Bipartite Graphs and Absolute Difference Tolerances

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Abstract

An abdiff-tolerance competition graph, G = (V, E), is a graph for which each vertex i can be assigned a non-negative integer t_i and at most |V| subsets S_j of V can be found such that xy ∈ E if and only if x and y lie in at least $|t_x-t_y|$ of the sets S_j. If G is not an abdiff-tolerance competition graph, it still is possible to find r > |V| subsets of V having the above property. The integer r - |V| is called the abdiff-tolerance competition number. This paper determines those complete bipartite graphs which are abdiff-tolerance competition graphs and finds an asymptotic value for the abdiff-tolerance competition number of $K_{1,n}$.

1. Introduction

In 1968 Cohen [4] defined the competition graph of a food chain as a directed graph, D, in which the vertices represent species and there is an arc from u to v if species u preys on species v. The *competition graph, G,* of arbitrary digraph D has the same vertex set as D, and vertices x and y are adjacent if and only if in D they compete for the same prey, that is, there is a vertex z such that xz and yz are arcs of D. After some time a body of research on the properties of competition graphs appeared [5, 9, 11, 12] and before long related concepts were defined. In particular, there has been interest in the *common enemy graph* [10, 14], G, of D in which vertices x and y are adjacent in G if and only if there is a vertex w such that wx and wy are arcs of D, the *competition-common enemy graph* [6, 7, 13] in which adjacencies exist when both of the conditions for competition and common enemy graphs are met, and the *niche graph* [3] for which there are adjacencies when at least one of those conditions is met.

Related to each of the above concepts is a determination of how far an arbitrary graph deviates from a competition graph, a common enemy graph, a competition-common enemy graph, or a niche graph. In most cases it can be shown that the addition of a sufficient number of isolated vertices to any graph can produce a graph in any of the four categories [11]. The one exception concerns niche graphs. There are some graphs which can not be turned into niche graphs by adding a finite number of vertices [3].

A different type of generalization of competition graphs was given in 1989 by Kim, McKee, McMorris, and Roberts [8]. They defined the *p-competition graph* G of digraph D as the graph with the same vertex set as D and two vertices adjacent if and only if they compete in D for at least p distinct species.

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When $p = 1$, this concept agrees with the conventional competition graph discussed above. It was observed (1) that if $S_i = \{x: xi \text{ is an arc of } D\}$ for vertex $i = 1, 2, \ldots, |V(D)|$, any intersection of p of the S_i induces a (possibly empty) complete subgraph of the p-competition graph G, and (2) the collection of all such p-intersections forms an edge clique cover of G. In this case the S_i are said to be a *p-edge clique cover* of G.

The most recent generalization is the *ϕ-tolerance competition graph* defined by [1]. Here ϕ is a non-negative valued symmetric function whose two arguments are usually assumed (but are not required) to be non-negative integers. A graph $G = (V, E)$ is a ϕ-tolerance competition graph if each vertex x can be assigned a value (tolerance) t_x such that there exists a collection of at most $|V|$ subsets of V having the property that edge xy is in G if and only if x and y lie together in at least $\phi(t_x, t_y)$ of the subsets. If such an assignment of tolerances allows such a collection of sets, the collection is said to be a ϕ-tolerance edge clique cover, or ϕ-T-ECC, of G. Brigham, McMorris, and Vitray [1, 2] investigated such graphs when all tolerances are non-negative integers and ϕ is "maximum," "minimum," or "sum." Notice that a p-competition graph is both a maximum- and a minimum-tolerance competition graph where every vertex is given tolerance p. It is known [1] that any graph can be transformed into a ϕ-tolerance competition graph by adding isolated vertices, and the minimum number of such vertices required to accomplish this is known as the *ϕ-tolerance competition number.* Of course, this number is 0 if the graph is a ϕ-tolerance competition graph.

Competition and p-competition graphs have application, of course, to any situation in which two or more entities compete for a common prize. Preliminary work indicates that ϕ-tolerance competition graphs may be useful in the area of resource allocation among users subject to inter-user relationships defined by the relative tolerances.

Unfortunately, although study of these graphs seems warrented, even simple classification problems appear to be extremely difficult. This paper attempts to establish some proof techniques applicable when ϕ is the absolute difference function, abbreviated *abdiff,* that is, $\text{abdiff}(s, t) = |s - t|$. It uses these techniques to determine which complete bipartite graphs are abdiff-tolerance competition graphs and to find the asymptotic order for the abdiff-tolearance competition number of $K_{1,n}$. It will be convenient to say that a collection of sets *forces* an edge uv of the graph G if u and v have tolerances t_u and t_v, respectively, and are together in at least $\phi(t_u, t_v) = |t_u - t_v|$ of the sets.

2. $K_{m,n}$ which are Abdiff-Tolerance Competition Graphs, with $2 \le m \le n$

In this section we consider complete bipartite graphs, $K_{m,n}$, where $2 \le m \le n$. As indicated above, if $\{u, v\} \in E$, then u and v must be in at least $|t_u - t_v|$ sets together, and, if $\{u, v\} \notin E$, then $|t_u - t_v| \ge 1$, that is, nonadjacent vertices must have distinct tolerances. Recall that, for a graph to be an abdiff-tolerance competition graph, the number of sets available is $|V|$. Showing that a graph is not an abdiff-tolerance competition graph involves proving that, under any feasible assignment of tolerances, the number of sets needed to force the edges is more than $|V|$. The flexibility in assigning tolerances makes this quite difficult in general. However, the following lemma does give a lower bound on the required number of sets.

Lemma 2.1. Let x and y be non adjacent vertices which are both adjacent to v. Then the number of sets which contain v but do not contain x is at least $|t_v - t_y| - |t_x - t_y| + 1$.

Proof: Let M be the number of sets which contain v but do not contain x. Since y and v are adjacent, they are together in at least $|t_v - t_y|$ sets. At most M of these sets do not contain x; hence, v, x, and y are together in at least $|t_v - t_y| - M$ sets. If $M \le |t_v - t_y| - |t_x - t_y|$, then x and y are together in at least $|t_v - t_y| - (|t_v - t_y| - |t_x - t_y|) = |t_x - t_y|$ sets which contradicts the assumption that x and y are non adjacent. ❑

Corollary 2.2. Let x and y be non adjacent vertices which are both adjacent to v. If $t_x \le t_y \le t_v$ or $t_x \ge t_y \ge t_v$, then v is in at least $2|t_y - t_v| + 1$ sets.

Proof: The number of sets containing v is the number of sets containing v and x plus the number of sets which contain v but do not contain x. The first of these numbers is at least $|t_v - t_x|$ and the second is greater than or equal to $|t_v - t_y| - |t_x - t_y| + 1$ by Lemma 2.1. Thus, v is in at least $|t_v - t_x| + |t_v - t_y| - |t_x - t_y| + 1$ sets, which equals $2|t_v - t_y| + 1$ because of the assumed inequality conditions. ❑

Consider $K_{m,n}$ where $2 \le m \le n$. We adopt the following notation. The vertices on one side of the bipartition are labeled with $v_1, v_2, \ldots, v_m$ and those on the other with $x_1, x_2, \ldots, x_n$. Without loss of generality, we assume that $t_{v_i} < t_{v_{i+1}}$ and $t_{x_j} < t_{x_{j+1}}$ for $1 \le i \le m - 1$ and $1 \le j \le n - 1$. Note that this assumption implies that $t_{v_i} - t_{v_j} \ge i - j$ and $t_{x_i} - t_{x_j} \ge i - j$ for all i and j such that $i \ge j$.

Theorem 2.3. $K_{2,n}$ is an abdiff-tolerance competition graph if and only if $n \le 4$.

Proof: We first show that, if $n \ge 5$, $K_{2,n}$ is not an abdiff-tolerance competition graph. Since $t_{v_2} > t_{v_1}$, either $t_{v_2} \ge t_{x_2}$ or $t_{v_1} \le t_{x_{n-1}}$. We assume $t_{v_2} \ge t_{x_2}$ and note that a similar argument can be given if $t_{v_1} \le t_{x_{n-1}}$. We have that $t_{x_1} < t_{x_2} \le t_{v_2}$; hence, by Corollary 2.2, v_2 must be in at least $2t_{v_2} - 2t_{x_2} + 1$ sets.

The total number of available sets is n + 2; hence $2t_{v_2} - 2t_{x_2} + 1 \le n + 2$, or $t_{v_2} \le \frac{n+1}{2} + t_{x_2}$. Thus $t_{v_1} \le \frac{n-1}{2} + t_{x_2}$. When $n \ge 5$, this inequality and the increasing tolerances for the vertices $x_1, x_2, \ldots, x_n$ imply that $t_{v_1} \le t_{x_{n-1}}$. Thus, $t_{v_1} \le t_{x_{n-1}} < t_{x_n}$ and, by Corollary 2.2, v_1 is in at least $2t_{x_{n-1}} - 2t_{v_1} + 1$ sets.

The two vertices v_1 and v_2 are nonadjacent, so they are in at most $t_{v_2} - t_{v_1} - 1$ sets together. Hence we need at least $(2t_{v_2} - 2t_{x_2} + 1) + (2t_{x_{n-1}} - 2t_{v_1} + 1) - (t_{v_2} - t_{v_1} - 1)$ sets altogether. This implies $t_{v_2} - t_{v_1} + 2t_{x_{n-1}} - 2t_{x_2} + 3 \leq n + 2$. On the other hand, since $n \geq 5$, $t_{x_{n-1}} - t_{x_2} \geq n - 3$ and $t_{v_2} - t_{v_1} \geq 1$; so, $t_{v_2} - t_{v_1} + 2(t_{x_{n-1}} - t_{x_2}) + 3 \geq 2(n - 3) + 4 = 2n - 2$. However, for $n \geq 5$, $2n - 2 > n + 2$, a contradiction.

Next, we will show by an assignment of tolerances to vertices and vertices to sets that, if n = 2, 3, or 4, $K_{2,n}$ is an abdiff-tolerance competition graph.

Case 1: $n = 2$. We assign $t_{v_1} = t_{x_1} = 0$ and $t_{v_2} = t_{x_2} = 1$. Then we need only two sets: $\{v_1, x_2\}$ and $\{v_2, x_1\}$.

Case 2: $n = 3$. We assign $t_{v_1} = t_{x_1} = 0$, $t_{v_2} = t_{x_3} = 2$, and $t_{x_2} = 1$. We need five sets: $\{v_2, x_1\}$, $\{v_2, x_1\}$, $\{v_1, x_3\}$, $\{v_1, x_3\}$, and $\{x_2, v_1, v_2\}$.

Case 3: $n = 4$. We assign x_1 through x_4 the tolerances 0 through 3, $t_{v_1} = 1$, and $t_{v_2} = 2$. We employ all six allowable sets as follows: $\{v_1, x_1, x_4\}$, $\{v_1, x_4\}$, $\{v_1, x_3\}$, $\{v_2, x_1, x_4\}$, $\{v_2, x_1\}$, and $\{v_2, x_2\}$. ❑

For the remainder of this section we focus attention on $K_{m,n}$, where $3 \leq m \leq n$. The next lemma is an easy technical result.

Lemma 2.4. For any v in $\{v_1, v_2, ..., v_m\}$, either

a) $t_v - t_{x_2} \geq \left\lfloor \frac{n}{2} \right\rfloor - 1$ or

b) $t_{x_{n-1}} - t_v \geq \left\lceil \frac{n}{2} \right\rceil - 1$.

Proof: Suppose $t_v - t_{x_2} < \left\lfloor \frac{n}{2} \right\rfloor - 1$, so that $t_v \leq t_{x_2} + \left\lfloor \frac{n}{2} \right\rfloor - 2$. Also, since $n \geq 3$, $t_{x_{n-1}} - t_{x_2} \geq (n - 1) - 2$. Hence,

$$t_{x_{n-1}} - t_v \geq t_{x_{n-1}} - \left(t_{x_2} + \left\lfloor \frac{n}{2} \right\rfloor - 2\right) \geq (n - 1) - 2 - \left\lfloor \frac{n}{2} \right\rfloor + 2 = \left\lceil \frac{n}{2} \right\rceil - 1. \quad ❑$$

We now use Lemma 2.4 in the proof of a slightly more difficult technical result which is used to divide the rest of the argument into two cases.

Lemma 2.5. Either $t_{x_{n-1}} - t_{v_2} \geq \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil - 3$ or

$t_{v_{m-1}} - t_{x_2} \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor - 2$.

Proof: Let $i = \left\lceil \frac{m}{2} \right\rceil$. By Lemma 2.4, either (a) $t_{v_i} - t_{x_2} \geq \left\lfloor \frac{n}{2} \right\rfloor - 1$, or (b) $t_{x_{n-1}} - t_{v_i} \geq \left\lceil \frac{n}{2} \right\rceil - 1$. We proceed by cases.

Case 1: $t_{v_i} - t_{x_2} \geq \left\lfloor \frac{n}{2} \right\rfloor - 1$. For $m \geq 3$, $m - 1 \geq \left\lceil \frac{m}{2} \right\rceil$ so, as always,

$t_{v_{m-1}} - t_{v_i} \geq m - 1 - \left\lceil \frac{m}{2} \right\rceil = \left\lfloor \frac{m}{2} \right\rfloor - 1$. Hence,

$t_{v_{m-1}} - t_{x_2} = (t_{v_{m-1}} - t_{v_i}) + (t_{v_i} - t_{x_2}) \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor - 2.$

Case 2: $t_{x_{n-1}} - t_{v_i} \geq \left\lceil \frac{n}{2} \right\rceil - 1$. Similar to above, $t_{v_i} - t_{v_2} \geq \left\lceil \frac{m}{2} \right\rceil - 2$, and we conclude $t_{x_{n-1}} - t_{v_2} = (t_{x_{n-1}} - t_{v_i}) + (t_{v_i} - t_{v_2}) \geq \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil - 3$. ❑

The next lemma determines the consequences of the first inequality in Lemma 2.5.

Lemma 2.6. If $3 \leq m \leq n$, $t_{x_{n-1}} - t_{v_2} \geq \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil - 3$, and $K_{m,n}$ is an abdiff-tolerance competition graph, then $m = n = 4$.

Proof: Since m and n are both at least three, the conditions of Lemma 2.5 imply that $t_{x_{n-1}} > t_{v_2}$. By Corollary 2.2, noting $t_{v_1} < t_{v_2} < t_{x_{n-1}}$, we have that x_{n-1} is in at least $2t_{x_{n-1}} - 2t_{v_2} + 1$ sets. Similarly, x_n is in at least $2t_{x_n} - 2t_{v_2} + 1$ sets. Also, x_n and x_{n-1} are not adjacent so they can be together in at most $t_{x_n} - t_{x_{n-1}} - 1$ sets. It follows that the number of sets which contain at least one of x_n or x_{n-1} is greater than or equal to:

$(2t_{x_n} - 2t_{v_2} + 1) + (2t_{x_{n-1}} - 2t_{v_2} + 1) - (t_{x_n} - t_{x_{n-1}} - 1) =$
$(t_{x_n} - t_{v_2}) + 3(t_{x_{n-1}} - t_{v_2}) + 3.$

By assumption, $t_{x_{n-1}} - t_{v_2} \geq \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil - 3$ which implies $t_{x_n} - t_{v_2} \geq \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil - 2$. Therefore, the number of sets which contain at least one of x_n or x_{n-1} is at least $4\left(\left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil\right) - 8$. Since m + n is the number of vertices, we must have

$m + n \geq 4\left(\left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil\right) - 8$, or $8 \geq 4\left(\left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{m}{2} \right\rceil\right) - (m + n)$.

Case 1. m and n are both even. Then, $8 \geq m + n$ and, since m and n are both at least 3, we obtain that $m = n = 4$.

Case 2. m is odd and n is even (the m even and n odd case is similar). Here $8 \geq 4\left(\frac{m+1}{2}+\frac{n}{2}\right)$ - (m + n) = m + n + 2. This is not possible if n is even and m and n are both at least 3.

Case 3. m and n are both odd. Here, $8 \geq 4\left(\frac{m+1}{2}+\frac{n+1}{2}\right)$ - (m + n) = m + n + 4 which is not possible when m and n are at least 3. ❑

The next lemma determines the consequences of the second inequality in Lemma 2.5.

Lemma 2.7. If $3 \leq m \leq n$, $t_{v_{m-1}} - t_{x_2} \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor -2$, and $K_{m,n}$ is an abdiff-tolerance competition graph, then either m = n = 3 or m = 3 and n = 5.

Proof: By an argument similar to that in Lemma 2.6 we can show that the number of sets which contain at least one of v_m or v_{m-1} is greater than or equal to $(t_{v_m} - t_{x_2}) + 3(t_{v_{m-1}} - t_{x_2}) + 3$. By assumption,

$t_{v_{m-1}} - t_{x_2} \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor - 2$ which implies that $t_{v_m} - t_{x_2} \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor - 1$.

It follows as in the proof to Lemma 2.6 that $m + n \geq 4\left(\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor\right) - 4$, or

$4 \geq 4\left(\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor\right) - (m + n)$.

Case 1. m and n are both even. Then, $4 \geq m + n$ which is impossible since m and n are both at least 3.

Case 2. m is odd and n is even (the case of m even and n odd is similar). In this case, $4 \geq 4\left(\frac{m-1}{2}+\frac{n}{2}\right)$ - (m + n) = m + n - 2. This is not possible if n is even and both m and n are at least 3.

Case 3. m and n are both odd. Here, $4 \geq 4\left(\frac{m-1}{2}+\frac{n-1}{2}\right)$ - (m + n) = m + n - 4 which can occur only if m = n = 3 or m = 3 and n = 5. ❑

Lemma 2.8. $K_{4,4}$ is not an abdiff-tolerance competition graph.

Proof: We may assume without loss of generality that $t_{v_2} \geq t_{x_2}$. Using Corollary 2.2 with x_1, x_2 and each of the vertices v_2, v_3, and v_4 in turn, we find that the number of sets containing at least one of v_2, v_3, and v_4 is greater than or equal to: $(2t_{v_2} - 2t_{x_2} + 1) + (2t_{v_3} - 2t_{x_2} + 1) + (2t_{v_4} - 2t_{x_2} + 1) -$ $(t_{v_3} - t_{v_2} - 1) - (t_{v_4} - t_{v_2} - 1) - (t_{v_4} - t_{v_3} - 1) = 4t_{v_2} + 2t_{v_3} - 6t_{x_2} + 6$.

Using the inequality $t_{v_3} \geq t_{v_2} + 1$, we obtain:

$$4t_{v_2} + 2t_{v_3} - 6t_{x_2} + 6 \geq 6t_{v_2} - 6t_{x_2} + 8. \qquad (*)$$

Since $t_{v_2} \geq t_{x_2}$ and there are at most eight sets available if $K_{4,4}$ is an abdiff-tolerance competition graph, it follows that $8 \geq 6t_{v_2} - 6t_{x_2} + 8$, which implies $6t_{v_2} - 6t_{x_2} + 8 = 8$ and that $t_{x_2} = t_{v_2}$. Further, t_{v_3} must equal $t_{v_2} + 1$ or else we would have a strict inequality in (*). Also, since $t_{x_2} = t_{v_2}$, we may reverse the roles of the x's and v's in the argument to obtain that $t_{x_3} = t_{x_2} + 1$. Using Corollary 2.2, we have that the number of sets containing at least one of x_3 and x_4 is greater than or equal to $(2t_{x_3} - 2t_{v_2} + 1) + (2t_{x_4} - 2t_{v_2} + 1) - (t_{x_4} - t_{x_3} - 1)$ which, by the above equalities, is equal to $t_{x_4} + 6 - t_{v_2}$. Hence, $2 = 8 - 6 \geq t_{x_4} - t_{v_2} = t_{x_4} - t_{x_2}$ and, since $t_{x_4} - t_{x_2} \geq 2$, we have $t_{x_4} = t_{x_2} + 2$. Similarly, $t_{v_4} = t_{v_2} + 2$. Replacing each tolerance by its negative and employing similarity, we conclude $t_{x_1} = t_{v_1} = t_{v_2} - 1$. It follows without loss of generality that $t_{v_i} = i - 1 = t_{x_i}$ for $i = 1, 2, 3, 4$. To generate the edges between v_1, v_2, x_3, and x_4, each of the pairs $\{v_1, x_4\}$, $\{v_1, x_4\}$, $\{v_1, x_4\}$, $\{v_1, x_3\}$, $\{v_1, x_3\}$, $\{v_2, x_4\}$, $\{v_2, x_4\}$, $\{v_2, x_3\}$ must be subsets of distinct sets. Just as v_1 must be in five of these sets, so must v_4 to generate its adjacencies with x_1 and x_2. But then v_4 shares at least three sets with one of v_1 or v_2, a contradiction. ❑

Lemma 2.9. $K_{3,5}$ is not an abdiff-tolerance competition graph.
Proof: Without loss of generality we take $t_{v_2} = 0$. Either $t_{x_4} - t_{v_2} \geq 1$ or $t_{v_2} - t_{x_2} \geq 2$. We treat the former case. The latter is similar and simpler. Thus $t_{x_5} - t_{v_2} \geq 2$. Using Corollary 2.2 twice, for x_5 and then x_4 with v_1 and v_2, and recognizing x_4 and x_5 can share at most $t_{x_5} - t_{x_4} - 1$ sets, we see that the number of sets containing at least one of x_4 and x_5 is greater than or equal to $[2(t_{x_5} - t_{v_2}) + 1] + [2(t_{x_4} - t_{v_2}) + 1] - (t_{x_5} - t_{x_4} - 1) =$ $t_{x_5} + 3t_{x_4} - 4t_{v_2} + 3 = (t_{x_5} - t_{v_2}) + 3(t_{x_4} - t_{v_2}) + 3 \geq 2 + 3 + 3 = 8$. Since only eight sets are available, $t_{x_5} - t_{v_2} = 2$ and $t_{x_4} - t_{v_2} = 1$, that is, $t_{x_4} = 1$ and $t_{x_5} = 2$ since $t_{v_2} = 0$.

The tolerances determined thus far show v_2 is in two sets with x_5 and a different set with x_4. Thus, there are at most five sets which contain v_1 but do not contain v_2. On the other hand, v_1 must be in at least $t_{x_5} - t_{v_1}$ sets with x_5 and $t_{x_4} - t_{v_1}$ different sets with x_4. Of these, $t_{v_2} - t_{v_1} - 1$ may also contain v_2, meaning there are at least $(t_{x_5} - t_{v_1}) + (t_{x_4} - t_{v_1}) - (t_{v_2} - t_{v_1} - 1)$ sets required which contain v_1 but not v_2. Hence, $4 - t_{v_1} \leq 5$, implying $t_{v_1} = -1$. Now v_1 must be in at least three sets with x_5 and two different ones with x_4. Furthermore, v_3, again by Corollary 2.2, is in at least $2(t_{v_3} - t_{x_2}) + 1 \geq 5$ sets, since $t_{x_2} \leq -1$ and $t_{v_3} \geq 1$. Thus it must be in at least two sets with v_1. This means $t_{v_3} \geq 2$, so in reality v_3 is in at least seven sets. Here, the overlap with sets containing v_1 is at least four, forcing $t_{v_3} \geq 4$ which imnplies v_3 is in at least 11 sets, a contradiction. ❑

Lemma 2.10. $K_{3,3}$ is an abdiff-tolerance competition graph.
Proof: Let $t_{x_i} = i = t_{v_i}$ for i = 1, 2, 3; and let the sets be$\{x_2, v_1, v_3\}$, $\{v_2, x_1, x_3\}$, $\{x_1, v_3\}$, $\{x_1, v_3\}$, $\{v_1, x_3\}$, and $\{v_1, x_3\}$. ❑

The results for $K_{m,n}$ with $2 \leq m \leq n$ are summarized in the following theorem.

Theorem 2.11. For $2 \leq m \leq n$, $K_{m,n}$ is an abdiff-tolerance competition graph if and only if m = 2 and $n \leq 4$ or m = n = 3.
Proof: The result follows from Theorem 2.3 and Lemmas 2.6 through 2.10. ❑

3. $K_{1,n}$ Which are Abdiff-Tolerance Competition Graphs

In this section we will determine those $K_{1,n}$ which are abdiff-tolerance competition graphs. Surprisingly, this case is more complex than when $2 \leq m \leq n$.

Suppose $K_{1,n}$ is an abdiff-tolerance competition graph and let S be the collection of sets of an abdiff-T-ECC. Let x be the vertex of degree n with neighbors labeled $v_1, v_2, \ldots, v_n$. We standardize the tolerance assignment by subtracting the tolerance of x from all tolerances to obtain an abdiff representation still using S but with the tolerance of x equal to 0. If fewer than $\left\lfloor \frac{n}{2} \right\rfloor$ vertices have positive tolerances, at least $\left\lfloor \frac{n}{2} \right\rfloor$ vertices have negative tolerances. In this case we negate all tolerances to obtain an abdiff representation in which at least $\left\lfloor \frac{n}{2} \right\rfloor$ vertices have positive tolerances. We label the vertices with positive tolerances $v_1, v_2, \ldots, v_m$. None of the independent vertices can have the same tolerance so we assign the labels in such a way that $0 < t_1 < t_2 < \ldots < t_m$ where, in a slight abuse of notation, t_i is the tolerance of vertex v_i. Also, all the edges of the graph are incident with x, so there is no need to have any sets in S which do not contain x. Thus we will assume x is in every set of S. Finally, there is no need for a vertex v_i to be in more than t_i sets although each v_i must be in at least t_i sets; hence we will assume each vertex v_i is in exactly t_i sets of S.

The following lemma applies to any sequence of consecutive independent vertices.

Lemma 3.1. The number of sets of S which contain v_{i-r} but do not contain any vertex in $\{v_{i-r+1}, v_{i-r+2}, \ldots, v_i\}$ is greater than or equal to

$$t_{i-r} - \sum_{k=i-r+1}^{i} (t_k - t_{i-r} - 1).$$

Proof: The vertex v_{i-r} is in exactly t_{i-r} sets. If $k > i-r$, then v_{i-r} and v_k which are not adjacent, can be in at most $t_k - t_{i-r} - 1$ sets together and the lemma follows immediately. ❑

The preceding result can be used to obtain a lower bound for the total number of sets required for a sequence of consecutive independent vertices.

Lemma 3.2. For $r \geq 0$, the number of sets of S which contain at least one vertex in $\{v_{i-r}, v_{i-r+1}, \ldots, v_i\}$ is greater than or equal to $t_i + \sum_{j=i-r}^{i-1}[t_j - \sum_{k=j+1}^{i}(t_k - t_j - 1)]$.

Proof: Exactly t_i sets contain v_i. Using Lemma 3.1 and moving through the list from highest to lowest index (i - 1 to i - r), we add in sets that could not have yet been counted to obtain the result. ❑

For any given i and r, with $i > r \geq 0$, we define T(i, r) to be the expression from Lemma 3.2, that is,

$$T(i, r) = t_i + \sum_{j=i-r}^{i-1}[t_j - \sum_{k=j+1}^{i}(t_k - t_j - 1)].$$

Here and elsewhere we assume a sum equals zero if the upper index is smaller than the lower index. We need one more easy technical result before proceeding to the central lemma of this section.

Lemma 3.3. If $K_{1,n}$ is an abdiff-tolerance competition graph with $n \geq 5$, then $2\left\lfloor \frac{n}{2} \right\rfloor - 2 \leq 2t_{m-1} \leq n$.

Proof: The condition on n ensures that there are at least two vertices of positive tolerance. By Lemma 3.2, S must contain at least $T(m, 1) = 2t_{m-1} + 1$ sets. On the other hand, S contains at most n +1 sets, so $T(m, 1) \leq n+1$, which yields the upper inequality. Also, as the tolerances are strictly increasing and at least $\left\lfloor \frac{n}{2} \right\rfloor$ vertices have positive tolerances, $t_{m-1} \geq \left\lfloor \frac{n}{2} \right\rfloor - 1$ which implies the lower inequality. ❑

The value of T(i, r) is a lower bound on the number of sets in S. We use this bound in the following lemma to obtain an upper bound on n when $K_{1,n}$ is an abdiff-tolerance competition graph.

Lemma 3.4. If $K_{1,n}$ is an abdiff-tolerance competition graph, then either $n \leq 17$ or $n = 19$.

Proof: Suppose $K_{1,n}$ is an abdiff-tolerance competition graph and that $n \geq 10$, so there are at least five vertices of positive tolerance. We divide the proof into two cases depending on whether n is even or odd.

Case 1. n is even. In this case Lemma 3.3 implies that $n - 2 \leq 2t_{m-1} \leq n$. Since $2t_{m-1}$ is even, one of the inequalities must actually be equality, that is, $t_{m-1} = \frac{n-2}{2}$ or $t_{m-1} = \frac{n}{2}$. Suppose that $t_{m-1} = \frac{n-2}{2}$. There are at least $\left\lfloor \frac{n}{2} \right\rfloor - 1 = \frac{n-2}{2}$ distinct positive tolerances less than or equal to t_{m-1}. Hence, the positive tolerances from t_1 to t_{m-1} must be the consecutive positive integers from 1 to $\frac{n-2}{2}$. We have assumed that $n \geq 10$, so that the vertices v_{m-1}, v_{m-2}, v_{m-3}, and v_{m-4} must exist and we compute $T(m-1, 3) = 2n - 14$. Suppose on the other hand that $t_{m-1} = \frac{n}{2}$. This implies that the positive tolerances from t_1 to t_{m-1} are taken from the integers from 1 to $\frac{n}{2}$, so that these tolerances are consecutive except for possibly one j for which $t_{j+1} = t_j + 2$. It is easy to check that in this case the value of $T(m-1, 3)$ is minimized if we take $t_{m-2} = t_{m-1} - 2$, and then $T(m-1, 3) = 2n - 16$. This latter expression is smaller than the previous one, and as the total number of sets is at least $T(m-1, 3)$ and at most $n + 1$, we have that $n + 1 \geq 2n - 16$ or $17 \geq n$. This implies $n \leq 16$ since n is an even number.
Case 2. n is odd. In this case Lemma 3.3 implies $n-3 \leq 2t_{m-1} \leq n$ and therefore, since n is odd, $t_{m-1} = \frac{n-1}{2}$ or $t_{m-1} = \frac{n-3}{2}$. Applying the same reasoning as in Case 1, we find that $T(m-1, 3)$ is minimized when $t_{m-1} = \frac{n-1}{2} = t_{m-2} + 2$ and, in this event, we compute $T(m-1, 3) = 2n - 18$. Therefore, $n + 1 \geq 2n - 18$ or $n \leq 19$. ❑

The above lemma yields 18 possible values of n for which $K_{1,n}$ could be an abdiff-tolerance competition graph. Each of these graphs must be considered in turn, which we have done to obtain the following theorem which completely settles the question.

Theorem 3.5. The graph $K_{1,n}$ is an abdiff-tolerance competition graph if and only if $n \leq 17$ or $n = 19$.
Proof: The forward implication is Lemma 3.4. The converse follows from the assignments and sets given in the appendix for the graphs in question. ❑

Observe that Theorem 3.5 shows that $K_{1,19}$ is a counterexample to the possible conjecture that, if G is an abdiff-tolerance competition graph, so is G - v for any vertex v.

4. An Asymptotic Lower Bound for the Abdiff-Tolerance Competition Number of $K_{1,n}$

In this section we find an asymptotic lower bound on the abdiff-tolerance competition number for $K_{1,n}$. Let k be the minimum number of isolated vertices which must be added to $K_{1,n}$ to obtain an abdiff-tolerance competition graph, and let $G = K_{1,n} \cup kK_1$. We will restrict our attention to values of n for which $K_{1,n}$ is not an abdiff-tolerance competition graph, in which case $k \geq 1$. Let S be the collection of sets of an abdiff-T-ECC for G and let N be the size of S, so that $N = n + k + 1$.

We will maintain the notation from the previous section and note that many of the observations made at the beginning of that section are still valid. Thus, x will be the vertex of degree n, and once again we may assume that x is in every element of S and the tolerance of x is 0. Also, $v_1, v_2, ..., v_m$ will be the degree one vertices of positive tolerance, and we assign those tolerances so that $0 < t_1 < t_2 < t_3 < ... < t_m$. As before, we may assume $m \geq \left\lfloor \frac{n}{2} \right\rfloor$.

It is easy to see that Lemmas 3.1 and 3.2 still hold for the vertices in $\{v_1, v_2, ..., v_m\}$. We begin this section with a rather technical lemma involving T(i, r).

Lemma 4.1. For $0 \leq q \leq r\text{-}1 \leq i$, T(i, r) =

$$\sum_{p=0}^{q}(2p+1-r)t_{i-p} + (q+2)\sum_{j=i-r}^{i-q-1} t_j + \sum_{p=0}^{q}(r-p) - \sum_{j=i-r}^{i-q-2}\sum_{k=j+1}^{i-q-1}(t_k - t_j - 1).$$

Proof: We employ induction on q. Using the definition of T(i,r) we have

$$T(i,r) = t_i + \sum_{j=i-r}^{i-1}[t_j - \sum_{k=j+1}^{i}(t_k - t_j - 1)] = t_i + \sum_{j=i-r}^{i-1} t_j - \sum_{j=i-r}^{i-1}\sum_{k=j+1}^{i}(t_k - t_j - 1).$$

By taking out all terms which correspond to k = i of the inner sum in the double sum, we obtain $T(i,r) = t_i + \sum_{j=i-r}^{i-1} t_j - \sum_{j=i-r}^{i-1}(t_i - t_j - 1) - \sum_{j=i-r}^{i-2}\sum_{k=j+1}^{i-1}(t_k - t_j - 1)$.

Taking out the terms not dependent on j in the second sum yields

$$T(i,r) = t_i + \sum_{j=i-r}^{i-1} t_j - rt_i + r + \sum_{j=i-r}^{i-1} t_j - \sum_{j=i-r}^{i-2}\sum_{k=j+1}^{i-1}(t_k - t_j - 1).$$ Hence

$$T(i,r) = (1-r)t_i + 2\sum_{j=i-r}^{i-1} t_j + r - \sum_{j=i-r}^{i-2}\sum_{k=j+1}^{i-1}(t_k - t_j - 1),$$ which is the stated result when q = 0.

Assume the lemma is true for some fixed q with $0 \leq q \leq r\text{-}2$, so that T(i, r) =

$$\sum_{p=0}^{q}(2p+1-r)t_{i-p} + (q+2)\sum_{j=i-r}^{i-q-1} t_j + \sum_{p=0}^{q}(r-p) - \sum_{j=i-r}^{i-q-2}\sum_{k=j+1}^{i-q-1}(t_k - t_j - 1).$$ We

proceed as above by taking the j = i-q-1 term out of the second sum and taking the k = i-q-1 terms out of the inner sum of the double sum to obtain

$$T(i,r)=\sum_{p=0}^{q}(2p+1-r)t_{i-p}+(q+2)t_{i-q-1}+(q+2)\sum_{j=i-r}^{i-q-2}t_j+\sum_{p=0}^{q}(r-p)$$

$$-\sum_{j=i-r}^{i-q-2}(t_{i-q-1}-t_j-1)-\sum_{j=i-r}^{i-q-3}\sum_{k=j+1}^{i-q-2}(t_k-t_j-1)$$. The sum in the fifth term has $(i - q - 2) - (i - r) + 1 = r - q - 1$ terms. Thus moving $t_{i-q-1} - 1$ (which is not dependent on j) out of the sum yields

$$T(i,r)=\sum_{p=0}^{q}(2p+1-r)t_{i-p}+(q+2)t_{i-q-1}+(q+2)\sum_{j=i-r}^{i-q-2}t_j+\sum_{p=0}^{q}(r-p)$$

$$-(r-q-1)t_{i-q-1}+(r-q-1)+\sum_{j=i-r}^{i-q-2}t_j-\sum_{j=i-r}^{i-q-3}\sum_{k=j+1}^{i-q-2}(t_k-t_j-1)$$. Combining the second and fifth terms gives us $[2(q+1)+1-r]t_{i-q-1}$ which can be incorporated into the first sum by increasing the upper index from q to q+1. Similarly the sixth term can be included in the sum of the fourth term by increasing the upper index by one. Furthermore, the third and seventh terms, which contain the same sum, can also be combined into one term. All of which leads to $T(i, r) =$

$$\sum_{p=0}^{q+1}(2p+1-r)t_{i-p}+(q+3)\sum_{j=i-r}^{i-q-2}t_j+\sum_{p=0}^{q+1}(r-p)-\sum_{j=i-r}^{i-q-3}\sum_{k=j+1}^{i-q-2}(t_k-t_j-1),$$ which is the result of the lemma for q + 1. ❑

The result of Lemma 4.1 can be simplified, as shown by the following.

Lemma 4.2. For $0 \le r < i$, $T(i,r)=\sum_{p=0}^{r}(2p+1-r)t_{i-p}+\frac{r(r+1)}{2}$.

Proof: It is easy to check the equality for r = 0 and r = 1. If r > 1, we set q equal to r - 2 in Lemma 4.1 and obtain

$$T(i,r)=\sum_{p=0}^{r-2}(2p+1-r)t_{i-p}+r\sum_{j=i-r}^{i-r+1}t_j+\sum_{p=0}^{r-2}(r-p)-\sum_{j=i-r}^{i-r}\sum_{k=j+1}^{i-r+1}(t_k-t_j-1)$$. The second term reduces to $r(t_{i-r} + t_{i-r+1})$ and the double sum becomes $t_{i-r+1} - t_{i-r} - 1$, so we have $T(i, r) =$

$$\sum_{p=0}^{r-2}(2p+1-r)t_{i-p}+r(t_{i-r}+t_{i-r+1})+\sum_{p=0}^{r-2}(r-p)-(t_{i-r+1}-t_{i-r}-1)$$. We increase the upper index on the sum in the first term on the right hand side of the previous equation from r-2 to r and subtract the two additional terms from the second term to obtain

$$T(i,r) = \sum_{p=0}^{r}(2p+1-r)t_{i-p} - t_{i-r} + t_{i-r+1} + \sum_{p=0}^{r-2}(r-p) - (t_{i-r+1} - t_{i-r} - 1).$$

Combining the terms outside the sums, we arrive at the equation

$T(i,r) = \sum_{p=0}^{r}(2p+1-r)t_{i-p} + \sum_{p=0}^{r-2}(r-p) + 1$. Since the last two terms equal the sum of the first r positive integers, the result follows. ❑

Our next task will be to find a lower bound on k for a given n, or, equivalently, a lower bound on N. This can be accomplished by obtaining an upper bound on n for a given N. As always, the number of vertices of positive tolerance is greater than or equal to $\left\lfloor \frac{n}{2} \right\rfloor$; hence we can attain our upper bound on n by finding an upper bound on m, the number of vertices of positive tolerance. The largest such tolerance must not exceed N, the number of available sets. Our strategy is to bound the number of such tolerances, and hence the number of vertices. With this in mind we establish the following notation. For a fixed i, let $x_r = t_i - t_{i-r}$ for $0 \le r < i$. The next lemma provides a lower bound on x_r.

Lemma 4.3. For a fixed i and any r such that $0 \le r < i$,

$$x_r \ge \max\left\{ r, t_i + \frac{r}{2} - \frac{1}{r+1}\left[\sum_{p=1}^{r-1}(2p+1-r)x_p + N\right]\right\}.$$

Proof: By definition, $x_0 = 0$ and $t_j > t_{j-1}$ for all j. Thus $x_r \ge r$ for $0 \le r < i$ which gives the first lower bound. Using Lemma 4.2 and the fact that $t_{i-p} = t_i - x_p$, we have

$$T(i,r) = \sum_{p=0}^{r}(2p+1-r)t_{i-p} + \frac{r(r+1)}{2} = \sum_{p=0}^{r}(2p+1-r)(t_i - x_p) + \frac{r(r+1)}{2}.$$

Splitting the sum yields

$T(i,r) = \sum_{p=0}^{r}(2p+1-r)t_i - \sum_{p=0}^{r}(2p+1-r)x_p + \frac{r(r+1)}{2}$. Again splitting the first sum and factoring out the t_i, which does not depend on p, we find

$T(i,r) = t_i\sum_{p=0}^{r}(2p+1) - t_i\sum_{p=0}^{r} r - \sum_{p=0}^{r}(2p+1-r)x_p + \frac{r(r+1)}{2}$. The first sum is the sum of the first r+1 positive odd integers and the second sum is simply (r+1)r; hence $T(i,r) = t_i(r+1)^2 - t_i(r+1)r - \sum_{p=0}^{r}(2p+1-r)x_p + \frac{r(r+1)}{2}$.

Separating the term involving x_r from the remaining sum, we obtain

$$T(i,r) = t_i(r+1)^2 - t_i(r+1)r - (2r+1-r)x_r - \sum_{p=0}^{r-1}(2p+1-r)x_p + \frac{r(r+1)}{2}.$$

Combining the first two terms and simplifying the coefficient of x_r yields

$T(i,r) = t_i(r+1) - (r+1)x_r - \sum_{p=0}^{r-1}(2p+1-r)x_p + \frac{r(r+1)}{2}$. Using the fact that

$N \geq T(i, r)$, we obtain $N \geq t_i(r+1) - (r+1)x_r - \sum_{p=0}^{r-1}(2p+1-r)x_p + \frac{r(r+1)}{2}$.

Solving for x_r and recalling $x_0 = 0$ yields the second lower bound of the lemma. ❑

We define $L(i, r) = t_i + \frac{r}{2} - \frac{1}{r+1}\left[\sum_{p=1}^{r-1}(2p+1-r)x_p + N\right]$, the second lower bound of Lemma 4.3. The question we are dealing with here is "For a fixed i, how small can x_r be?" Thus, we are interested in finding a lower bound for L(i, r), a result which is supplied by the following lemma.

Lemma 4.4. For a fixed i, and any r such that $0 \leq r < i$, and any choice of tolerances,

$$L(i,r) \geq t_i + \frac{r}{2} - \frac{1}{r+1}\left[N + \frac{(r-1)r(r+1)}{6} + (x_r - r)\left\{r^2 - \left\lfloor\frac{r+1}{2}\right\rfloor^2 - r\left\lfloor\frac{r}{2}\right\rfloor\right\}\right].$$

Proof: In the expression for L(i, r), the values of t_i, r, and N are fixed, so the only items that may vary are the x_p. It is apparent that the value of L(i,r) is minimized if x_p is as small as possible for those p for which $2p+1-r < 0$ and if x_p is as large as possible for those p for which $2p+1-r > 0$. By definition, $x_{p+1} - x_p = t_{i-p} - t_{i-(p+1)} > 0$, so $x_{p+1} > x_p$. Also, by Lemma 4.3, $x_p \geq p$. It follows that a lower bound for L(i, r) is obtained if we take values for x_p as follows:

$x_p = \begin{cases} p & \text{if } 1 \leq p \leq \frac{r-1}{2} \\ x_r - (r-p) & \text{if } \frac{r-1}{2} < p \leq r-1 \end{cases}$. With these values for the x_p's, we obtain the following lower bound for L(i, r):

$$L(i,r) \ge t_i + \frac{r}{2} - \frac{1}{r+1}\left[N + \sum_{p=1}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (2p+1-r)p + \sum_{p=\left\lceil \frac{r+1}{2} \right\rceil}^{r-1} (2p+1-r)(x_r - r + p)\right].$$

Splitting the second sum and factoring out the x_r - r, which does not depend on p, yields $L(i,r) \ge t_i + \frac{r}{2} -$

$$\frac{1}{r+1}\left[N + \sum_{p=1}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (2p+1-r)p + (x_r - r) \sum_{p=\left\lceil \frac{r+1}{2} \right\rceil}^{r-1} (2p+1-r) + \sum_{p=\left\lceil \frac{r+1}{2} \right\rceil}^{r-1} (2p+1-r)p\right].$$

Combining the first and last sums, we obtain

$$L(i,r) \ge t_i + \frac{r}{2} - \frac{1}{r+1}\left[N + \sum_{p=1}^{r-1} (2p+1-r)p + (x_r - r) \sum_{p=\left\lceil \frac{r+1}{2} \right\rceil}^{r-1} (2p+1-r)\right].$$

Splitting both sums leads to $L(i,r) \ge t_i + \frac{r}{2} -$

$$\frac{1}{r+1}\left[N + 2\sum_{p=1}^{r-1} p^2 - (r-1)\sum_{p=1}^{r-1} p + (x_r - r) \sum_{p=\left\lceil \frac{r+1}{2} \right\rceil}^{r-1} (2p+1) - (x_r - r)r \sum_{p=\left\lceil \frac{r+1}{2} \right\rceil}^{r-1} 1\right],$$

where terms independent of p have been factored out of the summations. The number of terms in the last two sums is $r - 1 - \left\lceil \frac{r+1}{2} \right\rceil + 1 = r - \left\lceil \frac{r+1}{2} \right\rceil = \left\lfloor \frac{r}{2} \right\rfloor$.

Noting that $\sum_{p=\left\lceil \frac{r+1}{2} \right\rceil}^{r-1} (2p+1) = \sum_{p=1}^{r-1} (2p+1) - \sum_{p=1}^{\left\lceil \frac{r+1}{2} \right\rceil - 1} (2p+1)$ and using the formulas for the sum of the first t positive integers, the sum of the first t positive odd integers, and the sum of the squares of the first t positive integers, we are able to rewrite our inequality as $L(i,r) \ge t_i + \frac{r}{2} -$

$$\frac{1}{r+1}\left[N + 2\frac{(r-1)r(2r-1)}{6} - (r-1)\frac{(r-1)r}{2} + (x_r - r)\left\{r^2 - \left\lceil \frac{r+1}{2} \right\rceil^2 - r\left\lfloor \frac{r}{2} \right\rfloor\right\}\right].$$

It is easy to see that $2\frac{(r-1)r(2r-1)}{6}-(r-1)\frac{(r-1)r}{2}=\frac{(r-1)r(r+1)}{6}$ which implies our result. ❑

The lower bound for L(i, r) given in the preceding lemma is unwieldy, but we can make it a bit simpler and use it to prove the following important lemma which gives the hoped for lower bound on x_r.

Lemma 4.5. For a fixed i and any r such that $1 \le r \le i-1$, $x_r > \frac{4[(r+1)t_i - N]}{(r+2)^2}$

Proof: Let $f(r)=r^2-\left\lfloor\frac{r+1}{2}\right\rfloor^2-r\left\lfloor\frac{r}{2}\right\rfloor$. Thus, if r is even, $f(r)=\frac{r^2}{4}$ and, if r is odd, $f(r)=\frac{r^2-1}{4}$. Rewriting the bound for L(i, r) found in Lemma 4.4 using f(r) and employing Lemma 4.3 gives
$x_r \ge L(i,r) \ge t_i+\frac{r}{2}-\frac{1}{r+1}\left[N+\frac{(r-1)r(r+1)}{6}+x_r f(r)-rf(r)\right]$. Solving this inequality for x_r, we obtain $x_r \ge \frac{(r+1)t_i+\frac{(r+1)r}{2}-\frac{r(r^2-1)}{6}+rf(r)-N}{r+1+f(r)}$. We know that $\frac{r^2-1}{4} \le f(r) \le \frac{r^2}{4}$. By replacing f(r) by $\frac{r^2-1}{4}$ in the numerator and by $\frac{r^2}{4}$ in the denominator, we obtain

$x_r \ge \frac{(r+1)t_i+\frac{(r+1)r}{2}-\frac{r(r^2-1)}{6}+r\frac{r^2-1}{4}-N}{r+1+\frac{r^2}{4}}$. Multiplying the top and bottom of the right hand side of this expression by 12 and expanding terms we obtain $x_r \ge \frac{12(r+1)t_i+6r^2+6r-2r^3+2r+3r^3-3r-12N}{12r+12+3r^2}=$

$\frac{12[(r+1)t_i-N]+r^3+6r^2+5r}{12r+12+3r^2}=\frac{12[(r+1)t_i-N]+r(r+1)(r+5)}{3(r+2)^2}$. We discard the r(r+1)(r+5) term from the numerator to obtain the result of the lemma. ❑

A major difficulty in any analysis of abdiff-tolerance is that there are many possible choices for the positive tolerances $t_1, t_2, ..., t_m$. To overcome this obstacle we use a sequence $s_0, s_1. ...$ of numbers which will dominate any of the choices. We define our sequence inductively as follows: $s_0 = N$ and $s_{i+1} = s_i - \frac{4[(i+2)s_i - N]}{(i+3)^2}$ for $i \geq 0$. For convenience, we let $\sigma(k)$ be the sum of the first k + 1 nonnegative integers. We prove a lemma which relates the s_i's to any collection of positive tolerances $t_1, t_2, \ldots, t_m$.

Lemma 4.6. For any fixed i and for any valid set of tolerances, if $m - \sigma(i) > 0$, then $s_i \geq t_{m-\sigma(i)}$.

Proof: We proceed by induction on i. For $i = 0$, $\sigma(0) = 0$ and $s_0 = N \geq t_{m-0}$. Assume $s_{i-1} \geq t_{m-\sigma(i-1)}$. We know

$t_{m-\sigma(i-1)} - t_{m-\sigma(i)} = t_{m-\sigma(i-1)} - t_{m-\sigma(i-1)-i} = x_i$. Note that $m - \sigma(i) > 0$ implies $m - \sigma(i-1) - 1 \geq i$; hence, by Lemma 4.5,

$t_{m-\sigma(i-1)} > t_{m-\sigma(i)} + \frac{4[(i+1)t_{m-\sigma(i-1)} - N]}{(i+2)^2}$. Let $d = s_{i-1} - t_{m-\sigma(i-1)}$, so $d \geq 0$ and we have

$$\frac{4[(i+1)s_{i-1} - N]}{(i+2)^2} = \frac{4[(i+1)[d + t_{m-\sigma(i-1)}] - N]}{(i+2)^2} =$$

$$\frac{4(i+1)d}{(i+2)^2} + \frac{4[(i+1)t_{m-\sigma(i-1)} - N]}{(i+2)^2}.$$ Therefore,

$$s_i = s_{i-1} - \frac{4[(i+1)s_{i-1} - N]}{(i+2)^2}$$

$$= t_{m-\sigma(i-1)} + d - \left[\frac{4(i+1)d}{(i+2)^2} + \frac{4[(i+1)t_{m-\sigma(i-1)} - N]}{(i+2)^2}\right].$$ It follows that

$$s_i > t_{m-\sigma(i)} + d - \frac{4(i+1)d}{(i+2)^2} = t_{m-\sigma(i)} + d\left[1 - \frac{4(i+1)}{(i+2)^2}\right] \geq t_{m-\sigma(i)}. \quad \square$$

Although the s_i's may not be among the possible choices of tolerances for any of the vertices $v_1, v_2, \ldots, v_m$, by using Lemma 4.6 we can achieve our desired bound on the number of t_i's by obtaining a bound on the number of s_i's. We will do this, but first we establish the following lemma which provides a recursive formula for the s_i's.

Lemma 4.7. For a fixed $i \geq 2$ and any q where $0 \leq q \leq i-2$,

$$s_i = \frac{(i-q)^2(i-1-q)^2}{(i+2)^2(i+1)^2} s_{i-2-q} + \frac{4N}{(i+2)^2(i+1)^2} \sum_{j=i-q}^{i+1} j^2.$$

Proof: We proceed by induction on q. From the definition of the s_i's we have

$s_i = s_{i-1} - \frac{4[(i+1)s_{i-1} - N]}{(i+2)^2} = \frac{i^2}{(i+2)^2} s_{i-1} + \frac{4N}{(i+2)^2}$. Iterating by substituting for s_{i-1} in the right hand side of the above equality yields

$$s_i = \frac{i^2}{(i+2)^2}\left[\frac{(i-1)^2}{(i+1)^2}s_{i-2} + \frac{4N}{(i+1)^2}\right] + \frac{4N}{(i+2)^2}.$$ Hence

$$s_i = \frac{i^2(i-1)^2}{(i+2)^2(i+1)^2}s_{i-2} + \frac{4N}{(i+2)^2(i+1)^2}\left[i^2 + (i+1)^2\right],$$ which is the result for $q = 0$. Suppose the equality holds for q where $0 \le q < i-2$. Again, by the definition of the s_i's, $s_{i-2-q} = \frac{(i-2-q)^2}{(i-q)^2}s_{i-3-q} + \frac{4N}{(i-q)^2}$. By the inductive hypothesis

$$s_i = \frac{(i-q)^2(i-1-q)^2}{(i+2)^2(i+1)^2}s_{i-2-q} + \frac{4N}{(i+2)^2(i+1)^2}\sum_{j=i-q}^{i+1} j^2.$$ Substituting the equality for s_{i-2-q} yields $s_i =$

$$\frac{(i-q)^2(i-1-q)^2}{(i+2)^2(i+1)^2}\left[\frac{(i-2-q)^2}{(i-q)^2}s_{i-3-q} + \frac{4N}{(i-q)^2}\right] + \frac{4N}{(i+2)^2(i+1)^2}\sum_{j=i-q}^{i+1} j^2.$$

Performing the multiplication and canceling the $(i-q)^2$, we obtain $s_i =$

$$\frac{(i-1-q)^2(i-2-q)^2}{(i+2)^2(i+1)^2}s_{i-3-q} + \frac{(i-1-q)^2}{(i+2)^2(i+1)^2}4N + \frac{4N}{(i+2)^2(i+1)^2}\sum_{j=i-q}^{i+1} j^2.$$

Factoring $\frac{4N}{(i+2)^2(i+1)^2}$ out of the last two terms gives

$$s_i = \frac{(i-1-q)^2(i-2-q)^2}{(i+2)^2(i+1)^2}s_{i-3-q} + \frac{4N}{(i+2)^2(i+1)^2}\left[(i-1-q)^2 + \sum_{j=i-q}^{i+1} j^2\right].$$

Rewriting in terms of (q+1) and combining the terms in brackets, we obtain

$$s_i = \frac{[i-(q+1)]^2[i-1-(q+1)]^2}{(i+2)^2(i+1)^2}s_{i-2-(q+1)} + \frac{4N}{(i+2)^2(i+1)^2}\left[\sum_{j=i-(q+1)}^{i+1} j^2\right],$$

which is the result of the lemma for q+1. □

We now proceed to the central theorem of this section.

Theorem 4.8. If k is the abdiff tolerance competition number of $K_{1,n}$, for $n \ge 18$, and $N = n + k + 1$, then $n \le 4\sqrt[3]{N^2} + 7$.

Proof: For $i \ge 2$, setting $q = i - 2$ in Lemma 4.7 gives

$$s_i = \frac{4}{(i+2)^2(i+1)^2}s_0 + \frac{4N}{(i+2)^2(i+1)^2}\sum_{j=2}^{i+1} j^2.$$ Using the formula for the

squares of the first i+1 positive integers, we see that

$$\frac{4N}{(i+2)^2(i+1)^2}\sum_{j=2}^{i+1} j^2 = \frac{4N}{(i+2)^2(i+1)^2}\left[\frac{(i+1)(i+2)(2i+3)}{6} - 1\right].$$ We increase the last expression by discarding the -1 and replacing the 2i + 3 in the numerator with 2i + 4. After simplification we obtain $\frac{4N}{(i+2)^2(i+1)^2}\sum_{j=2}^{i+1} j^2 < \frac{4N}{3(i+1)}$. It follows that $s_i < \frac{4}{(i+2)^2(i+1)^2}s_0 + \frac{4N}{3(i+1)}$. By construction, $s_0 = N$. We choose I to be the unique integer such that $\sqrt[3]{N}+1 > I+1 \geq \sqrt[3]{N}$. Note that I is at least 2 since $N \geq n + 1 \geq 19$. Hence, by the preceding, we have

$s_I < \frac{4N}{\left(\sqrt[3]{N}\right)^4} + \frac{4N}{3\sqrt[3]{N}} < 2 + \frac{4\sqrt[3]{N^2}}{3}$. If $m - \sigma(I) \leq 0$, then

$m \leq \frac{(I+1)(I+2)}{2} < \frac{(\sqrt[3]{N}+1)(\sqrt[3]{N}+2)}{2} < \sqrt[3]{N^2}$ for sufficiently large N.

Otherwise $m - p\sigma(I) > 0$ and we may use Lemma 4.6. If $t_1, t_2, ..., t_m$ is any actual sequence of tolerances, by Lemma 4.6, $s_I \geq t_{m-\sigma(I)}$; hence,

$t_{m-\sigma(I)} < 2 + \frac{4\sqrt[3]{N^2}}{3}$. Since the t_i's must be strictly increasing, fewer than

$2 + \frac{4\sqrt[3]{N^2}}{3}$ of the vertices $v_1, v_2, ..., v_m$ can have tolerances less than or equal to $t_{m-\sigma(I)}$. The remaining tolerances are selected from $t_{m-\sigma(I)+1}, t_{m-\sigma(I)+2}, ..., t_m$ and so the number of such possibilities is

$\sigma(I) = \frac{I(I+1)}{2} < \frac{(\sqrt[3]{N})(\sqrt[3]{N}+1)}{2} = \frac{2\sqrt[3]{N^2}}{3} + \frac{3\sqrt[3]{N}}{6} - \frac{\sqrt[3]{N^2}}{6} < \frac{2\sqrt[3]{N^2}}{3} + 1$, where the last inequality is a consequence of the fact that $N \geq 19$. It follows that

$m \leq 3 + \frac{4\sqrt[3]{N^2}}{3} + \frac{2\sqrt[3]{N^2}}{3} = 3 + 2\sqrt[3]{N^2}$ (actually, for large values of N our bound is more like $\frac{11\sqrt[3]{N^2}}{6}$). By construction, $2m + 1 \geq n$ and the result follows. ❑

Finally, we rephrase the result of Theorem 4.8 in terms of a lower bound on k.

Corollary 4.9. If k is the abdiff-tolerance competition number of $K_{1,n}$, for $n \geq 18$, then $k \geq \frac{1}{8}\left(\sqrt{n}-7\right)^3(1-\varepsilon)$ where ε goes to 0 as n grows large.

Proof: By Theorem 4.8 and the definition of N we have $n \le 4\sqrt[3]{(n+k+1)^2} + 7$. Solving for k yields $k \ge \frac{1}{8}(n-7)^{3/2} - (n+1) = \frac{(n-7)^{3/2}}{8}\left(1 - \frac{8(n+1)}{(n-7)^{3/2}}\right)$ and the result follows by letting ε equal $\frac{8(n+1)}{(n-7)^{3/2}}$. ❑

Corollary 4.9 shows that the abdiff-tolerance competition number k is bounded below asymptotically by $c_1 n^{\frac{3}{2}}$ for some positive constant c_1.

5. An Asymptotic Upper Bound for the Abdiff-Tolerance Competition Number of $K_{1,n}$

In this section we show that the abdiff-tolerance competition number k of $K_{1,n}$ is bounded above asymptotically by $c_2 n^{\frac{3}{2}}$ for some positive constant c_2. Combining this with the result of Section 4 shows that $n^{\frac{3}{2}}$ is indeed the correct asymptotic order for this competition number. Our goal is achieved by defining, for a fixed value of N, a collection of n' vertices, a tolerance for each vertex, and an assignment of the vertices to a collection of N sets such that the sets form an abdiff-T-ECC of $K_{1,n'}$. The value of n' will be so large that it will be possible to derive the desired upper bound on k. Observe that if we have m vertices of positive tolerance placed in the sets of an abdiff-T-ECC, we can also have m vertices of negative tolerance. This can be achieved by creating, for each vertex u of positive tolerance t_u, a vertex u' of negative tolerance $-t_u$, and placing u' in every set which contains u. We can also add a vertex of tolerance 0. Thus we may assume n' = 2m + 1, and we will describe the proper assignment of sets and tolerances for the m vertices of positive tolerance.

The N sets are labeled $S_1, S_2, \ldots, S_N$. The m vertices with positive tolerance are identified as v_{ij} with $1 \le j \le i$. Vertex v_{ij} is assigned tolerance $t_{ij} = \left\lceil \frac{N}{i} \right\rceil - \frac{(i-1)i}{2} - j + 1$. The index i ranges from 1 to the largest value such that $\left\lceil \frac{N}{i} \right\rceil - \frac{(i-1)i}{2} - i + 1 > 0$ (the largest i such that $t_{ii} > 0$). Finally, vertex v_{ij} is placed in set S_k if and only if $k = j + is$, for $0 \le s \le t_{ij} - 1$. The following facts follow easily from these definitions.

Observation 5.1. The assignments described above lead to the following results:

(a) vertex v_{ij} is in exactly t_{ij} sets by definition and the fact that $t_{ij} < \left\lceil \frac{N}{i} \right\rceil$ when i > 1,

(b) $t_{ii} > t_{i+1,1}$ so the t_{ij} form a strictly decreasing sequence when viewed in the lexicographical ordering implied by the subscripts, and

(c) vertices v_{ij} and $v_{ij'}$, $j \neq j'$, are together in no sets.

The following sequence of lemmas prepares the way for the main theorem by establishing that v_{ij} and $v_{i-k,j'}$, $k \geq 1$, are together in fewer sets than $|t_{ij} - t_{i-k,j'}|$. The first two of the lemmas are technical in nature. The greatest common divisor of positive integers s and t will be denoted gcd(s, t).

Lemma 5.2. If p is an integer such that S_p contains $v_{i,j}$ and $v_{i-k,j'}$, then there exists an integer r such that $p = j + ir$ and $j' = (j + ir) \bmod (i-k)$.

Proof: By construction, since $v_{i,j}$ is in S_p, there exists an integer r such that $p = j + ir$. Similarly, there exists an integer r' such that $p = j' + (i-k)r'$. Hence, $j' = j + ir - (i-k)r' = (j + ir) \bmod (i-k)$. ❑

Lemma 5.3. Let $d = \gcd(i-k, k)$. For any two integers r and s,

$(j + ir) = (j + is) \bmod (i-k)$ if and only if $r = s \bmod \left(\frac{i-k}{d}\right)$.

Proof: Suppose $(j + ir) = (j + is) \bmod (i-k)$. Then $(r - s)i = 0 \bmod (i-k)$. We have that $d \mid (i - k)$ and $d \mid k$, so $d \mid i$, meaning $i = td$ for some integer t. Therefore, $(r - s)td = 0 \bmod (i-k)$, so $(r - s)t = 0 \bmod \left(\frac{i-k}{d}\right)$. Let $a = \gcd\left(t, \frac{i-k}{d}\right)$. Then ad divides i-k. Since $i = td$ and a divides t, ad divides i and, therefore, ad divides k. By the choice of d, a = 1. Therefore, $(r - s)t = 0 \bmod \left(\frac{i-k}{d}\right)$ and $\gcd\left(t, \frac{i-k}{d}\right) = 1$. Hence, $r - s = 0 \bmod \left(\frac{i-k}{d}\right)$, so $r = s \bmod \left(\frac{i-k}{d}\right)$.

Conversely, suppose $r = s \bmod \left(\frac{i-k}{d}\right)$. Then $r = s + b\left(\frac{i-k}{d}\right)$ for some integer b. Therefore,

$(j + ir) = \left\{j + i\left[s + b\left(\frac{i-k}{d}\right)\right]\right\} = \left[j + is + ib\left(\frac{i-k}{d}\right)\right]$. It follows, since d divides i, that $j + ir = (j + is) \bmod (i-k)$. ❑

The following lemma places an upper bound on the number of sets containing both v_{ij} and $v_{i-k,j'}$. This is the first step in showing they don't share as many as $|t_{ij} - t_{i-k,j'}|$ sets.

Lemma 5.4. If I is the number of sets containing both $v_{i,j}$ and $v_{i-k,j'}$, where $1 \leq k \leq i - 1$, then $I \leq \left\lceil \frac{t_{ij}}{\left(\frac{i-k}{d}\right)} \right\rceil$, where $d = \gcd(i-k, k)$.

Proof: The result is trivial if I = 0. Otherwise let S_p be the set of smallest index which contains both v_{ij} and $v_{i-k,j'}$. By Lemma 5.2 there is an integer r such that p

$= j + ir$ and $j' = (j + ir) \bmod (i-k)$. Let S_q be an arbitrary set (perhaps S_p) which also contains v_{ij} and $v_{i-k,j'}$. Again, by Lemma 5.2, there exists an integer s such that $q = j + is$ and $j' = (j + is) \bmod (i-k)$. By Lemma 5.3, $r = s \bmod \left(\frac{i-k}{d}\right)$. In other words, $q = j + i\left(r + t\frac{i-k}{d}\right)$ where, by the choice of p, $t \geq 0$. By construction, $0 \leq s < t_{ij}$, so that, $0 \leq r + t\frac{i-k}{d} < t_{ij}$. Hence,

$0 \leq \frac{r}{\left(\frac{i-k}{d}\right)} + t < \frac{t_{ij}}{\left(\frac{i-k}{d}\right)}$. It follows that $0 \leq t < \left\lceil \frac{t_{ij}}{\left(\frac{i-k}{d}\right)} \right\rceil$. But S_q was chosen arbitrarily and, since S_p is fixed, j, i, r, i-k, and d also are fixed, that is, q depends only on t. Therefore, there are at most $\left\lceil \frac{t_{ij}}{\left(\frac{i-k}{d}\right)} \right\rceil$ possible values for q corresponding to the at most $\left\lceil \frac{t_{ij}}{\left(\frac{i-k}{d}\right)} \right\rceil$ possible choices for the value of t. ❑

The next lemma completes the proof that the assigned tolerances behave appropriately by showing that the bound of Lemma 5.4 is smaller than $|t_{ij} - t_{i-k,j'}|$.

Lemma 5.5. For $i > 1$ and $1 \leq k \leq i - 1$, the number of sets containing both v_{ij} and $v_{i-k,j'}$ is less than $|t_{ij} - t_{i-k,j'}|$.

Proof: Let $N = i(i-k)p + s$ where p is a nonnegative integer and $0 \leq s < i(i-k)$. Let I be the number of sets which contain both v_{ij} and $v_{i-k,j'}$. By Lemma 5.4, $I \leq$

$$\left\lceil \frac{t_{ij}}{\left(\frac{i-k}{d}\right)} \right\rceil = \left\lceil \frac{\left\lceil \frac{N}{i} \right\rceil - \frac{(i-1)\,i}{2} - j + 1}{\left(\frac{i-k}{d}\right)} \right\rceil$$ where $d = \gcd(i - k, k)$. Here

$\left\lceil \frac{N}{i} \right\rceil = (i-k)p + \left\lceil \frac{s}{i} \right\rceil$, so

$$I \leq \left\lceil \frac{(i-k)p + \left\lceil \frac{s}{i} \right\rceil - \frac{(i-1)\,i}{2} - j + 1}{\left(\frac{i-k}{d}\right)} \right\rceil = pd + \left\lceil \frac{\left\lceil \frac{s}{i} \right\rceil - \frac{(i-1)\,i}{2} - j + 1}{\left(\frac{i-k}{d}\right)} \right\rceil.$$ Examining the last term and using $s < i(i - k)$, we see

$$\left\lceil \frac{\left\lceil \frac{s}{i} \right\rceil - \frac{(i-1)\,i}{2} - j + 1}{\left(\frac{i-k}{d}\right)} \right\rceil \leq \left\lceil d - \frac{\frac{(i-1)\,i}{2}}{\left(\frac{i-k}{d}\right)} - \frac{j-1}{\left(\frac{i-k}{d}\right)} \right\rceil \leq \left\lceil d - \frac{\frac{(i-1)\,i}{2}}{i-k} \right\rceil$$

$\leq \left\lceil d - \frac{\frac{(i-1)\,i}{2}}{i-1} \right\rceil = \left\lceil d - \frac{i}{2} \right\rceil \leq d - 1$ since $i \geq 2$. Consequently, $I \leq pd + d - 1$. For the difference of the tolerances, we compute

$$t_{i-k,j'} - t_{ij} = \left\lceil \frac{N}{i-k} \right\rceil - \frac{(i-k-1)(i-k)}{2} - (j'-1) - \left\lceil \frac{N}{i} \right\rceil + \frac{(i-1)i}{2} + (j-1)$$

$\geq \left\lceil \frac{N}{i-k} \right\rceil - \left\lceil \frac{N}{i} \right\rceil + (i-1) + 1 - 1 + j - j'$ since $k \geq 1$. By construction, $j \geq 1$ and $j' \leq i-k$, so that $t_{i-k,j'} - t_{ij} \geq \left\lceil \frac{N}{i-k} \right\rceil - \left\lceil \frac{N}{i} \right\rceil + k$. Also $\left\lceil \frac{N}{i-k} \right\rceil = ip + \left\lceil \frac{s}{i-k} \right\rceil$ and $\left\lceil \frac{N}{i} \right\rceil = (i-k)p + \left\lceil \frac{s}{i} \right\rceil$, and, by substituting, we have

$t_{i-k,j'} - t_{ij} \geq kp + \left\lceil \frac{s}{i-k} \right\rceil - \left\lceil \frac{s}{i} \right\rceil + k \geq kp + k$. However, $d = \gcd(i-k, k)$, so d divides k which implies $d \leq k$. Therefore $kp + k \geq dp + d > dp + d - 1$ which completes the proof. ❑

It follows from the preceding that the defined sets S_i form an abdiff-T-ECC for the given vertices. We are now in a position to prove the main theorem of this section.

Theorem 5.6. If k is the abdiff-tolerance competition number of $K_{1,n}$, then $k < c\,n^{3/2}$ where c is a positive constant independent of n.

Proof: Let m be the number of vertices v_{ij} created as described in the beginning of this section and let M be the largest value of i used, that is, $t_{M,M} > 0$ but $t_{M+1,M+1} \leq 0$. Using the definition of t_{ij}, we have

$t_{M+1,M+1} = \left\lceil \frac{N}{M+1} \right\rceil - \frac{M(M+1)}{2} - M \leq 0$. Therefore, $M^3 + 4M^2 + 3M \geq 2N$ and so $M^3 > c_1 N$ for an appropriately chosen c_1. Thus, $M > c_2 N^{1/3}$. In our construction there are exactly i vertices with i as the first index, so the total number of vertices of positive tolerance is computed as

$m = \sum_{i=1}^{M} i = \frac{M(M+1)}{2} > c_3 N^{2/3}$. Thus, n, which is at least $n' = 2m + 1$, obeys the relation $n \geq 2m + 1 > c_4 N^{2/3}$ or $n^{3/2} > c_5 N$. Since $N = n + k + 1$, we have $c_5 k < n^{3/2} - c_5 n - c_5$ or $k < c\,n^{3/2}$ for an appropriately chosen c. ❑

6. References

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Appendix

The following lists the tolerances and sets which show $K_{1,n}$ is an abdiff-tolerance competition graph for $n \leq 17$ and $n = 19$. The single vertex in one bipartite set is given tolerance 0. For vertices in the other bipartite set, only those with positive tolerances are shown. In addition, there is always one vertex of tolerance 0 and it never needs to be in a set. If n is odd, there are the same number of vertices with negative tolerances as there are with positive tolerances, and then those negative tolerances are the negative of the positive tolerances. Furthermore, a vertex with a negative tolerance will be placed in the same sets as contain the vertex with the corresponding positive tolerance. If n is even, there is one fewer vertex with negative tolerance. In view of this, if we show sets for n even, the same sets will work for the succeeding odd value n + 1, so we do not list them. The tolerances for the vertices 1 to $\lfloor n/2 \rfloor$ of positive tolerance are given as a vector $(t_1, t_2, \ldots, t_{\lfloor n/2 \rfloor})$.

n = 1: No sets required

n = 2: **(1)**
Set 1: 1

n = 4: **(1,2)**
Set 1: 1
Set 2: 2
Set 3: 2

n = 6: **(1,3,7)**
Set 1: 1,2,3
Set 2: 2,3
Set 3: 2,3
Set 4: 3
Set 5: 3
Set 6: 3
Set 7: 3

n = 8: **(1,2,4,9)**
Set 1: 1,3,4
Set 2: 2,3,4
Set 3: 3,4
Set 4: 3,4
Set 5: 2,4
Set 6: 4
Set 7: 4
Set 8: 4
Set 9: 4

n = 10: **(1,2,3,5,11)**
Set 1: 1,3,4,5
Set 2: 2,4,5
Set 3: 2,4,5
Set 4: 4,5
Set 5: 4,5
Set 6: 3,5
Set 7: 3,5
Set 8: 5
Set 9: 5
Set 10: 5
Set 11: 5

n = 12: **(1,2,3,4,6,13)**
Set 1: 1,3,5,6
Set 2: 2,4,5,6
Set 3: 3,5,6
Set 4: 2,5,6
Set 5: 5,6
Set 6: 5,6
Set 7: 4,6
Set 8: 4,6
Set 9: 4,6
Set 10: 3,6
Set 11: 6
Set 12: 6
Set 13: 6

n = 14: **(1,2,3,4,5,7,15)**
Set 1: 1,3,5,6,7
Set 2: 2,4,6,7
Set 3: 4,6,7
Set 4: 3,6,7
Set 5: 3,6,7
Set 6: 2,6,7
Set 7: 6,7
Set 8: 5,7
Set 9: 5,7
Set 10: 5,7
Set 11: 5,7
Set 12: 4,7
Set 13: 4,7
Set 14: 7
Set 15: 7

n = 16: **(1,2,3,4,5,6,8,17)**
Set 1: 1,3,5,7,8
Set 2: 2,4,6,7,8
Set 3: 2,5,7,8
Set 4: 4,7,8
Set 5: 4,7,8
Set 6: 3,7,8
Set 7: 3,7,8
Set 8: 7,8
Set 9: 6,8
Set 10: 6,8
Set 11: 6,8
Set 12: 6,8
Set 13: 6,8
Set 14: 5,8
Set 15: 5,8
Set 16: 5,8
Set 17: 4,8

n = 19:
(1,2,3,4,5,6,7,9,20)
Set 1: 4,7,8,9
Set 2: 3,6,8,9
Set 3: 3,6,8,9
Set 4: 3,5,8,9
Set 5: 2,5,8,9
Set 6: 2,5,8,9
Set 7: 1.4,8,9
Set 8: 4,8,9
Set 9: 4,8,9
Set 10: 5,7,9
Set 11: 7,9
Set 12: 7,9
Set 13: 7,9
Set 14: 7,9
Set 15: 7,9
Set 16: 6,9
Set 17: 6,9
Set 18: 6,9
Set 19: 6,9
Set 20: 5,9