \leq \lceil\frac{\Delta}{2}\rceil$, the result of the theorem follows. □

Using the result of Theorem 3.9, we obtain

$$\begin{aligned} 2m &= \sum_{k=2}^{\Delta} kn_k = \sum_{k=2}^{\Delta-1} kn_k + cn_\Delta + (\Delta-c)n_\Delta \\ &\geq \sum_{k=2}^{\Delta-1} kn_k + cn_\Delta + (\Delta-c)\Bigg[2n_2 + \sum_{k=3}^{\lfloor\sqrt{\Delta-1}+2\rfloor} \frac{k}{k-1} n_k \\ &\qquad + \sum_{k=\lfloor\sqrt{\Delta-1}+2\rfloor+1}^{\lceil\frac{\Delta}{2}\rceil} \left(\frac{2}{k-1} + \frac{\Delta-k+1}{\Delta-2} \right) n_k \\ &\qquad + \sum_{k=\lceil\frac{\Delta}{2}\rceil+1}^{\Delta-3} \frac{2}{k-1} n_k + \frac{1}{\Delta-3} n_{\Delta-2} \Bigg] \end{aligned}$$

$$= [2+2(\Delta - c)]n_2 + \sum_{k=3}^{\lfloor \sqrt{\Delta-1}+2 \rfloor} \left[k + \frac{k}{k-1}(\Delta - c)\right] n_k$$
$$+ \sum_{k=\lfloor \sqrt{\Delta-1}+2\rfloor+1}^{\lceil \frac{\Delta}{2} \rceil} \left[k + \left(\frac{2}{k-1} + \frac{\Delta-k+1}{\Delta-2}\right)(\Delta - c)\right] n_k$$
$$+ \sum_{k=\lceil \frac{\Delta}{2}\rceil+1}^{\Delta-3} \left[k + \frac{2}{k-1}(\Delta - c)\right] n_k + \left[\Delta - 2 + \frac{\Delta - c}{\Delta - 3}\right] n_{\Delta-2}$$
$$+(\Delta-1)n_{\Delta-1} + cn_\Delta. \tag{3.7}$$

The next lemma is proved in [2].

Lemma 3.10 *Let $f_2(c)$, $f_3(c), \ldots, f_{\Delta-1}(c)$ be positive, decreasing linear functions on* $[0,\ \Delta]$ *and $f_\Delta(c) = c$. Set $g(c) = \min_{2\le k\le \Delta}\{f_k(c)\}$ for $c \in [0,\ \Delta]$. Then g is continuous on* $[0,\ \Delta]$ *and*
$$\max_{0\le c\le\Delta} g(c) = \min_{2\le k\le\Delta-1}\{c : f_k(c) = f_\Delta(c)\}. \qquad \square$$

By Lemma 3.10, the value of c that gives an optimal value for the coefficient of n in (3.7) is $\min\{c(k) : 2 \le k \le \Delta - 2\}$, where $c(k)$ is the point where $f_k(c) = c$ with

$f_2(c) = 2(\Delta + 1 - c)$;

$f_k(c) = \frac{k[k+\Delta-1-c]}{k-1}$ if $3 \le k \le \lfloor\sqrt{\Delta-1}+2\rfloor$;

$f_k(c) = k + \left(\frac{2}{k-1} + \frac{\Delta-k+1}{\Delta-2}\right)(\Delta - c)$ if $\lfloor\sqrt{\Delta-1}+2\rfloor + 1 \le k \le \lceil\frac{\Delta}{2}\rceil$;

$f_k(c) = k + \frac{2}{k-1}(\Delta - c)$ if $\lceil\frac{\Delta}{2}\rceil + 1 \le k \le \Delta - 3$;

$f_{\Delta-2}(c) = \frac{\Delta^2-4\Delta+6-c}{\Delta-3}$.

Moreover,

$$c(2) = \frac{2(\Delta+1)}{3} \tag{3.8}$$
$$c(k) = \frac{k[k+\Delta-1]}{2k-1} \quad \text{if } 3 \le k \le \lfloor\sqrt{\Delta-1}+2\rfloor \tag{3.9}$$
$$c(k) = \frac{k[\Delta^2+\Delta+2] - 2k^2 + \Delta^2 - 5\Delta}{2k\Delta - k^2 - 3} \quad \text{if } \lfloor\sqrt{\Delta-1}+2\rfloor + 1 \le k \le \lceil\tfrac{\Delta}{2}\rceil \tag{3.10}$$
$$c(k) = \frac{k^2 - k + 2\Delta}{k+1} \quad \text{if } \lceil\tfrac{\Delta}{2}\rceil + 1 \le k \le \Delta - 3 \tag{3.11}$$

$$c(\Delta-2) = \frac{\Delta^2-4\Delta+6}{\Delta-2} \qquad (3.12)$$

Letting c_2, c_3, c_4, c_5 and $c_{\Delta-2}$ to be the minimum values of the functions in (3.8) to (3.12), respectively, we obtain $c_2 = (2\Delta+2)/3$,

$$c_3 = \begin{cases} \frac{3}{5}(\Delta+2) & \text{if } 9 \le \Delta \le 13; \\ \frac{1}{4}(\sqrt{2\Delta-1}+1)^2 & \text{if } \Delta \ge 14, \end{cases}$$

$$c_4 \ge (\Delta+2)/2 + \sqrt{(\Delta-2)^3/2(\Delta^2-3)},$$

$$c_5 = \begin{cases} \dfrac{\Delta(\Delta+10)}{2(\Delta+4)} & \text{if } \Delta \text{ is even;} \\ \dfrac{\Delta^2+12\Delta+3}{2(\Delta+5)} & \text{if } \Delta \text{ is odd,} \end{cases}$$

and $c_{\Delta-2} = (\Delta^2 - 4\Delta + 6)/(\Delta - 2)$. Analysis of each $\Delta \in \{9, 11, 13\}$ gives $c = c_3 = \frac{3}{5}(\Delta+2)$. Observe that $c_{\Delta-2} > c_2 > c_5$ for $\Delta \ge 10$. For $\Delta \ge 10$ & $\Delta \ne 11, 13, 15$, we have $c_3 > c_5$ and $c_4 > c_5$. Moreover, at $\Delta = 15$, we have $c_4 > c_5 > c_3$. Consequently, we have the following result.

Theorem 3.11 *Let G be a Δ-critical graph of order n and size m. Then*

$$m \ge f(\Delta)n$$

where

$$f(\Delta) = \begin{cases} \frac{3}{10}(\Delta+2) & \text{for } \Delta = 9, 11, 13; \\ \frac{1}{8}(\sqrt{2\Delta-1}+1)^2 & \text{for } \Delta = 15; \\ \dfrac{\Delta(\Delta+10)}{4(\Delta+4)} & \text{for } \Delta \ge 10,\ \Delta \text{ is even;} \\ \dfrac{\Delta^2+12\Delta+3}{4(\Delta+5)} & \text{for } \Delta \ge 17,\ \Delta \text{ is odd.} \end{cases}$$ □

Vizing [9] conjectured that a simple planar graph of maximum degree equal to 6 or 7 is class one. On the assumption that it is not easy to prove this conjecture, various restriction on graphs were considered to solve the problem at least partially. One such a result is due to Yap [12] who proved that if a 6- or 7-critical planar graphs G exist, then G has quite a few minor vertices. In fact, he showed that

Theorem 3.12 [12] *(i) If G is a 7-critical graph, then*

$$2n_3 + \frac{4}{3}n_4 + \frac{1}{2}n_5 \ge 12 + \frac{2}{5}n_6 \quad \& \quad n_7 \ge 6 + 2n_2 + \frac{1}{4}n_5 + \frac{3}{5}n_6$$

(ii) If G is a 6-critical graph, then

$$2n_2 + 3n_3 + 2n_4 + n_5 \geq 12 \qquad \& \qquad n_6 \geq 4 + \frac{4}{3}n_2 + \frac{1}{6}n_5.$$

Using VAL and Theorem 2.8, we slightly improve the results of Theorem 3.12 and prove that if a 7-critical planar graph G exists then the number of major vertices of G is at least 12.

Theorem 3.13 *Let G be a 7-critical planar graph. Then*

$$\frac{3}{2}n_3 + n_4 + \frac{1}{4}n_5 \geq 12 + \frac{1}{4}n_6 \quad \textit{and} \quad n_7 \geq 12 + 2n_2 + \frac{1}{2}n_5 + \frac{1}{2}n_6.$$

Proof. Clearly, $G \setminus A_2$ is planar. Every vertex in A_2, by VAL, is adjacent to exactly two major vertices in G. Thus in deleting a vertex of A_2 we lose a face and two edges. Now since each face in $G \setminus A_2$ is bounded by at least three edges and since each edge bounds at most two faces, we have

$$3(f - n_2) \leq 2(m - 2n_2) \quad \Longleftrightarrow \quad 3f \leq 2m - n_2. \tag{3.13}$$

Using (3.13), VAL and the Euler's Polyhedral formula we obtain

$$\begin{aligned} 12 + 6m - 6n &\leq 4m - 2n_2 \\ \Longrightarrow 12 + \sum_{k=2}^{7} kn_k - \sum_{k=2}^{7} 6n_k &\leq -2n_2 \\ \Longrightarrow 12 + n_7 &\leq 2n_2 + 3n_3 + 2n_4 + n_5. \end{aligned}$$

Then using Theorem 2.8, we have

$$\begin{aligned} n_7 &\geq 2n_2 + \frac{3}{2}n_3 + n_4 + \frac{3}{4}n_5 + \frac{1}{4}n_6 \\ \Longrightarrow 2n_2 + 3n_3 + 2n_4 + n_5 &\geq 12 + 2n_2 + \frac{3}{2}n_3 + n_4 + \frac{3}{4}n_5 + \frac{1}{4}n_6 \\ \Longrightarrow \frac{3}{2}n_3 + n_4 + \frac{1}{4}n_5 &\geq 12 + \frac{1}{4}n_6, \end{aligned}$$

and hence

$$n_7 \geq 2n_2 + \frac{3}{2}n_3 + n_4 + \frac{3}{4}n_5 + \frac{1}{4}n_6 \geq 12 + 2n_2 + \frac{1}{2}n_5 + \frac{1}{2}n_6. \qquad \square$$

For $\Delta = 7$, by VAL the sets A_3, A_4 and A_{52} are independent. Using the notations introduced in section 2, we have

$$\begin{aligned} a_{32} + 2a_{42} + a_{43} + 3a_{52} &\leq a_{65} + 2a_{64} + 3a_{63} + a_{54} \\ a_{32} + 2a_{42} + a_{43} &\leq a_{65} + 2a_{64} + a_{54} \\ a_{32} &\leq a_{65} \end{aligned}$$

Now we use an argument similar to the one used in the proof of Theorem 3.9. Multiply the above three inequalities by α, β and γ, respectively. We choose these three multipliers in such a way that $1+\alpha+\beta+\gamma = \frac{3}{2}$, $\frac{2}{3}+2\alpha+2\beta = \frac{4}{3}$, $1+\alpha+\beta = \frac{4}{3}$, $\frac{1}{2}+3\alpha = 1-\alpha-\beta$, and moreover $1-\alpha-\beta-\gamma \geq 0$, $\frac{4}{5}-2\alpha-2\beta \geq 0$, $\frac{3}{5}-3\alpha \geq 0$. Thus, we have $\alpha = \frac{1}{18}$, $\beta = \frac{5}{18}$ and $\gamma = \frac{1}{6}$. Using these values and taking the linear combination of the above three inequalities, we get

$$\frac{1}{2}a_{32} + \frac{2}{3}a_{42} + \frac{1}{3}a_{43} + \frac{1}{6}a_{52} \leq \frac{1}{2}a_{65} + \frac{2}{3}a_{64} + \frac{1}{6}a_{63} + \frac{1}{3}a_{54} \qquad (3.14)$$

Now using (3.14) in (2.1), we have

$$n_7 \geq 2n_2 + \frac{3}{2}n_3 + \frac{4}{3}n_4 + \frac{2}{3}n_5 + \frac{2}{15}n_6.$$

By an argument similar to the one used in the proof of Theorem 3.13, we therefore obtain

$$\frac{3}{2}n_3 + \frac{2}{3}n_4 + \frac{1}{3}n_5 \geq 12 + \frac{2}{15}n_6 \quad \text{and} \quad n_7 \geq 12 + 2n_2 + \frac{2}{3}n_4 + \frac{1}{3}n_5 + \frac{4}{15}n_6.$$

Corollary 3.14 *If G is a 7-critical planar graph, then*

$$n_3 + n_4 + n_5 \geq 8 \qquad \textit{and} \qquad n_7 \geq 12 + 2n_2. \qquad \square$$

Similarly we have

Theorem 3.15 *If G is a 6-critical planar graph, then*

$$n_6 \geq 5 + \frac{1}{3}n_2 + \frac{1}{4}n_3 + \frac{1}{6}n_4. \qquad \square$$

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