Compositions With m Distinct Parts

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ABSTRACT. We study $F(n,m)$, the number of compositions of n in which repetition of parts is allowed, but exactly m *distinct* parts are used. We obtain explicit formulas, recurrence relations, and generating functions for $F(n,m)$ and for auxiliary functions related to F. We also consider the analogous functions for partitions.

Introduction

One of the problems considered by Wilf in [7] involves the number of different sizes of parts in a partition of the integer n. This paper investigates the function $F(n,m)$, which gives the number of compositions of n in which repetition of parts is allowed, but m distinct parts are used in all. For example, in the table below we note $F(4,2) = 5$, from the five compositions of 4 with 2 distinct parts: 3+1, 1+3, 2+1+1, 1+2+1, and 1+1+2.

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The first observation to make about $F(n,m)$ is that for $m=1$ there is a composition of n with one distinct part k if and only if k is a divisor of n. Hence $F(n,1)=d(n)$, the divisor counting function. We extend this sum over divisors to the second column in the next section.

On the right hand boundary of the table, we note that the first non-zero entry in column m is at $n=m(m+1)/2$, where $F(m(m+1)/2,m)=m!$ This is because there are $m!$ arrangements of the summands in the first integer with m distinct parts: $1+2+\ldots+m$.

In order to understand $F(n,m)$ we provide explicit formulas, recurrences, and generating functions for functions related to F. Some of the formulas we derive have immediate analogues to formulas for partitions, especially those in which the ordered structure of compositions manifests itself in a general summand involving a binomial coefficient. We develop the partition formulas in the fourth section.

Table 1. Compositions of n with m distinct parts, $F(n,m)$, $1 \le n \le 16$, $1 \le m \le 5$

$n\backslash m$	1	2	3	4	5
1	1				
2	2				
3	2	2			
4	3	5			
5	2	14			
6	4	22	6		
7	2	44	18		
8	4	68	56		
9	3	107	146		
10	4	172	312	24	
11	2	261	677	84	
12	6	396	1358	288	
13	2	606	2666	822	
14	4	950	5012	2226	
15	4	1414	9542	5304	120
16	5	2238	17531	12514	480

Basic recurrences

Since $F(n,1)=d(n)$, it is natural to look for an interpretation of later columns involving divisors of n. We begin by offering an explicit formula for the case $m=2$.

Theorem 1. *For $n \geq 2$,*

$$F(n,2) = \sum_{j=2}^{\lfloor n/3 \rfloor} \sum_{k=1}^{\lfloor n/j \rfloor - 1} \sum_{\substack{d \mid (n-jk) \\ k \neq d < \frac{n-jk}{j-1}}} \binom{j + (n-jk)/d}{j}$$
$$- \sum_{\substack{j=1 \\ j \mid n}}^{\lfloor n/3 \rfloor} \binom{2j}{j} \left\lfloor \frac{n/j - 1}{2} \right\rfloor + \sum_{k=1}^{n-1} \sum_{\substack{d \mid (n-k) \\ k \neq d}} (1 + (n-k)/d).$$

Proof: Consider a composition of n into two distinct parts. Write one of the parts as k, where $1 \leq k \leq n-1$ and count the number of compositions according to the number of occurrences of the part k. If k occurs exactly once then the remainder of the composition is just a composition of $n-k$ with one distinct part, say d, with $d \neq k$. For this to be possible we must have $d \mid n-k$ and then the composition of n consists of $(n-k)/d$ parts equal to d and one part k which can be inserted in any of $(n-k)/d+1$ places. Thus the number of compositions of n with exactly two distinct parts in which one of these parts occurs once only is

$$\sum_{k=1}^{n-1} \sum_{\substack{d \mid (n-k) \\ d \neq k}} (1 + \frac{n-k}{d}) - 2 \left\lfloor \frac{n-1}{2} \right\rfloor.$$

The reason for the subtracted term is that compositions into exactly two parts are counted twice (e.g. for $n = 6$ and $k = 2$ we count the compositions $2+4$ and $4+2$, and again when $k = 4$).

Next suppose the distinct part k occurs twice. That leaves a composition of $n-2k$ consisting of $(n-2k)/d$ copies of part d, where $k \neq d$, $d \mid (n-2k)$, and $d < n-2k$ since the case in which either distinct part occurs only once has already been covered. The number of different orderings of two k's and $(n-2k)/d$ d's is $\binom{2+(n-2k)/d}{2}$. Since the part d appears at least twice we need $n-2k \geq d \geq 2$, whence $1 \leq k \leq \lfloor n/2 \rfloor - 1$. Thus compositions with two distinct parts in which one part occurs twice and the other part occurs two or more times are enumerated by

$$\sum_{k=1}^{\lfloor n/2 \rfloor - 1} \sum_{\substack{d \mid (n-2k) \\ d \neq k \\ d < n-2k}} \binom{2 + (n-2k)/d}{2} - \begin{cases} \binom{4}{2} \left\lfloor \frac{(n/2-1)}{2} \right\rfloor & , n \text{ even} \\ 0 & , n \text{ odd.} \end{cases}$$

The subtracted term here deals with compositions of n into two distinct parts which each appear twice (e.g. $6 = 1+1+2+2$), which are counted twice in the left sum. Such compositions are possible only if n is even. In this case we have $\binom{4}{2}$ orderings of the composition.

In general, suppose the part k occurs j times and the other part d occurs j or more times. In the same manner as above we require $d \mid (n-jk)$, $k \neq d$, and $d < (n-jk)/(j-1)$. There are $\binom{j+(n-jk)/d}{j}$ ordered arrangements of j k's and $(n-jk)/d$ d's, and $n-jk \geq jd \geq j$ implies $k \leq \lfloor n/j \rfloor - 1$. In the case that $j \mid n$ there are $\binom{2j}{j}$ compositions with exactly j k's and j d's that are counted twice for given values of k and d. The number of different possibilities for k and d is given by the number of solutions to $n/j = k+d$ which just interchange k and d, this number being $\lfloor (n/j-1)/2 \rfloor$.

Finally, summing all these cases over j yields the formula. Since $jk+jd \leq n$, $j \leq n/(k+d) \leq n/(1+2) = n/3$ which provides the limit for the outer sum.

In the next theorem and in later theorems we will use the auxiliary function $F(n,m,j)$ to represent the number of compositions of n into m distinct parts, using exactly j parts altogether.

Theorem 2. *For $j \geq 2$ and $n > j$,*

$$F(n,2,j) = \sum_{r=1}^{\lfloor j/2 \rfloor} \sum_{\substack{k=1 \\ (j-r) \mid (n-kr) \\ k \neq (n-kr)/(j-r)}}^{\lfloor n/r \rfloor - 1} \binom{j}{r} - \begin{cases} \binom{j}{\frac{j}{2}} \left\lfloor (\frac{2n}{j} - 1)/2 \right\rfloor, & \text{if } 2 \mid j \text{ and } j \mid 2n \\ 0, & \text{otherwise.} \end{cases}$$

Proof: We proceed as in the proof of the formula for $F(n,2)$ by counting the compositions according to the number of occurrences of one part k. If k occurs exactly once then $n-k$ must be given as a sum of $j-1$ equal numbers, each $(n-k)/(j-1)$, provided this is a positive integer. Thus we require $j-1 \mid n-k$, $k \neq (n-k)/(j-1)$, and there are j possible arrangements of the parts. In the case that $j=2$ we need to subtract off compositions into two parts which are counted twice. Thus compositions with two distinct parts, j parts in all, and one part occurring once are enumerated by

$$\sum_{\substack{k=1 \\ (j-1) \mid (n-k) \\ k \neq (n-k)/(j-1)}}^{\lfloor n/j \rfloor - 1} j - \begin{cases} 2 \left\lfloor \frac{n-1}{2} \right\rfloor & ,j=2 \\ 0 & ,j>2. \end{cases}$$

In general suppose part k occurs r times so that $n-rk$ must be a sum of $j-r$ equal numbers. Thus we require $j-r \mid n-kr$ and $k \neq (n-kr)/(j-r)$. There are $\binom{j}{r}$ ways of arranging the sequences of r summands of size k and $j-r$ summands of size $(n-kr)/(j-r)$. As in the earlier proof we require

$n - rk \geq r$ so $k \leq \lfloor n/r \rfloor - 1$. To ensure the other distinct part occurs at least r times we need $j - r \geq r$ so $r \leq j/2$ in the outer sum. Compositions are counted twice and must be subtracted off in the event that both distinct parts occur the same number $j/2$ of times. This is possible only if $2 \mid j$ and then $\frac{j}{2}(k+d) = n$ implies that $j \mid 2n$.

We note that from the relation valid for $n \geq 2$,

$$F(n,2) = \sum_{j=1}^{n-1} F(n,2,j),$$

we can recover $F(n,2)$ as a threefold sum using the formula for $F(n,2,j)$ from Theorem 2.

Perhaps these formulas can be extended to later columns, but the number of special cases to consider becomes forbidding. Another family of results enumerates the compositions by first considering possible values of the summands in the composition.

Lemma 3. *Denote by $F^*(n,m,j)$ the number of compositions of n with exactly m distinct parts, j parts in all, and at least one part being a 1. Then*

$$F(n,m,j) = F(n-j,m,j) + F^*(n,m,j). \tag{1}$$

Proof: Divide the compositions counted by $F(n,m,j)$ into two classes: those with at least one part a 1 and those having no 1's. Compositions in the first class are enumerated by $F^*(n,m,j)$. For compositions in the second class, subtract 1 from each of the j parts, to obtain a composition of $n-j$ into m distinct parts, still with j parts in all.

As an application of this lemma, we can fix m and j and consider the sequence of values $\{F(n,m,j)\}$, $n \geq 1$. Even though the sequence may fail to be monotone, each subsequence consisting of every jth term from an arbitrary starting point *will* be monotone.

Lemma 4.

$$F(n,m,j) = F(n-j,m,j) + \sum_k{}^* \binom{j}{k} F(n-j,m-1,j-k), \tag{2}$$

where $\sum^*$ indicates a sum over those k for which a composition of n into m distinct parts, j parts in all, can have exactly k 1's.

Proof: Consider a composition counted by $F^*(n,m,j)$ in (1). Decrease each part by 1. Then n is reduced to $n-j$, m is reduced to $m-1$, and

j is reduced by the number of 1's that were in the original composition. Summing over appropriate k provides

$$F^*(n,m,j) = \sum_k{}^* \binom{j}{k} F(n-j, m-1, j-k).$$

Surprisingly, the restriction on k imposed by $\sum^*$ is more complicated for $m = 2$ than for larger values of m. For k ones to be summands in a composition of n into j parts for $m = 2$, n must have a representation of the form $n = k \cdot 1 + (j-k) \cdot d$ for $d \geq 2$. Hence we must have $j - k \mid n - k$. We offer a brief table for $m = 2$ that makes the pattern clear in this case.

Table 2. Summands k for $m = 2$ in $\sum^*$, $3 \leq n \leq 17, 2 \leq j \leq 9$

$n \backslash j$	2	3	4	5	6	7	8	9
3	1							
4	1	2						
5	1	1,2	3					
6	1	2	2,3	4				
7	1	1,2	1,3	3,4	5			
8	1	2	2,3	2,4	4,5	6		
9	1	1,2	3	1,3,4	3,5	5,6	7	
10	1	2	1,2,3	4	2,4,5	4,6	6,7	8
11	1	1,2	3	2,3,4	1,5	3,5,6	5,7	7,8
12	1	2	2,3	4	3,4,5	2,6	4,6,7	6,8
13	1	1,2	1,3	1,3,4	5	1,4,5,6	3,7	5,7,8
14	1	2	2,3	2,4	2,4,5	6	2,5,6,7	4,8
15	1	1,2	3	3,4	3,5	3,5,6	1,7	3,6,7,8
16	1	2	1,2,3	4	1,4,5	4,6	4,6,7	2,8
17	1	1,2	3	1,2,3,4	5	2,5,6	5,7	1,7,8

For $m \geq 3$, eventually Σ^* is an unrestricted sum.

Proposition 5. *Let $m \geq 3$. Then for $n \geq \frac{m(m+1)}{2} + 2(j-m)$, $\sum_k^* = \sum_{k=1}^{j-(m-1)}$.*

Proof: Let $n = 1^{a_1} 2^{a_2} \ldots n^{a_n}$ denote the partition $n = a_1 1 + a_2 2 + \ldots + a_n n$, where a_i is the number of occurrences i in the partition of n. Consider $n_0 = m(m+1)/2 + 2(j-m)$. Then for any value of k, $1 \leq k \leq j-(m-1)$, we have the partition $1^k 2^b 3^1 \ldots (m-1)^1 (m-1+k)^1$, where $b = j-m-k+2$, which is a partition of $(k-1)1+(m-1)m/2+(m+k-1)+2(j-m-(k-1)) = m(m+1)/2 + 2(j-m) = n_0$ into m distinct parts, namely $1, 2, \ldots, m-1$ and $m+k-1$, with total number of parts $k+m-2+b = j$ as required. Note $b = j-m+2-k \geq j-m+2-(j-m+1) \geq 1$ so the part 2 does occur at least once. If $n > n_0$, say $n = n_0 + r$ for $r \geq 1$, then the corresponding partition with k 1's, $1 \le k \le j-m+1$, is $n = 1^k 2^b 3^1 \ldots (m-1)^1 \ldots (m-1+k+r)^1$. We note that n_0 is the smallest value of n for which $\sum_k{}^* = \sum_{k=1}^{j-m+1}$, since if $k = 1$ the smallest number with m different parts and j parts in all is $1+2+3+\ldots+m+2(j-m) = n_0$.

Corollary 6.

$$F(n,2) = \sum_{j=1}^{n-1} \sum_{l=1}^{\left\lfloor \frac{n-1}{j} \right\rfloor} \sum_{\substack{k \\ j-k \mid n-lj}} \binom{j}{k} = \sum_{\substack{n=a+b \\ 1 \le a,b < n}} \sum_{j \mid a} \sum_{\substack{k \mid b \\ k \ne j}} \binom{j}{k}$$

Proof: Observe that $F(n-j,1,j-k) = \begin{cases} 1 & ,j-k \mid n-j \\ 0 & ,\text{otherwise} \end{cases}$. Hence (2) gives

$$F(n,2,j) = \sum_{\substack{k \\ j-k \mid n-j}}^{*} \binom{j}{k} + F(n-2j,2,j) = \sum_{\substack{k \\ j-k \mid n-j}} \binom{j}{k} + F(n-2j,2,j). \tag{3}$$

The last equality is justified by the remark before Table 2, where the condition on k imposed in Σ^* is $j-k \mid n-k$. Since $n-k = (n-j)+(j-k)$, this condition already holds by the constrain $j-k \mid n-j$. Now (2) can be applied again to the last term, and iterated for all first arguments $n-lk$ as long as $lj < n$. This gives the limits $1 \le l \le \left\lfloor \frac{n-1}{j} \right\rfloor$. The first formula in the lemma then follows by writing $F(n,2) = \sum_{j=1}^{n-1} F(n,2,j)$. Now make the change of variables $a = jl$, $b = n-a$, $k = j-k$, and note $\binom{j}{j-k} = \binom{j}{k}$.

Corollary 6 is in the spirit of the sum over divisors of Theorems 1 and 2, but it admits a generalization to later columns.

Theorem 7.

$$F(n,m) = \sum_{\substack{n=a_1+a_2+\ldots+a_m \\ a_i \ne 0}} \sum_{j_1 \mid a_1} \sum_{\substack{j_2 \mid a_2 \\ j_1 \ne j_2}} \sum_{\substack{j_3 \mid a_3 \\ j_2 \ne j_3}} \cdots \sum_{\substack{j_m \mid a_m \\ j_{m-1} \ne j_m}} \binom{j_1}{j_2} \binom{j_2}{j_3} \cdots \binom{j_{m-1}}{j_m}. \tag{4}$$

Proof: The outer summation is over compositions of n into m parts. First we build from compositions of n into m parts certain partitions of n into m distinct parts, then permute the parts of the partitions to obtain all possible compositions of n into m distinct parts.

Begin by representing the summands in a composition of n as a rectangular array of dots, one row for each summand. This resembles the Ferrars graph of a partition, except the lengths of the rows do not have to be monotone decreasing. Now attempt to transform this graph into the Ferrars graph of a partition of n with m distinct parts by replacing some of the rows of a_i dots with rectangles of width a_i/j_i dots and height j_i for j_i a divisor of a_i. The rows of the rectangle represent the size of the new parts, a_i/j_i, that replace the old a_i, and the number of parts goes from 1 part of size a_i in the composition to j_i parts of size a_i/j_i in the partition.

For a particular composition $n = a_1 + a_2 + \ldots a_m$, we obtain a partition of n into m distinct parts exactly when there is a sequence of divisors j_i of a_i with $j_1 > j_2 > j_3 > \ldots > j_m$. Furthermore this process is reversible, with any partition of n into m distinct parts yielding a composition of n into m parts when the parts of the same size in the partition are combined into one summand in the composition, and with the two partitions yielding the same composition only when rectangular blocks of different dimensions representing successive rows in the two partitions contain the same number of dots.

The inner summations in (4) generate all sequences of divisors $\{j_i\}$ that yield partitions of n into m distinct parts (the conditions that $j_i \neq j_{i+1}$ are sufficient, since if $j_i > j_{i+1}$ one of the binomial coefficients is zero). With j_i parts of size a_i/j_i, $1 \leq i \leq m$, the total number of compositions of n that can be formed by rearranging the summands is given by the multinomial coefficient

$$\binom{j_1}{j_1 - j_2, j_2 - j_3, \ldots, j_{m-1} - j_m, j_m} = \binom{j_1}{j_2}\binom{j_2}{j_3}\cdots\binom{j_{m-1}}{j_m}.$$

Compositions with a fixed part

In [7], Wilf outlines a general technique to obtain mean values for the number of distinct part sizes in a combinatorial structure. The success of his method depends on the multiplicativity of the generating function for the total number of structures of size n. Many common combinatorial structures have such generating functions, so that in addition to partitions his results apply equally well to permutations [7], partitions of sets [4], and polynomials over finite fields [3]. Part of the interest concerning the number of part sizes in a composition stems from the fact that the familiar generating function for compositions is not of the above type. Below we use a different technique based on a simple counting argument to obtain a generating function for the mean value.

Suppose we wish to guarantee that a particular part l (perhaps repeated) occurs in a composition of n. Denote by $C_l(n)$ the number of compositions of n in which at least one l occurs. It will also be necessary to keep track of $C_l(n,j)$, the number of compositions of n into j parts in all, in which at least one part is l. This notation extends the more standard use of $C(n)$ to represent the total number of compositions of n, which is 2^{n-1}, and $C(n,j)$ to represent the number of compositions with exactly j parts, $\binom{n-1}{j-1}$.

For a composition π of n, let $\delta(\pi)$ be the number of distinct parts of π, and let

$$\chi(r,\pi) = \begin{cases} 1 & , r \text{ is a part of } \pi \\ 0 & , \text{ otherwise.} \end{cases}$$

Then

$$\sum_{m=1}^{\infty} mF(n,m) = \sum_{\pi} \delta(\pi) = \sum_{\pi}\sum_{l\geq 1} \chi(l,\pi) = \sum_{l\geq 1}\sum_{\pi} \chi(l,\pi) = \sum_{l\geq 1} C_l(n)$$

Thus the numbers in row Σ in the table below represent $\sum_{l\geq 1} C_l(n)$. They are of special interest because of the above connection with the average number of distinct parts in a composition of n.

Table 3. Compositions of n into parts in which at least one part is an l, $C_l(n)$, $1 \leq n \leq 12, 1 \leq l \leq 6$

$l\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	3	6	13	27	56	115	235	478	969	1959
2		1	2	4	9	20	43	91	191	398	824	1697
3			1	2	5	11	25	55	120	258	550	1163
4				1	2	5	12	27	61	135	295	639
5					1	2	5	12	28	63	141	311
6						1	2	5	12	28	64	143
							$\ddots$					$\vdots$
Σ	1	2	6	13	30	66	144	308	655	1380	2891	6024

There is a general recurrence satisfied by the rows of Table 3, which we will recover from the generating functions established in the next theorem.

Theorem 8.

$$\sum_{n=1}^{\infty} C_l(n)t^n = \frac{t}{1-2t} - \frac{t-t^l+t^{l+1}}{1-2t+t^l-t^{l+1}}. \tag{5}$$

Proof: Write $C_l^*(n)$ to be the number of compositions of n with *no* part equal to l, and $C_l^*(n,m)$ to be the number of such compositions with m parts in all. Thus

$$C(n) - C_l^*(n) = C_l(n). \tag{6}$$

Observe that the generating function for $C_l^*(n,m)$ is

$$\sum_{n=0}^{\infty} C_l^*(n,m)t^n = (t+t^2+\ldots+t^{l-1}+t^{l+1}+\ldots)^m = (\frac{t}{1-t}-t^l)^m.$$

Now account for all possible numbers of parts m via

$$\sum_{n=0}^{\infty} C_l^*(n)t^n = \sum_{m=1}^{\infty}\left(\frac{t}{1-t}-t^l\right)^m = \frac{t-t^l+t^{l+1}}{1-2t+t^l-t^{l+1}}.$$

The last step is to recall that the generating function for arbitrary compositions is $\frac{t}{1-2t}$. Thus the generating function for the table entries in row l follows from (6).

Corollary 9. *For $l \geq 1, n \geq l+2$,*

$$C_l(n) = 2C_l(n-1) - C_l(n-l) + C_l(n-l-1) + 2^{n-l-2}.$$

Proof: From the generating function for $C_l^*(n)$ above we obtain

$$\begin{aligned}
t-t^l+t^{l+1} &= (1-2t+t^l-t^{l+1})\sum_{n=0}^{\infty} C_l^*(n)t^n \\
&= \sum_{n=0}^{\infty} C_l^*(n)t^n - 2\sum_{n=0}^{\infty} C_l^*(n)t^{n+1} + \sum_{n=0}^{\infty} C_l^*(n)t^{n+l} - \sum_{n=0}^{\infty} C_l^*(n)t^{n+l+1} \\
&= \sum_{n=0}^{\infty} C_l^*(n)t^n - 2\sum_{n=1}^{\infty} C_l^*(n-1)t^n + \sum_{n=l}^{\infty} C_l^*(n-l)t^n - \sum_{n=l+1}^{\infty} C_l^*(n-l-1)t^n.
\end{aligned}$$

Equating coefficients of t^n for $n \geq l+2$ gives

$$C_l^*(n) = 2C_l^*(n-1) - C_l^*(n-l) + C_l^*(n-l-1). \qquad (7)$$

The last step is to note

$$\begin{aligned}
C_l(n) &= 2^{n-1} - C_l^*(n) \\
&= 2^{n-1} - (2C_l^*(n-1) - C_l^*(n-l) + C_l^*(n-l-1)) \\
&= 2(2^{n-2} - C_l^*(n-1)) - (2^{n-l-1} - C_l^*(n-l)) \\
&\quad + (2^{n-l-2} - C_l^*(n-l-1)) + 2^{n-l-2} \\
&= 2C_l(n-1) - C_l(n-l) + C_l(n-l-1) + 2^{n-l-2}.
\end{aligned}$$

The first row therefore satisfies the recurrence

$$C_1(n) = C_1(n-1) + C_1(n-2) + 2^{n-3},$$

valid for $n \geq 3$. From this we observe that $C_1(n) = 2^{n-1} - F_{n-1}$, where F_{n-1} is the $n-1$th Fibonacci number.

A combinatorial proof of (7) is of independent interest. Write

$$\begin{aligned} C_l^*(n) &= \#\,\{\text{compositions of } n \text{ with no part } l\} \\ &= \sum_{\substack{k=1 \\ k \neq l}}^{n} \#\{\text{compositions of } n \text{ with no part } l \text{ and first part } k\} \\ &= \sum_{\substack{k=1 \\ k \neq l}}^{n} C_l^*(n-k) \end{aligned}$$

Similarly

$$C_l^*(n-1) = \sum_{\substack{k=1 \\ k \neq l}}^{n-1} C_l^*(n-1-k)$$

Thus

$$\begin{aligned} C_l^*(n) &= C_l^*(n-1) + \sum_{\substack{k=2 \\ k \neq l}}^{n} C_l^*(n-k) \\ &= C_l^*(n-1) + \sum_{\substack{j=1 \\ j+1 \neq l}} C_l^*(n-1-j) \\ &= 2C_l^*(n-1) - C_l^*(n-l) + C_l^*(n-l-1) \end{aligned}$$

Corollary 10. *The generating function for the last row of Table 3, labelled Σ, is*

$$\left(1 + \frac{t^2}{1-2t}\right) \sum_{l=1}^{\infty} \frac{t^l}{1-2t+t^l-t^{l+1}}. \tag{8}$$

Proof: Write the summands in

$$\sum_{l=1}^{\infty} \left(\frac{t}{1-2t} - \frac{t-t^l+t^{l+1}}{1-2t+t^l-t^{l+1}} \right)$$

over a common denominator and then remove the common factor.

Hwang and Yeh have derived from the generating function (8) the asymptotic mean value

$$\frac{\sum_{m=1}^{n} mF(n,m)}{2^{n-1}} = \log_2 n - \frac{3}{2} + \frac{\gamma}{\log 2} - \varpi(\log_2 n) + O(n^{-1}\log n),$$

where $\varpi(u)$ is a periodic function of small amplitude. This result and others are contained in [2].

It is interesting to observe that the first l (nonzero) terms of row l in Table 3 are also the first l terms of every subsequent row of the table. Thus there is a sequence beginning $1, 2, 5, 12, 28, \ldots$ which we will call an *envelope* for the rows of the table. We account for this series in the next result.

Corollary 11. *The envelope* $1, 2, 5, 12, 28, 64, 144, \ldots$ *of Table 3 has generating function* $\left(\frac{1-t}{1-2t}\right)^2$.

Proof: In the generating function in Corollary 10 the significance of the t^l factor is only that row l is offset. For the envelope, we first consider $1/(1-2t+t^l-t^{l+1})$. Note (by long division) that the series expansion of this function of t matches the series expansion of $1/(1-2t)$ for l terms. Thus the first l terms of

$$\left(1+\frac{t^2}{1-2t}\right)\frac{1}{1-2t+t^l-t^{l+1}}$$

are provided by the simpler function

$$\left(1+\frac{t^2}{1-2t}\right)\frac{1}{1-2t}=\left(\frac{1-t}{1-2t}\right)^2$$

From the generating function we deduce that the nth entry of the envelope sequence (numbered from $n=0$) equals $2^{n-2}(n+3)$ for $n \geq 1$.

By studying the generating function of Corollary 10 we can provide a family of "nested recurrences" for the numbers Σ.

Theorem 12. *Denote by* $S(n)$ *and* $D_i(n)$ *sequences defined for* $n \geq 1$ *by the initial conditions* $S(1)=1, S(2)=2$, *and for any* i $D_i(1)=D_i(2)=1$, *and by the recurrences*

$$S(n)=2S(n-1)-D_1(n-1)+D_1(n-2)+(2^{n-3}+2^{n-4}+\ldots)+1$$

$$D_1(n)=2D_1(n-1)-D_2(n-1)+D_2(n-2)+(2^{n-3}+2^{n-5}+\ldots)+\begin{cases}1, & \text{if } 2 \mid n-1\\ 0, & \text{otherwise}\end{cases}$$

$$D_2(n)=2D_2(n-1)-D_3(n-1)+D_3(n-2)+(2^{n-3}+2^{n-6}+\ldots)+\begin{cases}1, & \text{if } 3 \mid n-1\\ 0, & \text{otherwise}\end{cases}$$

$$D_3(n)=2D_3(n-1)-D_4(n-1)+D_4(n-2)+(2^{n-3}+2^{n-7}+\ldots)+\begin{cases}1, & \text{if } 4 \mid n-1\\ 0, & \text{otherwise}\end{cases}$$

$$\vdots$$

Then $S(n)$ *is the* n*th entry of row* Σ *of Table 3. The sums of powers of 2 include terms as long as the exponents remain non-negative.*

Proof: The proof will be by induction on k, the subscript of D. Summing the entries in column n, we apply the recurrence of Corollary 9 summand by summand to obtain

$$S(n) = 2S(n-1) - \sum_{l=1}^{\infty} C_l(n-l) + \sum_{l=1}^{\infty} C_l(n-l-1) + \sum_{l=1}^{n-2} 2^l.$$

We denote $\sum_{l=1}^{\infty} C_l(n-l)$ by $D_1(n)$, where D is chosen mnemonically to represent a sum over a diagonal. The diagonal sum $D_1(n)$ steps through Table 3, repeatedly moving down one entry and to the left one entry from entry $C_1(n)$ in the first row. There is an extra summand of 1 because $C_n(n) = 1$ did not arise from a recurrence but from an initial condition. This explains the first recurrence.

Now consider the recurrence appropriate for D_{k+1}. Recurrence (8) applies to the summands of D_k as well, which arose by stepping down one entry and to the left k entries from entry $C_1(n)$ in the first row. Recurrence (8) applied to the summands of D_k gives a shallower diagonal, stepping down one entry and to the left $k+1$ entries, with an extra summand of 1 arising for $k+1 \mid n-1$ because in every $k+1$st shallow diagonal the initial condition $C_{\lfloor n/(k+1) \rfloor}(n) = 1$ gives a term that does not arise in any recurrence.

The partition $\lambda_1 1 + \lambda_2 2 + \ldots + \lambda_n n = n$ (with λ_i occurrences of i) can be ordered in $(\lambda_1 + \lambda_2 + \ldots + \lambda_n)!/(\lambda_1!\lambda_2! \ldots \lambda_n!)$ ways. If we count compositions of n with j parts by taking ordered arrangements over partitions of n with j parts we get the well known identity

$$\binom{n-1}{j-1} = \sum_{\substack{\lambda_1 1 + \lambda_2 2 + \ldots + \lambda_n n = n \\ \lambda_1 + \lambda_2 + \ldots + \lambda_n = j}} \frac{j!}{\lambda_1!\lambda_2! \ldots \lambda_n!}.$$

Similarly we can count compositions of n with j parts of m different sizes by taking ordered arrangements of partitions of n into j parts with m different sizes. The result is expressed in the following proposition.

Proposition 13.

$$F(n, m, j) = \sum_{\substack{\lambda_1 1 + \lambda_2 2 + \ldots + \lambda_n n = n \\ \lambda_1 + \lambda_2 + \ldots + \lambda_n = j \\ \#\{i : \lambda_i > 0\} = m}} \frac{j!}{\lambda_1!\lambda_2! \ldots \lambda_n!}.$$

As this proposition shows, there is a close connection between partition identities and composition identities. Now we go the other way, and note some results for partitions analogous to the composition results we have derived.

Partitions with m distinct parts

Since partitions may be regarded as compositions with decreasing part size, and compositions may be generated from partitions by permuting the parts, it is not surprising that many of the formulas generated above have analogues for partition counting functions. We begin by recasting Table 1 as a table about partitions.

Table 4. Partitions of n with m distinct parts, $G(n,m)$, $1 \leq n \leq 16$, $1 \leq m \leq 5$

$n\backslash m$	1	2	3	4	5
1	1				
2	2				
3	2	1			
4	3	2			
5	2	5			
6	4	6	1		
7	2	11	2		
8	4	13	5		
9	3	17	10		
10	4	22	15	1	
11	2	27	25	2	
12	6	29	37	5	
13	2	37	52	10	
14	4	44	67	20	
15	4	44	97	30	1
16	5	55	117	52	2

The first column of Table 4 is again $d(n)$, the divisor counting function. The sums over divisors of the first two theorems have the following versions for partitions, obtained by counting partitions according to occurrences of the distinct part that occurs least often.

Theorem 14. *For $n \geq 2$,*

$$G(n,2) = \sum_{j=2}^{\lfloor n/3 \rfloor} \sum_{k=1}^{\lfloor n/j \rfloor - 1} \sum_{\substack{d \mid (n-jk) \\ k \neq d < \frac{n-jk}{j-1}}} 1 - \sum_{\substack{j=1 \\ j \mid n}}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n/j - 1}{2} \right\rfloor + \sum_{k=1}^{n-1} \sum_{\substack{d \mid (n-k) \\ k \neq d}} 1.$$

Theorem 15. *For $j \geq 2$ and $n > j$,*

$$G(n,2,j) = \sum_{r=1}^{\lfloor j/2 \rfloor} \sum_{\substack{k=1 \\ (j-r) \mid (n-kr) \\ k \neq (n-kr)/(j-r)}}^{\lfloor n/r \rfloor - 1} 1 - \begin{cases} \left\lfloor (\frac{2n}{j} - 1)/2 \right\rfloor, & \text{if } 2 \mid j \text{ and } j \mid 2n \\ 0, & \text{otherwise.} \end{cases}$$

Formula (4) in Theorem 7 has a version for partitions whose proof is immediate: From a composition of n into m parts one seeks the Ferrars graph of a partition into m distinct parts by replacing summands a_i with groups of j_i summands each of size a_i/j_i. All that is different is that, when an acceptable Ferrars graph is found, the partition counts once, instead of having its parts permuted to generate a family of compositions. What results is

Theorem 16.

$$G(n,m) = \sum_{\substack{n=a_1+a_2+\ldots+a_m \\ a_i\neq 0}} \sum_{j_1|a_1} \sum_{\substack{j_2|a_2 \\ j_1>j_2}} \sum_{\substack{j_3|a_3 \\ j_2>j_3}} \cdots \sum_{\substack{j_m|a_m \\ j_{m-1}>j_m}} 1.$$

In structure Table 4 resembles Table 1, but it has an envelope for its columns reminiscent of the envelope for the rows of Table 3.

Theorem 17. *The first $m+1$ non-zero entries in column m of Table 4 are the first $m+1$ non-zero entries of all subsequent columns. The envelope,*

$$1, 2, 5, 10, 20, 36, \ldots,$$

has generating function

$$\prod_{i=1}^{\infty}(1-t^i)^{-2}.$$

Proof: The first non-zero entry in column m occurs at the smallest n for which it is possible to have m distinct summands in a partition of n, which is at T_m, the mth triangular number. The farthest term in the envelope occurring in column m counts partitions of $T_m + m = T_{m+1} - 1$ into m distinct parts.

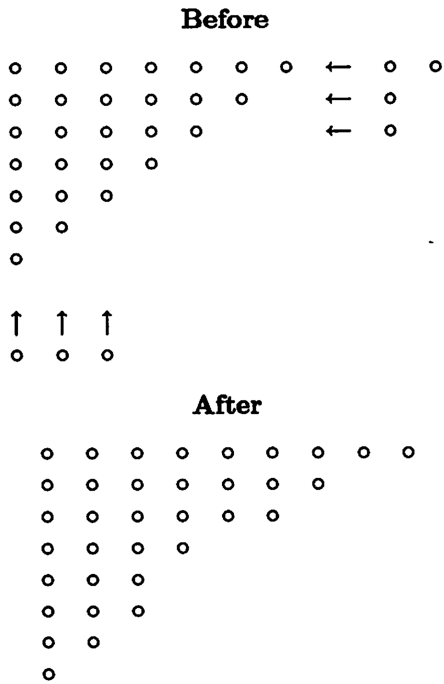
Consider the triangle of dots that is the Ferrars graph of the partition $1+2+\ldots+m$, and let $1 \leq a \leq m$ be given. Any partition of a can be represented as a Ferrars graph on its own, and appended to the triangular Ferrars graph by either of two different methods. One method is to adjoin dots representing the successive summands in the partition row by row to the rows of the triangular Ferrars graph, top to bottom. This results in a partition of T_m+a into m parts in all, all of them distinct. Another method is to adjoin dots representing the successive summands in the partition column by column to the columns of the triangular Ferrars graph, left to right. This results in a partition of $T_m + a$ in which the largest part is m, in which there are more than m parts in all, but in which there are only m distinct parts.

For any representation of $m = a + b$, any of the $p(a)$ partitions of a may be appended to the triangular Ferrars graph of $1+2+\ldots+m$ by the first method, and any of the $p(b)$ partitions of b may be appended by the second method. This results in a Ferrars graph for a partition of $T_m + m$ into exactly m distinct parts. Furthermore the process is reversible. Thus in the Ferrars graph of any partition of $T_m + m$ into m distinct parts, it is possible to strip off the first m dots in the first row, the first $m-1$ dots in the second, ..., the first dot in the mth row. This leaves m dots, in clusters of dots in the upper right and/or lower left, that can be interpreted as partitions of a (upper right) and b (lower left).

Overall the number of partitions of $T_m + m$ into m distinct parts is thus given by $\sum_{m=a+b} p(a)p(b)$. This sum allows a or b to be 0. This is the coefficient of t^m in the series expansion of $(1 + p(1)t + p(2)t^2 + \ldots)^2$, and hence the generating function is the square of the generating function for unrestricted partitions:

$$\prod_{i=1}^{\infty}(1 - t^i)^{-2}.$$

Figure 1. The Ferrars graph construction of Theorem 16 for $m = 7$, $a = 4 = 2 + 1 + 1$, and $b = 3 = 1 + 1 + 1$, yielding the partition of $T_7 + 7 = 35 = 9 + 7 + 6 + 4 + 3 + 3 + 2 + 1$ with 7 distinct parts.



The combinatorial approach of Theorem 17 also explains the values of the next term beyond the first $m+1$ terms of the envelope, $G(m(m+1)/2+$

$m+1, m)$, because in this case when the partitions of a and b consist of all 1's, the appended partitions span the $t+1$st diagonal to give a Ferrars graph of the partition of T_{m+1} into $m+1$ distinct parts. Excluding these $m+2$ cases gives the correct value of $G(m(m+1)/2+m+1, m)$.

Given the simplicity of the generating function, it is not surprising that the envelope has arisen in many enumeration problems. See, for example, [1, p. 90] in connection with partitions into parts of two kinds.

In analogy with Lemma 3 we have

Lemma 18. *Denote by $G(n, m, j)$ the number of partitions of n with exactly m distinct parts and j parts in all, and by $G^*(n, m, j)$ the number of compositions of n with exactly m distinct parts, j parts in all, and at least one part being a 1. Then*

$$G(n, m, j) = G(n-j, m, j) + G^*(n, m, j).$$

The parallel development continues.

Proposition 19.

$$G(n, m, j) = G(n-j, m, j) + \sum_k{}^* G(n-j, m-1, j-k),$$

where $\sum^*$ indicates a sum over those k for which a partition of n into m distinct parts, j parts in all, can have exactly k 1's.

The same results about the summands of Σ^* earlier, in Proposition 5 and the preceding remarks, apply here as well.

Corollary 20.

$$G(n, 2, j) = \sum_{l=1}^{\lfloor \frac{n-1}{j} \rfloor} \sum_{\substack{k \\ (j-k)|(n-lj)}} 1.$$

In conclusion we mention that the mean value for the number of distinct parts in a partition of n was obtained by Wilf [7]. He showed that

$$\frac{\sum_{m=1}^{n} mG(n, m)}{p(n)} = \sum_{i=0}^{n-1} p(i)/p(n) \sim \frac{\sqrt{6}}{\pi}\sqrt{n},$$

as $n \to \infty$.

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