The Existence of Self-Conjugate Self-Orthogonal Idempotent Diagonal Latin Squares

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ABSTRACT. In this paper we prove that there exists an $SCSOIDLS(v)$ if and only if $v \equiv 0, 1 \pmod 4$ other than $v = 5$, with 40 possible exceptions.

1 Introduction

A Latin square of order n is an $n \times n$ array such that every row and every column is a permutation of an n-set N. A transversal in a Latin square is a set of positions, one per row and one per column, among which the symbols occur precisely once each. A diagonal Latin square is a Latin square whose main diagonal and back diagonal are both transversals.

A Latin square is called *idempotent* if its leading diagonal is $(1, 2, \ldots, n)$. Clearly any diagonal Latin square can be converted into an idempotent diagonal Latin square by an appropriate permutation of the symbol names.

Two Latin squares of order n are orthogonal if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. A Latin square is self-orthogonal if it is orthogonal to its transpose. Clearly the main diagonal of a self-orthogonal Latin square can contain no repetitions, so a self-orthogonal Latin square can also be assumed to be idempotent (up to symbol permutation).

In an earlier paper, Danhof, Phillips and Wallis [5] considered a special type of self-orthogonal idempotent diagonal Latin square, one which is self-conjugate.

Given an orthogonal pair A, B of Latin squares of order n, we define the conjugate pair A^*, B^* as follows:

$$\text{for } i,j \in N, \quad A^*(A(i,j), B(i,j)) = i \text{ and } B^*(A(i,j), B(i,j)) = j.$$

A^*, B^* is again an orthogonal pair and its conjugate pair is A, B. Thus forming the conjugate pair is an involutory operation. If A is self-orthogonal, we define the conjugate of A to be A^* where A^*, $(A^T)^*$ is the conjugate pair of A, A^T. In this case we have the simpler formula $A^*(A(i,j), A(j,i)) = i$, and $(A^T)^* = (A^*)^T$. We call A self-conjugate if $A = A^*$, so that $A(A(i,j), A(j,i)) = i$ for all i and j. An example of a self-conjugate self-orthogonal Latin square is

$$\begin{matrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4. \end{matrix}$$

Any Latin square can be interpreted as the multiplication table of a quasigroup. A *Schroeder quasigroup* is one with the property that $(xy)(yx) = x$ for all x and y. This property is equivalent to the self-conjugacy condition above, so a $SCSOIDLS(n)$ is equivalent to a special type of Schroeder quasigroup.

Every Schroeder quasigroup has order congruent to 0 or 1 (mod 4) [7]. So

Theorem 1.1. *If there exists an $SCSOIDLS(v)$, then $v \equiv 0$ or 1 (mod 4).*

It is shown in [7] that there is no Schroeder quasigroup of order 5 and no idempotent Schroeder quasigroup of order 9, so there is no $SCSOIDLS(5)$ or $SCSOIDLS(5)$. It was conjectured in [5] that there is no self-conjugate self-orthogonal idempotent diagonal Latin square of order v for any $v \equiv 1$ (mod 4). In [6] the first author gave a self-conjugate self-orthogonal idempotent diagonal Latin square of order 25 and so showed that the conjecture is false. The purpose of this paper is to prove that there is a self-conjugate self-orthogonal idempotent diagonal Latin square of order v for any $v > 5$ and $v \equiv 1$ (mod 4) with 33 possible exceptions. At the same time we also consider the case $v \equiv 0$ (mod 4) and prove that there is a self-conjugate self-orthogonal idempotent diagonal Latin square of order v for any $v \equiv 0$ (mod 4) with 7 possible exceptions. We have therefore shown that the necessary condition is also sufficient for all $v \equiv 0$ or 1 (mod 4) but $v = 5$, with 40 possible exceptions.

Theorem 1.2. *There exists an $SCSOIDLS(v)$ if and only if $v \equiv 0$ or 1 (mod 4) with the exception of $v = 5$ and the possible exceptions of*

$v \in E_0 \cup E_1 \cup E_5 \cup E_8 \cup E_9$ *where*

$$E_0 = \{12n \mid n = 2\},$$
$$E_1 = \{12n + 1 \mid n = 1, 7, 11\},$$
$$E_5 = \{12n + 5 \mid n = 1, 3, 4, 5, 6, 8, 11, 12, 13, 14, 15, 20, 21, 27\},$$
$$E_8 = \{12n + 8 \mid n = 1, 3, 5, 7, 11, 13\},$$
$$E_9 = \{12n + 9 \mid n = 0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 12, 13, 17, 21, 24, 33\}.$$

□

For our purpose we need the concept of an incomplete self-conjugate self-orthogonal idempotent diagonal Latin square ($ISCSOIDLS$). This is defined formally as follows. Suppose $v = 2t + n$, where t and n are positive integers. Let S and N be disjoint sets with $|S| = 2t$ and $|N| = n$. An $ISCSOIDLS(v, n)$ is a $v \times v$ array A with cell (i, j) empty when $t < i \leq t+n$ and $t < j \leq t + n$; the other cells are filled with members of $S \cup N$ in such a way that:

- every row and every column contains every element of S exactly once;
- every row and every column contains every element of N once except for rows and columns $t + 1$, $t + 2$, $\ldots$, $t + n$;
- if A and A^T are superimposed, the resulting array contains every ordered pair in $(S \times S) \cup (S \times N) \cup (N \times S)$ exactly once;
- if there is an element in position (i, i) of A then $A(i, i) = i$.
- if there is an element in position (i, j) of A then $A(A(i, j), A(j, i)) = i$.

If an $SCSOIDLS$ of order v contains a sub-$SCSOIDLS$ of order n at the central position, removing the sub-$SCSOIDLS$ gives an incomplete $SCSOIDLS$, denoted by $ISCSOIDLS(v, n)$. It is easy to see that any $SCSOIDLS(v)$ is an $ISCSOIDLS(v, 1)$ when v is odd.

We shall assume for the rest of this paper that every diagonal Latin square is idempotent. (If not, one can replace the square by an idempotent one obtained by symbol permutation.)

2 Preliminaries

In this section we shall define some terminology and give some constructions. For more details on GDDs and related designs, the reader is referred to [2].

Let K and M be sets of positive integers. A group divisible design (GDD) $GD(K, M; v)$ is a triple $(\mathbf{X}, \mathcal{G}, \mathcal{B})$ where

(i) $\mathbf{X}$ is a v-set of points,

(ii) $\mathcal{G}$ is a collection of non-empty subsets of $\mathbf{X}$ (called groups) which partition $\mathbf{X}$; if $G \in \mathcal{G}$ then $|G| \in M$;

(iii) $\mathcal{B}$ is a collection of subsets of $\mathbf{X}$ (called blocks); if $B \in \mathcal{B}$ then $|B| \in K$ and $|B| \geq 2$

(iv) no block intersects any group in more than one point;

(v) each pair (x, y) of points not contained in a group is contained in exactly one block.

The group-type (or *type*) of $GDD(\mathbf{X}, \mathcal{G}, \mathcal{B})$ is the multiset of sizes $|G|$ of the $G \in \mathcal{G}$ and we usually use the "exponential" notation for its description: group-type $1^i 2^j 3^k \ldots$ denotes i groups of size 1, j groups of size 2, and so on.

Let$(\mathbf{X}, \mathcal{G}, \mathcal{B})$ be a $GD(K, M; v)$. A parallel class in $(\mathbf{X}, \mathcal{G}, \mathcal{B})$ is a collection of disjoint blocks of $\mathcal{B}$, the union of which equals $\mathbf{X}$. $(\mathbf{X}, \mathcal{G}, \mathcal{B})$ is called resolvable if the blocks of $\mathcal{B}$ can be partitioned into parallel classes.

We need establish some more notation. We shall simply write $GD(k, m; v)$ for $GD(\{k\}, \{m\}; v)$. If $m \notin M$, then $GD(K, M \cup \{m^*\}, v)$ denotes a $GD(K, M \cup \{m\}; v)$ which contains a unique group of size m and if $m \in M$, then a $GD(k, M \cup \{m^*\}; v)$ is a $GD(K, M; v)$ containing at least one group of size m. We shall sometimes refer to a $GDD(\mathbf{X}, \mathcal{G}, \mathcal{B})$ as a K-GDD if $|B| \in K$ for every block $B \in \mathcal{B}$. In every acronym we denote resolvability by a leading R.

For some of our recursive constructions of GDDs, we shall make use of Wilson's "Fundamental Construction" (see [9]). We define a weighting of a $GDD(\mathbf{X}, \mathcal{G}, \mathcal{B})$ to be any mapping $w : \mathbf{X} \to Z^+ \cup \{0\}$. We present a brief description of Wilson's construction relating to GDDs below.

Lemma 2.1. *Suppose that $(\mathbf{X}, \mathcal{G}, \mathcal{B})$ is a "master" GDD and let $w : \mathbf{X} \to Z^+ \cup \{0\}$ be a weighting of the GDD. For every $x \in \mathbf{X}$, let S_x be the multiset of $w(x)$ copies of x. For each block $B \in \mathcal{B}$, assume a $GDD(\underset{x \in B}{\rightarrow} \bigcup S_x, \{S_x | x \in B\}, \mathcal{B}_B)$ is given. Write $\mathbf{X}^* = \underset{x \in \mathbf{X}}{\rightarrow} \bigcup S_x$, $\mathcal{G}^* = \{\underset{x \in G}{\rightarrow} \bigcup S_x | G \in \mathcal{G}\}$, and $\mathcal{B}^* = \underset{B \in \mathcal{B}}{\rightarrow} \bigcup \mathcal{B}_B$. Then $(\mathbf{X}^*, \mathcal{G}^*, \mathcal{B}^*)$ is a GDD.*

The following lemma is our main construction.

Lemma 2.2. *Let K be a set of positive integers and $s \geq 0$. Suppose there exists a K-GDD of group-type $m_1 m_2 \ldots m_n$ and*

(1) *for every $k \in K$ there exits an $SCSOIDLS(k)$,*

(2) *for every $i < n$ there exists an $ISCSOIDLS(m_i + s, s)$ and m_i is even.*

Then there exists an $ISCSOIDLS(v, m_n + s)$, *where* $v = s + \sum_{1 \le i \le n} m_i$. *Moreover, if there exists an* $SCSOIDLS(m_n + s)$, *then there exists an* $SCSOIDLS(v)$.

Proof: Let us denote $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$, where $|G_i| = m_i$, and denote $|S| = s$. We first relabel the elements of G_i and S so that

$$G_1 = \{1, 2, \ldots, \frac{1}{2}m_1, v - \frac{1}{2}m_1 + 1, \ldots, v-1, v\},$$
$$G_2 = \{\frac{1}{2}m_1 + 1, \frac{1}{2}m_1 + 2, \ldots, \frac{1}{2}m_1 + \frac{1}{2}m_2, v - \frac{1}{2}m_1 - \frac{1}{2}m_2 + 1, \ldots, v - \frac{1}{2}m_1 - 1, v - \frac{1}{2}m_1\},$$

and so on, S contains the central s elements. Then the standard construction, outlined above, produces the required square, provided the Latin square used to correspond to each of the groups G_i is an $ISCSOIDLS(m_i + s, s)$ (for $i < n$) or an $SCSOIDLS(m_n + s)$ and each of the blocks size k is an $SCSOIDLS(k)$. □

In order to establish our GDD construction we shall need some "small" input designs.

Lemma 2.3. [1,3] *There exist* $\{4\}$ *or* $\{4, 8\}$*-GDDs of the following group-types:* (a) 3^8, (b) 3^9, (c) 3^{12}, (d) 4^7, (e) 4^{10}, (f) 4^8, (g) $4^6 1^1$, (h) $4^7 1^1$.

Lemma 2.4. [1] *Suppose there exists a* $GD(8, m; 8m)$ *and* $0 \le x, y, z \le m$, *where* $x + y = m$. *Then there exists a* $\{4, 8\}$*-GDD of group-type* $(4m)^6 (4x + y)^1 (4z)^1$.

The following result will be quite useful (see [4]).

Lemma 2.5. *A* $GD(8, m; 8m)$ *exists for all integers* $m \ge 76$.

We shall make use of the following lemma, which is a consequence of Lemma 2.5.

Lemma 2.6. *Define the sequence* $M = \{m_1, m_2, m_3, \ldots\}$ *by*

$$M = \{7, 13, 16, 19, 25, 31, 37, 43, 49, 61, 64, 67, 70, 73\} \cup \{x : x \equiv 1 \pmod 3, x \ge 76\}.$$

Then for each i

(1) $m_i \equiv 1 \pmod 3$,

(2) $m_{i+1} - m_i \le 12$, *and*

(3) *a* $GD(8, m_i; 8m_i)$ *exists.*

□

3 The case $v \equiv 0 \pmod 4$

3.1 The case $v \equiv 0 \pmod{12}$

Lemma 3.1.1. *There exists an $SCSOIDLS(12n)$ for $n = 1, 3, 4$ or 8.*

Proof: For $n = 1$, see [5]. For $n = 3$, we make use of the existence of a GD $(\{4,8\},\{4,8^*\};36)$ (see, for example, [1]) and then we obtain an $SCSOIDLS$ (36) from Lemma 2.2 with $s = 0$. For $n = 4$, we have a $GD(\{4,12\},4;48)$ from an $RGD(4,12;48)$. We then obtain an $SCSOIDLS$ (48) from Lemma 2.2 with $s = 0$. For $n = 8$, we take an RGD $(4,3;24)$ (see, for example [7]) and give every point weight 4. In our resulting GDD, we take a parallel class of blocks as groups to form a GD $(\{4,12\},4;96)$. We then obtain an $SCSOIDLS(96)$ from Lemma 2.2 with $s = 0$. We also notice that there exist $ISCSOIDLS(12n,4)$ for $n \neq 1$. □

Lemma 3.1.2. [3] *For every n, $n \geq 4$, there exists a $GD(4,12;12n)$.* □

We then have

Lemma 3.1.3. *For every n, $n \geq 4$, there exists an $SCSOIDLS(12n)$.*

Proof: Apply Lemma 2.2 with $m_i = 12$ and $s = 0$, we obtain the desired designs. □

Combining Lemmas 3.1.1 and 3.1.3, we have essentially proved the following result.

Theorem 3.1. *If $v \equiv 0 \pmod{12}$ and $v \notin E_0$, then there exists an $SCSOIDLS(v)$* □

3.2 The case $v \equiv 4 \pmod{12}$

Lemma 3.2.1. [3] *If $v \equiv 4 \pmod{12}$, then there exists a $GD(4,4;v)$.* □

We then have

Theorem 3.2. *If $v \equiv 4 \pmod{12}$, then there exists an $SCSOIDLS(v)$.*

Proof: Apply Lemma 2.2 with $m_i = 4$ and $s = 0$, we obtain the desired design. We also notice that there exists an $ISCSOIDLS(v,4)$. □

3.3 The case $v \equiv 8 \pmod{12}$

Lemma 3.3.1. *Let M be as defined in Lemma 2.6. If $m \in M$, then there exist $SCSOIDLS(v)$ for all $v \equiv 8 \pmod{12}$ in the interval $25m+4 \leq v \leq 32m$.*

Proof: We shall apply Lemma 2.4 with m ∈M. Since $m \equiv 1 \pmod 3$, we can choose $4x + y \equiv 4 \pmod{12}$, where $0 \leq x, y \leq m$, $x + y = m$, and $m \leq$

$4x + y \leq 4m$. We choose $4z \equiv 4 \pmod{12}$, where $\leq 4z \leq 4m$. Note that there exist $SCSOIDLS(4m)$, $SCSOIDLS(4x+y)$ and $SCSOIDLS(4z)$. Let $v = 24m + 4x + y + 4z$. Then it readily follows that there exists an $SCSOIDLS(v)$ from Lemma 2.2 with the resulting GDD and $s = 0$. □

Lemma 3.3.2. *If $v \equiv 8 \pmod{12}$, then there exist $SCSOIDLS(v)$ for all $v \geq 188$, $v \notin \{236, 248, 260, 272, 284, 296, 308, 320, 620\}$.*

Proof: We shall apply Lemma 3.1.1. If $m = 7, 13, 16, 18, 25$, then we obtain $SCSOIDLS(v)$ for all values of $v \equiv 8 \pmod{12}$ in the interval $188 \leq v \leq 800$; apart from the exceptions listed in the statement of the lemma. For $m \geq 25$, if we apply Lemma 3.3.1 repeatedly, then we find that the intervals for v overlap and we obtain $SCSOIDLS(v)$ for all $v \equiv 8 \pmod{12}$ where $v \geq 632$. This completes the proof of the lemma. □

Lemma 3.3.3. *There exists an $SCSOIDLS(v)$ for $v = 8$ and $v = 32$.*

Proof: For $v = 8$, see [5]. For $v \doteq 32$, we have an $RGD(4, 8; 32)$ and then we obtain an $SCSOIDLS(32)$ from Lemma 2.2 with $s = 0$. We notice that there exist an $ISCSOIDLS(32, 4)$ and an $ISCSOIDLS(32, 8)$. □

Lemma 3.3.4. *If $v \in \{56, 80, 104, 128, 152, 176, 248, 272, 296, 320\}$, then there exists an $SCSOIDLS(v)$.*

Proof: Write $v = 8 \cdot \frac{v}{8}$. We make use of the existence of a $GD(4, 8; v)$ (see, for example, [3]) and obtain an $SCSOIDLS(v)$ from Lemma 2.2 with $s = 0$. □

Lemma 3.3.5. *There exists an $SCSOIDLS(116)$.*

Proof: We have $GD(\{4, 8\}, 4; 116)$ from an $RGD(4, 29; 4 \cdot 29)$, and we use this to obtain an $SCSOIDLS(116)$ from Lemma 2.2 with $s = 0$. □

Lemma 3.3.6. *There exists an $SCSOIDLS(284)$.*

Proof: We first adjoin 8 infinite points to an $RGD(7, 9; 63)$ so as to form a $GD(\{7, 8, 10\}, \{7, 8^*\}; 71)$, and then give each point weight 4 to obtain $GD(\{4, 8\}, \{28, 32^*\}; 284)$, using $\{4, 8\}$-GDDs of type 4^7, 4^8 and 4^{10}. We obtain an $SCSOIDLS(284)$ from Lemma 2.2 with $s = 0$. □

Lemma 3.3.7. *There exists an $SCSOIDLS(308)$.*

Proof: Take an $RGD(4, 76; 304)$ and adjoin 4 infinite points to the groups and applying Lemma 2.1 with the fact that a $GD(\{4, 8\}, 1; 80)$ exists (see, for example, [1]). Consequently there exists a GD $(\{4, 8\}, 4; 308)$ and then there exists an $SCSOIDLS(308$, from Lemma 2.2 with $s = 0$. □

Lemma 3.3.8. *There exists an $SCSOIDLS(v)$ for $v = 236$, 260 or 620.*

Proof: Apply Lemma 2.4 with $m = 8$ or 9, $4x + y = 32$ and $z = 3$, we know the result is true for $v = 236$ or 260 from Lemma 2.2 with $s = 0$.

Apply Lemma 2.4 with $m = 23$, $4x + y = 32$ and $z = 8$. In our resulting GDD we apply Lemma 2.2 with $s = 4$. Notice that $ISCSOIDLS(96, 4)$ and $ISCSOIDLS(36, 4)$ both exist (see the proof of Lemma 3.1.1). Consequently we obtain an $SCSOIDLS(620)$. □

Combining Lemmas 3.3.2–3.3.8, we have essentially proved the following result.

Theorem 3.3. *If* $v \equiv 8 \pmod{12}$ *and* $v \notin E_8$, *then there exists an* $SCSOIDLS(v)$. □

4 The Case $v \equiv 1 \pmod 4$

4.1 The case $v \equiv 1 \pmod{12}$

Define

$$F = \{265, 457, 553, 661, 853, 865\}$$

Lemma 4.1.1. [4, p.191] *If* $v \equiv 0 \pmod{12}$ *and* $v + 1 \notin E_1 \cup F$, *then there exists an* $RGD(4, 3; v)$. □

We then have

Lemma 4.1.2. *If* $v \equiv 1 \pmod{12}$ *and* $v \notin E_1 \cup F$, *then there exists an* $SCSOIDLS(v)$.

Proof: We adjoin one infinite point to an $RGD(4, 3; v - 1)$ so as to form a $GD(4, \{4, 1^*\}; v)$, and then we can construct an $SCSOIDLS(v)$ from Lemma 2.2 with $s = 0$. □

Lemma 4.1.3. [3] *For every* $n > 4$, *there exists a* $GD(4, 6; 6n)$. □

We then have

Lemma 4.1.4. *For every* $n > 4$, *there exists an* $SCSOIDLS(24n + 1)$.

Proof: We first give every point of $GD(4, 6; 6n)$ weight 4. From Lemma 2.1 the resulting design is a $GD(4, 24; 24n)$, using $\{4\}$-GDD of type 4^4. We then adjoin one infinite point to the groups of this GDD and then, make use of the existence of a $GD(4, \{4, 1^*\}; 25)$ to obtain a $GD(4, \{4, 1^*\}; 24n + 1)$. From Lemma 2.2 with $s = 0$ we know there exists an $SCSOIDLS(24n + 1)$. □

We now have

Lemma 4.1.5. *If* $v \in \{265, 457, 553, 865\}$, *then there exists an* $SCSOIDLS(v)$. □

Lemma 4.1.6. *If* $v \in \{661, 853\}$, *then there exists an* $SCSOIDLS(v)$.

Proof: Apply Lemma 2.4 with $m = 25$, $4x+y = 25$ and $z = 9$: we see the result is true for $v = 661$ from Lemma 2.2 with $s = 0$.

Apply Lemma 2.4 with $m = 31$, $4x+y = 73$ and $z = 9$: we see the result is true for $v = 853$ from Lemma 2.2 with $s = 0$. □

Combining Lemmas 4.1.2, 4.1.5 and 4.1.6, we have essentially proved the following result.

Theorem 4.1. *If $v \equiv 1 \pmod{12}$ and $v \notin E_1$, then there exists an $SCSOIDLS(v)$.* □

4.2 The Case $v \equiv 5 \pmod{12}$

Lemma 4.2.1. *Let M be as defined in Lemma 2.6. If $m \in M$, then there exist $SCSOIDLS(v)$ for all $v \equiv 5 \pmod{12}$ in the interval $25m+4 \leq v \leq 32m-3$ (provided also that $v \geq 24m+29$, if $m = 7$, or 13).*

Proof: The proof is similar to that of Lemma 3.3.1. Here we also apply Lemma 2.4 with $m \in M$. We can choose $4x+y = 1 \pmod{12}$ such that the conditions $0 \leq x, y \leq m$, $x+y = m$, $m \leq 4x+y \leq 4m-3$ ($4x+y \geq 25$ when $m = 7$, or 13) all hold. We choose $4z \equiv 4 \pmod{12}$. Note that there exist $SCSOIDLS(4m)$, $SCSOIDLS(4x+y)$ (except for $4x+y = 85$ or 133, but we can write $24m+(4x+y)+4z = 24m+(4x+y\pm 12)+(4z\mp 12)$) and $SCSOIDLS(4z)$. From Lemma 2.2 with $s = 0$ the result is true for all $v \equiv 5 \pmod{12}$ in the interval $25m+4 \leq v \leq 32m-3 (v \geq 24m+29$, if $m = 7, 13)$. □

Lemma 4.2.2. *If $v \equiv 5 \pmod{12}$, then there exist $SCSOIDLS(v)$ for all $v \geq 197$ and $v \notin \{233, 245, 257, 269, 318, 293, 305, 317, 329, 617\}$.*

Proof: We apply Lemma 4.2.1 repeatedly. If $m = 7, 13, 16, 19, 25$, then we obtain $SCSOIDLS(v)$ for all values of $v = 5 \pmod{12}$ in the interval $197 \leq v \leq 797$, apart from the exceptions listed in lemma. For $m \geq 25$, the intervals of v overlap and we obtain $SCSOIDLS(v)$ for all $v \equiv 5 \pmod{12}$ where $v \geq 641$. This completes the proof of the lemma. □

Lemma 4.2.3. *If $v \in \{29, 89\}$, then there exists an $SCSOIDLS(v)$.*

Proof: Take an $RGD(4, 7, 28)$ and adjoint one infinite point to the groups so as to form a $GD(\{4, 8\}, \{4, 1^*\}; 29)$, and then there exists an $SCSOIDLS$ (29) from Lemma 2.2 with $s = 0$.

Take an $RGD(8, 11; 88)$ and adjoin one infinite point to the groups so as to form a $GD(\{8, 12\}; \{8, 1^*\}; 89\}$, and then there exists an $SCSOIDLS(89)$ from Lemma 2.2 with $s = 0$. □

Lemma 4.2.4. *If $v \in \{113, 281, 617\}$, then there exists an $SCSOIDLS(v)$.*

Proof: For $n = 1$, 3 or 7, take an $RGD(4, 1; 12n + 4)$ and give each point weight 7. In our resulting GDD, we give one infinite point to the groups and obtain a $GD(\{4, 8\}, \{4, 1^*\}; 84n + 29)$. We then obtain the desired designs from Lemma 2.2 with $s = 0$. □

Lemma 4.2.5. *If $v \in \{125, 317\}$, then there exists an $SCSOIDLS(v)$.*

Proof: For $n = 1$ or 3, take an $RGD(4, 24n + 7; 96n + 28)$ and adjoin one infinite point to the groups to obtain a $GD(\{4, 24n+8\}, \{4, 1^*\}; 96n+29)$. We then obtain the desired designs from Lemma 2.2 with $s = 0$. □

Lemma 4.2.6. *There exists an $SCSOIDLS(233)$.*

Proof: Apply Lemma 2.4 with $m = 8$, $4x + y = 29$ and $z = 3$, we obtain the desired design from Lemma 2.2 with $s = 0$. □

Lemma 4.2.7. *There exists an $SCSOIDLS(269)$.*

Proof: Apply Lemma 2.4 with $m = 9$, $4x+y = 21$ and $z = 7$. In our resulting GDD we apply Lemma 2.2 with $s = 4$. Notice that $ISCSOIDLS(40, 4)$ and $ISCSOIDLS(32, 4)$ both exist (see the proof of Theorem 3.2 and Lemma 3.3.3), we obtain the desired design. □

Lemma 4.2.8. *There exists an $SCSOIDLS(293)$.*

Proof: We first adjoin 7 infinite points to an $RGD(8, 11; 88)$ so as to form a $GD(\{8, 9, 12\}, \{8, 7^*\}; 95)$, where one of the infinite points is adjoined to the groups. In the resulting GDD, we give each point weight 3 to form a $GD(4, \{24, 21^*\}; 285)$, using $\{4\}$-GDDs of types 3^8, 3^9, 3^{12}. Finally, we adjoin 8 infinite points to this GDD, using Lemma 2.2 and the fact that an $ISCSOIDLS(32, 8)$ exists (see the proof of Lemma 3.3.3) to obtain the desired design. □

Lemma 4.2.9. *There exists an $SCSOIDLS(305)$.*

Proof: Take a $GD(8, 11; 88)$ and delete one block entirely to get a $GD(\{7, 8\}, 10; 80)$. In all but one of the groups, we give weight 4 to each point. In the last group, give weight 1 to five points and weight 4 to the remaining five points. This gives a $GD(\{4, 8\}, \{40, 25^*\}; 305)$, using $\{4, 8\}$–GDDs of type 4^7, 4^8, $4^7 1^1$. It follows that there exists an $SCSOIDLS(305)$ from Lemma 2.2 with $s = 0$.

Combining Lemmas 4.2.2–4.2.9, we have essentially proved the following result.

Lemma 4.2. *If $v \equiv 5 \pmod{12}$ and $v \notin \{5\} \cup E_5$ then there exists an $SCSOIDLS(v)$.* □

4.3 The Case $v \equiv 9 \pmod{12}$

Lemma 4.3.1. *Let M be as defined in Lemma 2.6. If $m \in M$, then there exist $SCSOIDLS(v)$ for all $v \equiv 9 \pmod{12}$ and $v \neq 765$ in the following interval:*

(1) $25m + 8 \leq v \leq 32m - 11$, *if* $m \notin \{7, 13, 19, 25, 37, 43\}$,

(2) $25m + 8 \leq v \leq 32m - 23$, *if* $m \notin \{7, 13, 19, 25, 37, 43\}(v \geq 24m + 33$, *if* $m = 7, 13)$.

Proof: We apply Lemma 2.4 with $m \in M$. In each of (1) and (2), we take $4x + y \leq 1 \pmod{12}$ such that the conditions $0 \leq x,\ y \leq m,\ x + y = m$, $m \leq 4x+y \leq 4m-3(4x+y \geq 25$, if $m = 7, 13)$ hold. Note that there exist $SCSOIDLS(4m)$ and $SCSOIDLS(4x+y)$ (except for $4x+y = 85$ or 133, but we can write $24m+(4x+y)+4z = 24m+(4x+y\pm 12)+((4z\mp 12))$. For (1) we choose $4z \equiv 8 = \pmod{12}$ such that $8 \leq 4z \leq 4m - 8$ and for (2), choose $4z \equiv 8 \pmod{12}$ such that $8 \leq 4z \leq 4m - 20$, where the existence of $SCSOIDLS(4z)$ form Theorem 3.3. The gap is at most 24 between consecutive values of v for which $SCSOIDLS(v)$ exist in Theorem 3.3. If we put $v = 24m + 4x + y + 4z$, then it is not difficult to check that we obtain $SCSOIDLS(v)$ for all $v \equiv 9 \pmod{12}$ in the specified interval from Lemma 2.2 with $s = 0$. □

Lemma 4.3.2. *If $v \equiv 9 \pmod{12}$, then there exist $SCSOIDLS(v)$ for all $v \geq 201$ and $v \notin$ {213, 225, 237, 249, 261, 273, 285, 297, 309, 321, 333, 405, 597, 609, 621, 765}.*

Proof: We apply Lemma 4.3.1. If $m = 7, 13, 16, 19, 25$, then we obtain $SCSOIDLS(v)$ for all $v \equiv 9 \pmod{12}$ in the interval $201 \leq v \leq 753$, apart from the exceptions listed in the lemma. If we choose $m \in M$, $m \geq 25$, and apply Lemma 4.3.1 repeatedly, then it is readily checked that the intervals for v overlap and we obtain $SCSOIDLS(v)$ for all $v \equiv 9 \pmod{12}$ where $v \geq 633$. □

Lemma 4.3.3. *If $v \in \{45, 141, 177, 189, 285\}$, then there exists an $SCSOIDLS(v)$.*

Proof: For $n = 11, 35, 44, 47$ or 71, take an $RGD(4, n; 4n)$ and adjoin one infinite point to the groups to obtain a $GD(\{4, n+1\}, \{4, 1^*\}, 4n+1)$, we then obtain an $SCSOIDLS(4n+1)$ from Lemma 2.2 with $s = 0$. □

Lemma 4.3.4. *If $v \in \{225, 237, 249, 765\}$, then there exists an $SCSOIDLS(v)$.*

Proof: Apply Lemma 2.4 with $m = 8$, $4x + y = 29$ and $z = 1$, 4 or 7, we know the result is true for $v = 225, 237$ or 249 from Lemma 2.2 with $s = 0$.

Apply Lemma 2.4 with $m = 27$, $4x + y = 45$ and $z = 18$, we know the result is true for $v = 765$ from Lemma 2.2 with $s = 0$. □

Lemma 4.3.5. *If $v \in \{273, 309, 321, 333, 597, 609, 621\}$, then there exists an $SCSOIDLS(v)$.*

Proof: Apply Lemma 2.4 with $m = 9$, $4x+y = 21$ and $z = 8$. In our resulting GDD we apply Lemma 2.2 with $s = 4$. Notice that $ISCSOIDLS(40, 4)$ and $ISCSOIDLS(36, 4)$ both exist (see the proof of Theorem 3.2 and Lemma 3.1.1), we obtain an $SCSOIDLS(273)$.

Apply Lemma 2.4 with $m = 11$ or 23, $4x + y = 41$ and $z = 0, 3$, or 6. In our resulting GDD we apply Lemma 2.2 with $s = 4$. Notice that $ISCSOIDLS(48, 4)$, $ISCSOIDLS(196, 4)$, $ISCSOIDLS(16, 4)$ and $ISCSOIDLS(28, 4)$ (see the proof of Lemma 3.1.1 and Theorem 3.2), we obtain the remaining designs. □

Combining Lemmas 4.3.2–4.3.5, we have essentially proved the following result.

Theorem 4.3. *If $v \equiv 9 \pmod{12}$ and $v \notin E_9$, then there exists an $SCSOIDLS(v)$.* □

Proof of Theorem 1.2: The result now follows from Theorems 3.1–3.3 and Theorems 4.1–4.3. □

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